Finding zero of a monotone operator that admits splitting into two or three monotone operators, i.e., we want to find an $x$ such that $0 \in (A+B)x$ or $0 \in (A+B+C)x$, where $A$, $B$, and $C$ are maximal monotone operators.

The key idea: Transform the problem into a fixed point equation. In the fixed point equation, we are interested in finding the fixed point of a resolvent and/or Cayley operator of the maximal monotone operators. Though in theory such iterations converge to the fixed point, it is useful only when computing the resolvent and/or Cayley operator is efficient.

Forward-Backward Splitting:

We want to find an $x$ such that

\[ x = (I - \alpha A)R(I - \alpha A)x. \]

How do we know that this iteration will converge?

Suppose,

- $A$ is a subdifferential operator with Lipschitz parameter $L$, and $\alpha \in (0, 2/L]$, or
- $A$ is strongly monotone and Lipschitz with parameter $m$, $L$ with $\alpha \in (0, 2m/L^2)$

Then forward step $(I - \alpha A)$: averaged and backward step $(I + \alpha B)^{-1}$: averaged.

Explanation: note that as $B$ is maximal monotone, $R$ is nonexpansive, so

\[ (I + \alpha B)^{-1} \text{ is nonexpansive.} \]

So the averaged operator convergence proof where we show that the damped iteration will converge, will not work.

Condition of convergence:

Forward-backward-forward splitting will converge when $A$: (Lipschitz with parameter $L$, $\alpha \in (0, 1/L)$)

# Note that in Forward-backward we required strongly monotone $A$ and Lipschitz, so in FBF we have better assumption.

Convergence Proof: May be later...

Example:

Extragradient method:

Consider finding the zero of a maximal monotone function $A$, i.e., $A(x) = 0$, we can write this as $A(x) + 0x = 0$ so,

We have $B = 0$ so resolvent is $R = (I + \alpha 0) = I$,

\[ x = R(x - \alpha A x) = x - \alpha A x. \]

Which converges when $A$ is Lipschitz with parameter $L$, and $\alpha \in (0, 1/L)$

So by using Ptseng's method we have:

\[ x = \frac{1}{1+\alpha} x + \frac{\alpha}{1+\alpha} R(x - \alpha A x) = \frac{1}{1+\alpha} x + \frac{\alpha}{1+\alpha} x - \frac{\alpha}{1+\alpha} A x. \]

In these cases it can be shown that $\frac{1}{1+\alpha} I + \frac{\alpha}{1+\alpha} A$ is an averaged operator # More explanation needed.

For details see Operator splitting, Douglas-Rachford splitting.
why maximality is needed?

# composition of nonexpansive (contraction) mapping is nonexpansive

\[ \text{statement: overloaded sum operator for relations has additivity} \]

\[ \text{distributive property of relations} \]

As a mnemonic (unverified), in a composite relation, the constituent relation $\&$ $\&$ $\&$ $\&$ $\&$ function, tader jonno ukto equation e functional operation chalano jabe $\lambda \in F(x) \implies \lambda \in \lambda \in F(x)$

\[ \text{Proof Strategy: At first we will prove R.H.S} \implies \text{M.H.S, then we will prove M.H.S} \implies \text{L.H.S}. \text{This is the computationally} \]

important part, as it tells us that if we solve $C(x)=z$, $x=R(z)$, then $x$ will be a zero of the maximal monotone relation $F=A+B$.

\[ \text{Proof of R.H.S} \implies \text{M.H.S} \]

\begin{align*}
\textbf{Thm: main theorem behind operator splitting} & \\
& \text{Proof Strategy: At first we will prove R.H.S} \implies \text{M.H.S, then we will prove M.H.S} \implies \text{L.H.S}. \text{This is the computationally} \\
& \text{important part, as it tells us that if we solve } C(x)=z, \text{ then } x=R(z), \text{ then } x \text{ will be a zero of the maximal monotone relation } F=A+B.
\end{align*}
Douglas-Rachford splitting

- Equivalent to many other optimization algorithms
- Equivalent to normal cone operator
- Equivalent to list of projections on simple convex sets
- Equivalent to proximal operator is the resolvent of subdifferential operator

ADMM by Douglas-Rachford splitting

- Resolvent of normal cone operator
- List of projections on simple convex sets

\[ \text{main theorem behind operator splitting} \]

\[ \text{proximal operator is the resolvent of subdifferential operator} \]

\[ \text{ADMM by Douglas-Rachford splitting} \]

\[ \text{normal cone operator} \]

\[ \text{Resolvent of normal cone operator} \]

\[ \text{List of projections on simple convex sets} \]