On seeking efficient Pareto optimal points in multi-player minimum cost flow problems

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Outline

- what happens if the **minimum cost flow problem** is extended to a multi-player setup
- what is a good solution concept in such a multi-player setup
  ⇒ **efficient Pareto optimal point**
- **under what conditions** these good solutions exist
  - one version always exists in any network
  - existence of a stricter version can be checked
- how to **compute** such solutions
E-commerce product transportation systems are dominated by Amazon, Walmart, Alibaba...

Amazon uses FedEx, UPS, AAR and other competing organizations for transportation services

⇒ multi-player minimum cost flow problem
Minimum cost flow problem

- Directed connected graph with nodes and arcs
- Integer-valued flow of some material on each arc
- Each arc incurs a cost
- Minimize the total cost of all flows subject to the network constraints
Minimum cost flow problem

\[
\text{minimize}_x \sum_{i \in \{1, \ldots, n\}} f_i(x_i) \quad \text{\* default: } f_i(x_i) = c_i x_i \quad \text{\*}
\]

subject to \( Ax = b \quad \text{\* flow conservation constraint \*} \)
\[
0 \leq x \leq u \quad \text{\* flow bound constraint \*}
\]
\[
x \in \mathbb{Z}^n \quad \text{\* flow is integer \*}
\]

▶ The network has \( n \) arcs, \( m + 1 \) nodes
▶ \( A \) : reduced node-arc incidence matrix, dimension \( m \times n \)
▶ \( b \) : represents supplies/demands
▶ \( u \) : upper bound on flow
Meaning of the constraints

- **The flow conservation constraint:** for any node, the outflow minus inflow must equal the supply/demand of the node
  - must be maintained

- **The flow bound constraint:** imposes direction and capacity limit on the flow
  - can often be relaxed or omitted in practice
  - in the relaxed case, flow direction is flexible and overflow is allowed
A multi-player extension

- With each arc of the network graph we associate one player
- Each player tries to minimize its nonconvex cost function, subject to the network flow constraints
- Our goal is to seek a good solution concept in this multi-player problem
The goal of the $i$th player for $i = 1, \ldots, n$, given other players’ strategies $x_{-i} \in \mathbb{Z}^{n-1}$, is to solve:

\[
\begin{align*}
\text{minimize}_{x_i} & \quad f_i(x_i) \quad \text{\(\text{\* nonconvex \*}\)} \\
\text{subject to} & \quad A(x_i, x_{-i}) = b \quad \text{\(\text{\* constraints\*}\)} \\
& \quad 0 \preceq (x_i, x_{-i}) \preceq u \quad \text{couple the players \(\text{\*}\)} \\
& \quad x \in \mathbb{Z}^n.
\end{align*}
\]
Back to the application

- **Arcs**: transportation links
- **Nodes**:
  - **Supply nodes**: warehouses
  - **Demand nodes**: retail centers
- **Players**: transportation organizations (FedEx, UPS, AAR etc)
- **Flow**: products transported
- **Each player’s goal**:
  - maximize its profit (nonlinear)
  - only controls its own flow
Solution concepts

- A **solution to the optimization problem** would always favor the dominant player ignoring the rest.

- A **vector optimal solution** is the best social solution:
  - It minimizes all the objectives simultaneously
  - **Problem:** violates flow conservation

- The celebrated **Nash equilibrium** is also not very efficient in our setup.

- A better solution solution concept is the **Pareto optimal point**.
Pareto optimal point

**Pareto optimal point:** none of the cost functions can be reduced without increasing some other cost function.

A “feasible” point $x_{\text{Pareto}}$ is Pareto optimal if it satisfies the following: there does not exist another “feasible” point $x$ such that for all $i = 1, \ldots, n$

$$f_i(x_i) \leq f_i(x_{i\text{Pareto}}),$$

with at least one index $j$ satisfying $f_j(x_j) < f_j(x_{j\text{Pareto}})$.

