Convergence of Nonconvex Douglas-Rachford Splitting and Nonconvex ADMM

Shuvomoy Das Gupta

Thales Canada

56th Allerton Conference on Communication, Control and Computing, 2018

What is this talk about?

this talk is about ADMM and Douglas-Rachford splitting for nonconvex problems

- the alternating direction method of multipliers (ADMM)
  - originally designed to solve convex optimization problem
- Douglas-Rachford splitting algorithm
  - ADMM is its special case in a convex setup
- both guaranteed to converge for convex problems

Motivation

- nonconvex ADMM (**NC-ADMM**) has become a popular heuristic to tackle nonconvex problems
- recently, NC-ADMM heuristic has been applied to [Erseghe, 2014] optimal power flow problem, [Takapoui *et al.*, 2017] mixed integer quadratic optimization, [Iyer *et al.*, 2014] submodular minimization with nonconvex constraints ...
- [Diamond *et al.*, 2018] Python package `NCVX` (extension of `CVXPY`) implements ADMM heuristic (NC-ADMM)
  - often produces lower objective values compared with exact solvers within a time limit
- nonconvex Douglas-Rachford splitting (**NC-DRS**): analogous nonconvex heuristic based on Douglas-Rachford splitting
- not much has been done to improve the theoretical understanding of such heuristics

Summary of the results

- NC-DRS
  - attacks the original problem directly
  - optimal solutions can be characterized via the NC-DRS operator
  - if deviation from a convex setup is bounded $\Rightarrow$ it will converge or oscillate in a compact connected set
- NC-ADMM
  - works on a modified dual problem, not the original nonconvex problem
  - not equivalent to NC-DRS, but there is a relationship between them
  - likely to produce a lower objective value

Problem in consideration

- minimize a convex cost function with nonconvex constraint set

$$\begin{array}{ll} \text{minimize}_x & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array} \qquad \text{(OPT)}$$

- f is closed, proper, and convex
- $\mathcal{C}$ is compact, but not necessarily convex

Reformulation through indicator function

- indicator function of set $\mathcal{C}$:

$$\delta_{\mathcal{C}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{C} \\ \infty, & \text{if } x \notin \mathcal{C} \end{cases}$$

- $\delta_{\mathcal{C}}$ is closed and proper, but not necessarily convex
- we can write

$$\left(\begin{array}{ll} \text{minimize}_x & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array} \right) = \text{minimize}_x \ f(x) + \delta_{\mathcal{C}}(x)$$

Proximal operator of f and projection onto $\mathcal{C}$

- both NC-DRS and NC-ADMM have same subroutines: first $\mathbf{prox}_{\gamma f}$, then $\tilde{\mathbf{\Pi}}_{\mathcal{C}}$ and finally $\sum$
- proximal operator of f evaluated at point x with parameter $\gamma > 0$:

$$\mathbf{prox}_{\gamma f}(x) = \mathrm{argmin}_y \left(f(y) + \frac{1}{2\gamma}\|y - x\|^2\right)$$

  - single-valued, continuous

- projection onto $\mathcal{C}$:

$$\mathbf{prox}_{\gamma \delta_{\mathcal{C}}}(x) = \mathbf{\Pi}_{\mathcal{C}}(x) = \mathrm{argmin}_{y \in \mathcal{C}} \left(\|y - x\|^2\right)$$

  - there can be multiple projections
  - one such projection is denoted by $\tilde{\mathbf{\Pi}}_{\mathcal{C}}(\cdot)$

NC-ADMM and NC-DRS:

- NC-ADMM:

$$x_{n+1} = \mathbf{prox}_{\gamma f}(y_n - z_n)$$
$$y_{n+1} = \tilde{\mathbf{\Pi}}_{\mathcal{C}}(x_{n+1} + z_n)$$
$$z_{n+1} = z_n - y_{n+1} + x_{n+1}$$

- NC-DRS:

$$x_{n+1} = \mathbf{prox}_{\gamma f}(z_n)$$
$$y_{n+1} = \tilde{\mathbf{\Pi}}_{\mathcal{C}}(2x_{n+1} - z_n)$$
$$z_{n+1} = z_n + y_{n+1} - x_{n+1}$$

