Multi-player minimum cost flow problems with nonconvex costs and integer flows

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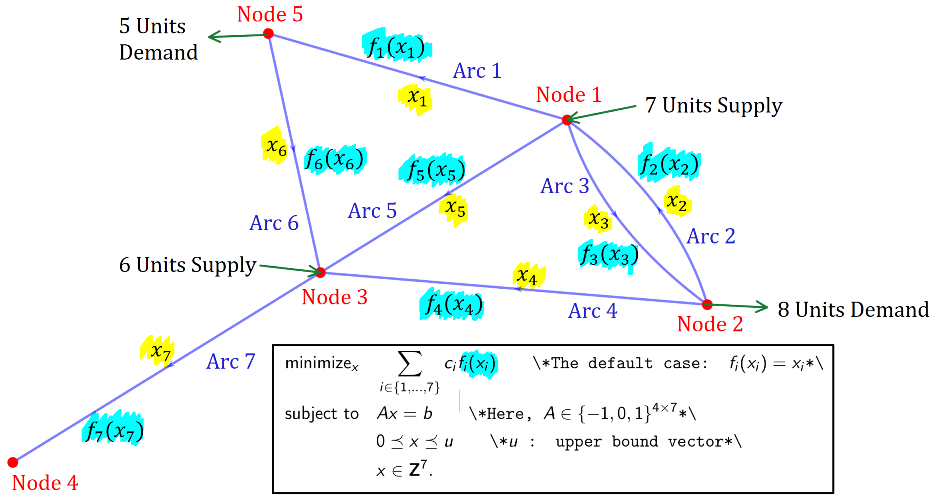
Network Flow Problems

- *Network flow problems*: optimization problems associated with underlying directed network.
- They arise in numerous application settings and in many forms.
- Some common application areas: communication networks, transportation system, social network, power system, electro-mechanical systems *etc.*
- *The minimum cost flow problem* is the most fundamental among network flow problems.

Minimum Cost Flow Problem



minimize$_x$ $\sum_{i\in\{1,\dots,7\}} c_i f_i(x_i)$ *The default case: $f_i(x_i) = x_i$*\

subject to $Ax = b$ *Here, $A \in \{-1,0,1\}^{4\times 7}$*\

$0 \preceq x \preceq u$ *u : upper bound vector*\

$x \in \mathbf{Z}^7$.

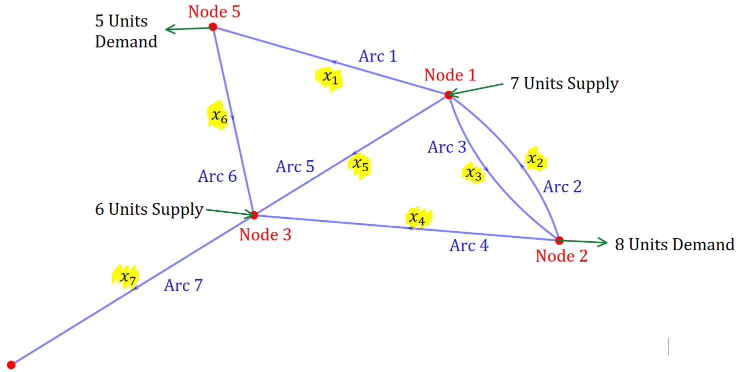
Minimum Cost Flow Problem

$$
\begin{aligned}
\text{minimize}_x \quad & \sum_{i\in\{1,\dots,n\}} c_i f_i(x_i) \qquad \texttt{\textbackslash *Default Case:} \quad f_i(x_i) = x_i \texttt{*\textbackslash} \\
\text{subject to} \quad & Ax = b \\
& 0 \preceq x \preceq u \\
& x \in \mathbf{Z}^n.
\end{aligned} \tag{1}
$$

- Here, the network has n arcs, $m+1$ nodes.
- A : reduced node-arc incidence matrix, dimension $m \times n$, full row rank.
- We re-index the m linearly independent columns of A as the first m columns.
- b : represents supplies to the nodes at different points.
- u : upper bound vector for the flow.

Extension of Min. Cost Flow to a Multi-player Setup

- With each arc of the network graph we associate one player.
- Each of the players is trying to minimize its nonconvex cost function, subject to the network flow constraints.
- Our goal is to seek an efficient solution concept in this multi-player problem.