**Problem:** There can be numerous such generic Pareto optimal points, some poor in quality or efficiency.
Efficient Pareto optimal point

- Finds a balance between *vector optimality* and the *generic Pareto optimality*
- It is a Pareto optimal point where
  - the maximum possible number of players minimize their cost functions simultaneously
  - flow is conserved
  - flow bound is maintained for the maximum possible number of arcs
The main result

For any multi-player minimum cost flow problem, there exists one efficient Pareto optimal point such that

- it is **Pareto optimal** and **$n - m$ vector optimal**
  - out of $n$ players, $n - m$ will minimize their cost functions simultaneously
  - the set of $n - m$ vector optimality is maximal (it cannot be made any larger)

- the flow conservation constraints are maintained

- at least $n - m$ of the flow bound constraints are maintained (possibly all)
An existence result

- Out of $n$ flow bound constraints, $m$ of them may not be maintained
- Can we check in advance if all of the flow bound constraints are maintained?
- We provide an existence theorem using algebraic geometry

Why need algebraic geometry?
The flow bound constraints over integers can be formulated as polynomials.
Some necessary concepts

- **The ideal** generated by polynomials \( f_1, f_2, \ldots, f_m \) is the set

\[
\text{ideal}\{f_1, \ldots, f_m\} = \left\{ \sum_{i=1}^{m} h_i f_i \mid h_1, \ldots, h_m \text{ are polynomials} \right\}.
\]

- analogous to **span of vectors**

- **Given an ideal** \( I \), **affine variety** is the set

\[
\text{variety}(I) = \{ x \mid f(x) = 0, \text{ for all } f \text{ in } I \}
\]

- analogous to **null-space of a matrix**

- **A Groebner basis** \( G \) is particular kind of generating set of an ideal \( I \)

- analogous to **basis of a span**

- **Reduced Groebner basis** \( G_{\text{reduced}} \) is the most compact Groebner basis for an ideal \( I \)

- analogous to **orthonormal basis** of a span
Statement of the existence theorem

From the structure of the network (after some pre-calculation), we can generate polynomials

\[ q_1, \ldots, q_m, r_1, \ldots, r_{n-m} \]

Exactly one of the following holds:
(a) There exists an efficient Pareto optimal point, where all the flow bound constraints are maintained.
(b) We have

\[ G_{\text{reduced}} = \{1\}, \]

where \( G_{\text{reduced}} \) is the reduced Groebner basis of ideal \( \{q_1, \ldots, q_m, r_1, \ldots, r_{n-m}\} \).

There are many computer algebra packages (Maple, Mathematica, FGb) that can compute \( G_{\text{reduced}} \)
Computing efficient Pareto optimal point

We propose an algorithm to compute efficient Pareto optimal points in two stages:

- **Stage 1:** We compute a larger set $\mathcal{F}$, that contains all the efficient Pareto optimal points
- **Stage 2:** From $\mathcal{F}$, we compute efficient Pareto optimal points using algebraic elimination theory
  - Need to solve only single-variable optimization problems
Stage 1

- For $i = 1, 2, \ldots, n - m$ calculate
  - Compute
    
    $$G_{n-m-i} = G_{\text{reduced}} \cap \mathbb{C}[z_{n-m-i+1}, z_{n-m-i+2}, \ldots, z_{n-m}]$$
  - Compute $\text{variety}(G_{n-m-i})$

- Each $G_{n-m-i}$ results in a single-variable polynomial system

- Finding $\text{variety}(G_{n-m-i})$ is just finding the roots of a single variable polynomial

- $\mathcal{F} = \text{variety}(G_0)$
Stage 2

\[
\text{for } i = 1, \ldots, m \\
X_i := d_i - h_i^T F \\
\text{end for} \\
\text{end if}
\]

/* The inverse operator is denoted } X_i^{-1} */

/* We can compute } d_i \text{ and } h_i \text{ from } A \text{ and } u */

Sort the elements of the } \{X_i\}_{i=1}^{m} \text{ with respect to cardinality of the elements in a descending order.}

Denote the index set of the sorted set by } \{s_1, \ldots, s_m\} \text{ such that } |X|_{s_1} \geq \cdots \geq |X|_{s_m}.

\[
\text{for } i = 1, \ldots, m \\
X^*_{s_i} := \arg\min_{x_{s_i} \in X_{s_i}} f_{s_i}(x_{s_i}) \quad \text{ /* Easy to solve single-variable optimization problem */} \\
F^*_{s_i} := \bigcup_{x_{s_i} \in X^*_{s_i}} (X^*)^{-1}(x_{s_i}) \quad \text{ /* Lemma. The set } F^*_{s_i} \text{ is nonempty */} \\
\text{if } i \leq m \\
X_{s_i+1} := \{d_{s_i+1} - h_{s_i+1}^T z \mid z \in F^*_{s_i}\}.
\text{end if}
\]

end for

return } F^*_{s_m} \quad \text{ /* Theorem. Any member of } F^*_{s_m} \text{ is an efficient Pareto optimal point */}
Concluding Remarks

- **Multi-player minimum cost flow problem**: a natural extension of the minimum cost flow problem
- **Efficient Pareto optimal point** is a desirable solution concept
  - A soft version always exists
  - A strict version can exist: existence can be checked
- **Algorithms** to compute efficient Pareto optimal points
Paper available at

- https://shuvomoy.github.io/site/

Thank You!