- both have same subroutines, but different inputs

Pretend $\mathcal{C}$ is convex

- f is closed, proper, convex
- pretend $\mathcal{C}$ is convex $\Rightarrow x_n, y_n$ converge to an optimal solution for any initial condition
- but $\mathcal{C}$ is not necessarily convex in our setup
  - convergence conditions are messy

Why are convergence conditions messy?

the convergence conditions are messy because:

- subdifferential operator of $\delta_{\mathcal{C}}$ is *monotone*, but ***not*** *maximally monotone*
- $\Rightarrow \tilde{\mathbf{\Pi}}_{\mathcal{C}}$: is *expansive i.e.,* ***not*** *nonexpansive*
- $\Rightarrow$ the underlying *reflection operator* is *expansive*

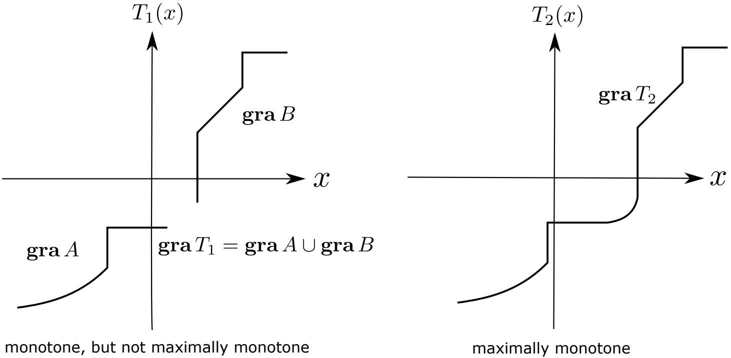
Little bit of review...

Monotone and maximally monotone operators

- T is monotone if for every $(x,u),(y,v) \in \mathbf{gra}T$

$$\langle x - y \mid u - v \rangle \geq 0$$

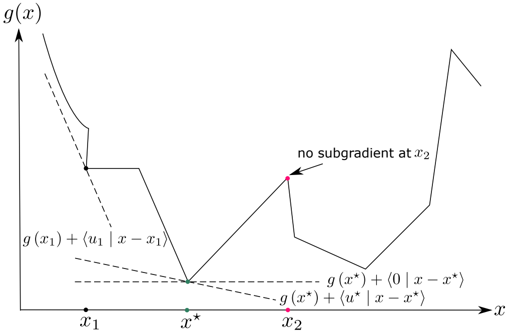
- T is maximally monotone if $\mathbf{gra}\,T$ is not properly contained by any other monotone operator's graph



Subdifferential operator for a nonconvex function

- g: closed, proper, but not necessarily convex
- ∂g: subdifferential of g is monotone, but not maximally monotone

$$\partial g(x) = \{u \in \mathbf{R}^n \mid (\forall y \in \mathbf{R}^n)\ g(y) \geq g(x) + \langle u \mid y - x \rangle\}$$



Why are convergence conditions messy?

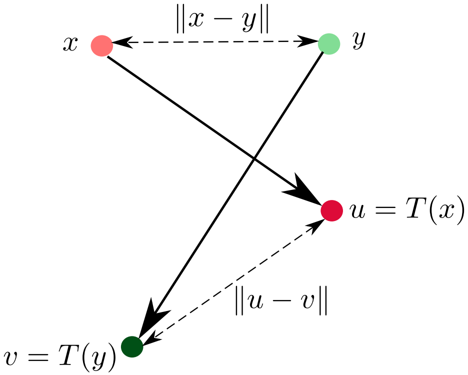
- our problem: $\mathrm{minimize}_x\, f(x) + \delta_{\mathcal{C}}(x)$
- ∂f: maximally monotone
- $\partial \delta_{\mathcal{C}}$: monotone, but **not** maximally monotone

  $\Rightarrow \tilde{\mathbf{\Pi}}_{\mathcal{C}}$ is *expansive (**not** nonexpansive)*
- What is a nonexpansive operator?

What is a nonexpansive operator?