Multi-player Extension

- The goal of the ith player for $i \in [n] = \{1, \ldots, n\}$, given other players' strategies $x_{-i} \in \mathbf{Z}^{n-1}$, is to solve:

$$
\begin{array}{lll}
\text{minimize}_{x_i} & f_i(x_i) \ \texttt{\textbackslash*proper: practically can be anything*\textbackslash} & \\
\text{subject to} & A(x_i, x_{-i}) = b \ \texttt{\textbackslash*The constraints} & \\
 & 0 \preceq (x_i, x_{-i}) \preceq u \quad \texttt{couple the players*\textbackslash} & (2) \\
 & x \in \mathbf{Z}^n. &
\end{array}
$$

Solution Concepts in Consideration

- A *vector optimal solution* that minimizes all the objectives simultaneously is unlikely to exist.
- The celebrated *Nash equilibrium* is also not very efficient in our setup because:
  - the constraint set of the problem has equality constraints, thus making any feasible point a Nash equilibrium, and
  - posteriori some of the players may decide to deviate from the Nash equilibrium in order to reduce their costs even more at the expense of the rest of the players.
- A more effective solution concept is the *Pareto optimal point.*

Pareto Optimal Point

A *Pareto optimal point* is a solution concept where none of the objective functions can be improved without worsening some of the other objective values.

Definition

(**Pareto Optimal Point**) In problem 2, a point $x^* \in P$ is Pareto optimal if it satisfies the following: there *does not* exist another point $\tilde{x} \in P$ such that

$$(\forall i \in [n]) \quad f_i(\tilde{x}_i) \leq f_i(x_i^*),$$

with at least one index $j \in [n]$ satisfying $f_j(\tilde{x}_j) < f_j(x_j^*)$.

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Related Work

- Our problem setup does not seem to be investigated in existing literature.
- Nonconvex network flow problems for very specific cost functions (Magnanti1984, Graves1985, Daskin2011, He2015).
- Integer multi-commodity flow problems (Brunetta2000, Ozdaglar2004).

Decoupling the Last $n-m$ Players

Theorem

(Existence of a decoupling variable) *Denote $B = [A_1|A_2|\cdots|A_m]$. The equality constraint set $Q = \{x \in \mathbf{Z}^n \mid Ax = b\}$ is nonempty and for any vector x, $x \in Q$ is equivalent to saying that there exists a $z \in \mathbf{Z}^{n-m}$ such that*

$$x = (d_1 - h_1^T z, \ldots, d_m - h_m^T z, z_1, \ldots, z_{n-m}), \tag{3}$$

where d_i is the ith component of $d = B^{-1}b$, and $h_i^T \in \mathbf{Z}^{n-m}$ is the ith row of $B^{-1}A_{[1:m,m+1:n]}$.

Decoupled Problems for the $n-m$ Players

- In z we can decouple the optimization problems for players $m+1, m+2, \ldots, n$ as follows

$$\begin{aligned} \text{minimize}_{z_i} \quad & f_i(z_i) \\ \text{subject to} \quad & 0 \leq z_i \leq u_i \\ & z_i \in \mathbf{Z}. \end{aligned} \tag{4}$$

- Set of different optimal solutions for player $m+i$ for $i \in [n-m]$ is

$$D_i = \{z_{i,1}, z_{i,2}, \ldots, z_{i,p_i}\}.$$

- Define, $D = \times_{i=1}^{n-m} D_i \neq \emptyset$

Consensus Reformulation for the First m players

- We provide each player $i \in [m]$ with its *own local copy* of z, denoted by $z^{(i)} \in \mathbf{Z}^{n-m}$, which acts as its decision variable.
- For any $i \in [m]$, $x_i = d_i - h_i^T z^{(i)}$.
- The copy $z^{(i)}$ has to be in *consensus* with the rest of the first m players:

$$z^{(i)} = z^{(j)} \quad \forall j \in [m] \setminus \{i\}.$$

- The copy $z^{(i)}$ has to satisfy the *flow bound constraints*, *i.e.*, $0 \leq d_i - h_i^T z^{(i)} \leq u_i$ for all $i \in [m]$.
- For the last $n-m$ players $z_i \in D_i$, so:

$$z^{(i)} \in D$$

Consensus Reformulation for the First m players

For all $i \in [m]$, the ith player's optimization problem in variable $z^{(i)}$ can be written as:

$$\begin{aligned} \text{minimize}_{z^{(i)}} \quad & \bar{f}_i\left(z^{(i)}\right) = f_i(d_i - h_i^T z^{(i)}) \\ \text{subject to} \quad & z^{(i)} = z^{(j)}, \quad j \in [m] \setminus \{i\} \\ & 0 \leq d_i - h_i^T z^{(i)} \leq u_i \\ & z^{(i)} \in D. \end{aligned} \tag{5}$$

$$q_i(z^{(i)}) = (d_i - h_i^T z^{(i)})(d_i - h_i^T z^{(i)} - 1) \cdots (d_i - h_i^T z^{(i)} - u_i) = 0, \tag{6}$$

$$r_j(z^{(i)}) = (z_j^{(i)} - z_{j,1})(z_j^{(i)} - z_{j,2}) \ldots (z_j^{(i)} - z_{j,p_i}) = 0, \qquad j \in [n-m]. \tag{7}$$

$$\mathscr{F} \quad = \quad \{z \in \mathbf{Z}^{n-m} \mid (\forall k \in [m])\ q_k(z) = 0, (\forall j \in [n-m])\ r_j(z) = 0\}.$$

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Consensus Reformulation for First m players

For $i \in [m]$, *i.e.*, each of these players are optimizing over a *common constraint set* $\mathscr{F}$:

$$\begin{aligned} \text{minimize}_{z^{(i)}} \quad & \bar{f}_i\left(z^{(i)}\right) \\ \text{subject to} \quad & z^{(i)} \in \mathscr{F}. \end{aligned} \tag{8}$$

- When $\mathscr{F}$ is nonempty? How to find the points in it?
- A little algebraic geometry will take us a long way...