Questions?
Numerical example

A multi-player transportation problem
- 5 nodes, 16 arcs
- Nodes 2, 4 are retail centers
- Nodes 1, 3 are warehouses
- Node 5 transshipment node
- Products need to be shipped from warehouses to retail centers
- Arcs represent different modes of shipment from one node to another and carried out by different organization
- Cost of a certain type of shipment depends on the number of products shipped

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\
\end{pmatrix}
\]

\[
b = (9, -13, 15, -11), \quad u = (5, 6, 6, 10, 10, 7, 11, 13, 16, 12, 4, 5, 6, 14, 13, 15)
\]
<table>
<thead>
<tr>
<th>Player</th>
<th>Cost function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{x^4}{30} - \frac{13x^3}{15} + \frac{259x^2}{30} - \frac{263x}{15} + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{77x^5}{120} - \frac{247x^4}{24} + \frac{471x^3}{8} - \frac{3365x^2}{24} + \frac{6779x}{60} + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{47x^4}{24} - \frac{133x^3}{4} + \frac{4897x^2}{24} - \frac{2123x}{4} + 485$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{323x^5}{3360} - \frac{2179x^4}{1120} + \frac{47393x^3}{3360} - \frac{48709x^2}{1120} + \frac{7885x}{168} + 5$</td>
</tr>
<tr>
<td>5</td>
<td>$(x_5 - 1)^2$</td>
</tr>
<tr>
<td>6</td>
<td>$-\frac{x^4}{8} + \frac{25x^3}{12} - \frac{71x^2}{8} + \frac{95x}{12} + 10$</td>
</tr>
<tr>
<td>7</td>
<td>$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{11x^7}{1260} - \frac{7x^6}{36} + \frac{119x^5}{72} - \frac{479x^4}{8} + \frac{4609x^3}{360} - \frac{803x^2}{72} + \frac{155x}{28} + 1$</td>
</tr>
<tr>
<td>9</td>
<td>$-\frac{15}{16}x_9^3 + \frac{365x_9^2}{16} - \frac{2865x_9}{16} + \frac{7315}{16}$</td>
</tr>
<tr>
<td>10</td>
<td>$(x_{10} - 10)^2$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{5x_{11}^4}{6} - \frac{35x_{11}^3}{3} + \frac{355x_{11}^2}{6} - \frac{370x_{11}}{3} + 90$</td>
</tr>
<tr>
<td>12</td>
<td>$\frac{5x_{12}^4}{6} - \frac{25x_{12}^3}{3} + \frac{175x_{12}^2}{6} - \frac{110x_{12}}{3} + 15$</td>
</tr>
<tr>
<td>13</td>
<td>$\frac{5x_{13}^4}{6} - \frac{15x_{13}^3}{3} + \frac{595x_{13}^2}{6} - 280x_{13} + 285$</td>
</tr>
<tr>
<td>14</td>
<td>$\frac{5x_{14}^4}{6} - \frac{85x_{14}^3}{3} + \frac{2155x_{14}^2}{6} - \frac{6020x_{14}}{3} + 4165$</td>
</tr>
<tr>
<td>15</td>
<td>$</td>
</tr>
<tr>
<td>16</td>
<td>$\begin{cases} x_{16} + 1, &amp; \text{if } 0 \leq x_{16} \leq 3 \ 0, &amp; \text{if } 4 \leq x_{16} \leq 6 \ (x_{16} + 1)^3, &amp; \text{if } 7 \leq x_{16} \leq 9 \ -\frac{x_{16}^3}{6} + \frac{13x_{16}^2}{2} - \frac{244x_{16}}{3} + 330, &amp; \text{else} \end{cases}$</td>
</tr>
</tbody>
</table>
Pareto Optimal Solutions

Our algorithm provides two efficient Pareto optimal points:

\[(1, 3, 5, 4, 11, 10, 2, 1, 3, 7, 7, 5)\]

and

\[(1, 3, 5, 6, 11, 10, 2, 1, 3, 7, 7, 6)\].