T : single-valued operator on $\mathbf{R}^n$

- T is nonexpansive on $\mathbf{R}^n$ if for every x, y we have $\|T(x) - T(y)\| \leq \|x - y\|$



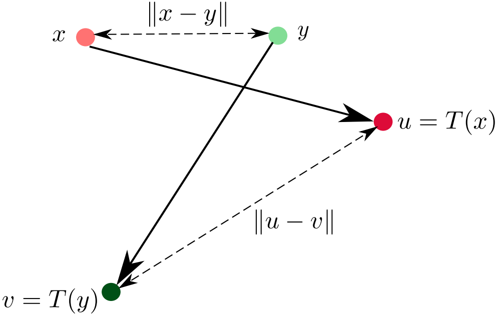
- $\mathbf{prox}_{\gamma f}$ is nonexpansive
- $\left(2\mathbf{prox}_{\gamma f} - I_n\right)$ is nonexpansive

Operators that are expansive

T is a single-valued operator on $\mathbf{R}^n$

- T is expansive if there exist x, y such that

$$\|T(x) - T(y)\| > \|x - y\|$$



- $\tilde{\mathbf{\Pi}}_{\mathcal{C}}$ is expansive

Characterization of minimizers: NC-DRS operator and its reflection

- $\tilde{T}$: NC-DRS operator

$$\tilde{T} = \tilde{\mathbf{\Pi}}_{\mathcal{C}}\left(2\mathbf{prox}_{\gamma f} - I_n\right) + I_n - \mathbf{prox}_{\gamma f}.$$

- $\tilde{R}$: reflection operator of $\tilde{T}$

$$\tilde{R} = 2\tilde{T} - I_n$$

- NC-DRS in compact form:

$$\boxed{z_{n+1} = \tilde{T} z_n = \frac{1}{2}\left(\tilde{R} + I_n\right) z_n}$$

- $\tilde{\mathbf{\Pi}}_{\mathcal{C}}$: expansive $\Rightarrow$ $\tilde{R}$: expansive $\Rightarrow$ root of all convergence issues

Characterization of minimizers

- $\mathrm{argmin}\,(f+\delta_{\mathcal{C}})$ is the set of minimizers of $\min._x\ f(x)+\delta_{\mathcal{C}}(x)$

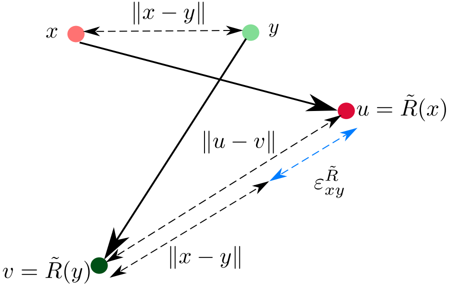
$$\mathbf{prox}_{\gamma f}(\mathbf{fix}\,\tilde{T}) \subseteq \mathrm{argmin}\,(f+\delta_{\mathcal{C}})$$

- underlying assumptions:
  1. $\mathbf{zer}\left(\partial f+\partial\delta_{\mathcal{C}}\right)$ is nonempty
  2. $\mathbf{fix}\,\tilde{T}$ is nonempty
  3. $\mathbf{fix}\left\{\left(2\mathbf{\Pi}_{\mathcal{C}}-I_n\right)\left(2\mathbf{prox}_{\gamma f}-I_n\right)\right\}$ is nonempty

Convergence of NC-DRS: setup

- $\varepsilon_{xy}^{\tilde{R}}$: expansiveness of $\tilde{R}$ at x, y

$$\varepsilon_{xy}^{\tilde{R}} = \begin{cases} \|\tilde{R}(x) - \tilde{R}(y)\| - \|x - y\|, & \text{if } \|x-y\| < \|\tilde{R}(x) - \tilde{R}(y)\| \\ 0, & \text{else} \end{cases}$$



- $\sigma_{xy}^{\tilde{R}} = \sqrt{\varepsilon_{xy}^{\tilde{R}}}\sqrt{\|\tilde{R}(x) - \tilde{R}(y)\| + \|x - y\|}$

Convergence of NC-DRS: conditions

- $(z_n)_{n\in\mathbf{N}}$: sequence of vectors generated for some chosen initial point z_0

if the following holds:

- there exists a $z \in \mathbf{fix}\,\tilde{T}$, such that $\sum_{n=0}^{\infty}\left(\sigma_{z_n z}^{\tilde{R}}\right)^2$ is bounded above, and $\|z_0 - z\|^2$ is finite
  - define $r := \sqrt{\|z_0 - z\|^2 + \sum_{n=0}^{\infty}\left(\sigma_{z_n z}^{\tilde{R}}\right)^2}$
  - $B(z;r)$: compact ball with center z and radius r

then...

Convergence of NC-DRS

then one of the following will happen:

1. **convergence to a point:** the sequence $(z_n)_{n\in\mathbf{N}}$ converges to a point $z^\star \in B(z;r)$
2. **cluster points form a continuum:** the set of cluster points of $(z_n)_{n\in\mathbf{N}}$ forms a nonempty compact connected set in $B(z;r)$

if situation 1 occurs and $\underline{\lim}_{n\to\infty}\left(\sigma_{z_n z^\star}^{\tilde{R}}\right)^2 = 0$, then $x_n = \mathbf{prox}_{\gamma f}(z_{n-1})$ converges to an optimal solution

Some comments on convergence

- for convergence total deviation of $\tilde{R}$ from being a nonexpansive operator over the sequence $\{(z_n, z)\}_{n\in\mathbf{N}}$ is bounded
- depends on the initial point
- in our case $\mathcal{C}$ is not necessarily convex
- **sanity check:** pretend $\mathcal{C}$ is convex $\Rightarrow$
  - total deviation of $\tilde{R}$ from being a nonexpansive operator over the sequence $\{(z_n, z)\}_{n\in\mathbf{N}}$ is zero
  - our convergence proof coincides with known convergence results for convex setup

Constructing NC-ADMM

- original problem: $\text{minimize}_{x \in \mathcal{C}}\ f(x)$
- take dual and apply NC-DRS to the dual
- resultant algorithm is relaxed NC-ADMM

$$
\begin{aligned}
x_{n+1} &= \mathbf{prox}_{\gamma f}(y_n - z_n) \\
y_{n+1} &= \mathbf{\Pi}_{\mathbf{conv}\mathcal{C}}(z_n + x_{n+1}) \\
z_{n+1} &= z_n - y_{n+1} + x_{n+1}
\end{aligned}
$$

- relaxed NC-ADMM solves $\text{minimize}_{x \in \mathbf{conv} C}\ f(x)$

Constructing NC-ADMM (continued)

- relaxed NC-ADMM

$$
\begin{aligned}
x_{n+1} &= \mathbf{prox}_{\gamma f}(y_n - z_n) \\
y_{n+1} &= \mathbf{\Pi}_{\mathbf{conv}\,\mathcal{C}}(z_n + x_{n+1}) \\
z_{n+1} &= z_n - y_{n+1} + x_{n+1}
\end{aligned}
$$

  - solves $\text{minimize}_{x \in \mathbf{conv} C}\ f(x)$
- replace $\mathbf{\Pi}_{\mathbf{conv}\,\mathcal{C}}$ with $\tilde{\mathbf{\Pi}}_{\mathcal{C}}$ to solve $\text{minimize}_{x \in \mathcal{C}}\ f(x)$
- resultant algorithm is NC-ADMM:

$$
\begin{aligned}
x_{n+1} &= \mathbf{prox}_{\gamma f}(y_n - z_n) \\
y_{n+1} &= \tilde{\mathbf{\Pi}}_{\mathcal{C}}(x_{n+1} + z_n) \\
z_{n+1} &= z_n - y_{n+1} + x_{n+1}
\end{aligned}
$$

Constructing NC-ADMM

$$\text{minimize}_{x\in\mathcal{C}} \;\; (f(x))$$

original problem

take dual →

$$\text{maximize}_{\nu} \;\; \left(-f^{\star\vee}(\nu) - \delta_{\mathcal{C}}^{\star}(\nu)\right)$$

apply **NC-DRS** ↓

$$\begin{aligned}
\zeta_{n+1} &= \mathbf{prox}_{\gamma\delta_{\mathcal{C}}^{\star}}(\psi_n)\\
\xi_{n+1} &= \mathbf{prox}_{\gamma f^{\star\vee}}(2\zeta_{n+1} - \psi_n)\\
\psi_{n+1} &= \psi_n + \xi_{n+1} - \zeta_{n+1}
\end{aligned}$$

$$\delta_{\mathcal{C}}^{\star\star} = \delta_{\mathbf{conv}\,\mathcal{C}}$$

$$\mathbf{prox}_{\gamma\delta_{\mathcal{C}}^{\star}}(x) = x - \gamma\mathbf{\Pi}_{\mathbf{conv}\,\mathcal{C}}(\gamma^{-1}x)$$

↓

$$\begin{aligned}
x_{n+1} &= \mathbf{prox}_{\tilde{\gamma}f}(y_n - z_n)\\
y_{n+1} &= \mathbf{\Pi}_{\mathbf{conv}\,\mathcal{C}}(z_n + x_{n+1})\\
z_{n+1} &= z_n - y_{n+1} + x_{n+1}
\end{aligned}$$

relaxed NC-ADMM: solves $\text{minimize}_{x\in\mathbf{conv}\,\mathcal{C}} \;\; (f(x))$

← replace $\mathbf{\Pi}_{\mathbf{conv}\,\mathcal{C}}$ with $\tilde{\mathbf{\Pi}}_{\mathcal{C}}$ to solve $\text{minimize}_{x\in\mathcal{C}} \;\; (f(x))$

$$\begin{aligned}
x_{n+1} &= \mathbf{prox}_{\gamma f}(y_n - z_n)\\
y_{n+1} &= \tilde{\mathbf{\Pi}}_{\mathcal{C}}(x_{n+1} + z_n)\\
z_{n+1} &= z_n - y_{n+1} + x_{n+1}
\end{aligned}$$

NC-ADMM

Pretend $\mathcal{C}$ is convex

pretend $\mathcal{C}$ is convex

- strong duality holds
- $\mathbf{\Pi}_{\mathbf{conv}\,\mathcal{C}} = \tilde{\mathbf{\Pi}}_{\mathcal{C}}$
- Moreau's decomposition can be applied to $\delta_{\mathcal{C}}$:

$$\mathbf{prox}_{\delta_{\mathcal{C}}} + \mathbf{prox}_{\delta_{\mathcal{C}}^{\star}} = I_n$$

then:

- no information loss in constructing NC-ADMM from NC-DRS
- NC-ADMM and NC-DRS are equivalent to each other

However...

$\mathcal{C}$ is nonconvex

- NC-ADMM and NC-DRS are not equivalent
- reasons for the loss of equivalency:
  1. strict duality gap
  2. $\mathbf{\Pi}_{\mathbf{conv}\,\mathcal{C}} \neq \tilde{\mathbf{\Pi}}_{\mathcal{C}}$
  3. Moreau's decomposition does not hold
- NC-ADMM in `NCVX-CVXPY` works on a modified dual, hence produces a lower objective value
- relationship between minimizers of the original problem and the NC-ADMM operator breaks down

What if?

- is it possible to establish convergence NC-ADMM ignoring NC-DRS?
- our convergence analysis is established for
  - nonempty, compact, but not necessarily convex constraint sets
  - it is equally applicable to the smaller subclass of problems with convex constraint sets
  - in this smaller subclass of convex problems NC-ADMM and NC-DRS have the same convergence properties

Summary

- NC-DRS and NC-ADMM are very similar looking, but very different heuristics
- the theoretical rationale to use NC-DRS could be stronger than NC-ADMM...

Limitations

- strong assumptions in minimizer characterization
- step size is 1, does not consider adaptive step sizes
- no numerical experiments in the paper

End of talk

- if you are working on nonconvex problems using ADMM/DRS, please talk to me!

Thank you!

Questions?

Diamond, S, Takapoui, R, & Boyd, S. 2018.
A general system for heuristic minimization of convex functions over non-convex sets.
Optimization Methods and Software, **33**(1), 165–193.

Erseghe, Tomaso. 2014.
Distributed optimal power flow using ADMM.
IEEE transactions on power systems, **29**(5), 2370–2380.

Iyer, Rishabh, Jegelka, Stefanie, & Bilmes, Jeff. 2014.
Monotone Closure of Relaxed Constraints in Submodular Optimization: Connections Between Minimization and Maximization.
Pages 360–369 of: Proceedings of the Thirtieth Conference on Uncertainty in Artificial Intelligence.
UAI'14.
Arlington, Virginia, United States: AUAI Press.

Takapoui, Reza, Moehle, Nicholas, Boyd, Stephen, & Bemporad, Alberto. 2017.
A simple effective heuristic for embedded mixed-integer quadratic programming.
International Journal of Control, 1–11.