A Little Algebraic Geometry

- The *ideal* generated by polynomials $f_1, f_2, \ldots, f_m \in \mathbf{C}[x]$ is the set

$$\textbf{ideal}\,\{f_1, \ldots, f_m\} = \left\{ \sum_{i=1}^{m} h_i f_i \mid (\forall i \in [m])\ h_i \in \mathbf{C}[x] \right\}.$$

  * analogous to span of vectors *\

- *A Groebner basis* $G_{\succ}$ is particular kind of generating set of an ideal I over a field $C[x]$ * analogous to basis set of a span*\
- *Reduced Groebner basis* $G_{\text{reduced},\succ}$ is the most compact Groebner basis for an ideal I. * analogous to orthonormal basis of a span *\
- There are many computer algebra packages to compute reduced Groebner basis such as Macaulry2, SINGULAR, FGb, Maple, Mathematica *etc.*

When is $\mathscr{F}$ Nonempty?

Theorem

The set $\mathscr{F}$ is nonempty if and only

$$G_{reduced,\succ} \neq \{1\},$$

where $G_{reduced,\succ}$ is the reduced Groebner basis of $\textbf{ideal}\{q_1, \ldots, q_m, r_1, \ldots, r_{n-m}\}$ *with respect to any ordering.*

Computing Points in $\mathscr{F}$

- *For $i = [n-m]$ calculate*

$$G_{n-m-i} = G_{reduced,\succ_{lex}} \cap \mathbf{C}\left[z_{n-m-i+1}, z_{n-m-i+2}, \ldots, z_{n-m}\right].$$

Theorem

$G_0 = \mathscr{F}$.

Finding the Pareto optimal points from $\mathscr{F}$

for $i \in [m]$
  $X_i := d_i - h_i^T \mathscr{F}$ `\* The inverse operator is denoted` X_i^{-1} `*\`
end for
  Sort the elements of the $\{X_i\}_{i=1}^m$s with respect to cardinality of the elements in a descending order.
  Denote the index set of the sorted set by $\{s_1, \ldots, s_m\}$ such that $|X|_{s_1} \geq \cdots \geq |X|_{s_m}$.
for $i \in [m]$
  $X_{s_i}^* := \operatorname{argmin}_{x_{s_i} \in X_{s_i}} f_{s_i}(x_{s_i})$ `\* Univariate optimization problem *\`
  $\mathscr{F}_{s_i}^* := \bigcup_{x_{s_i} \in X_{s_i}^*} (X_{s_i}^*)^{-1}(x_{s_i})$ `\* Lemma. The set` $\mathscr{F}_{s_i}^*$ `is nonempty *\`
  if $i \leq m$
    $X_{s_{i+1}} := \left\{ d_{s_{i+1}} - h_{s_{i+1}}^T z \mid z \in \mathscr{F}_{s_i}^* \right\}.$
  end if
end for
return $\mathscr{F}_{s_m}^*$ `\* Theorem. Any member of` $\mathscr{F}_{s_m}^*$ `is a Pareto optimal point *\`

Finding the Pareto optimal points from $\mathscr{F}$

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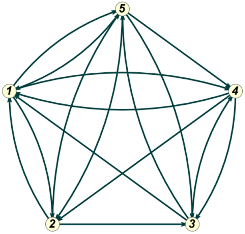
Numerical Example: 16 Players



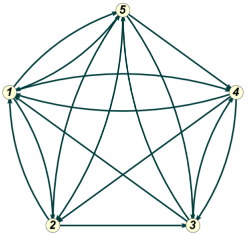
<u>A multi-player transportation problem</u>

- 5 nodes, 16 arcs
- Nodes 2, 4 are retail centers
- Nodes 1, 3 are warehouses
- Node 5 transshipment node
- Products need to be shipped from warehouses to retail centers
- Arcs represent different modes of shipment from one node to another and carried out by different organization
- Cost of a certain type of shipment depends on the number of products shipped

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \end{pmatrix}$$

$$b = (9, -13, 15, -11), \; u = (5, 6, 6, 10, 10, 7, 11, 13, 16, 12, 4, 5, 6, 14, 13, 15)$$

Pareto Optimal Solutions



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The Pareto optimal points are

$$(1, 3, 5, 4, 11, 10, 2, 1, 3, 7, 7, 5)$$

and

$$(1, 3, 5, 6, 11, 10, 2, 1, 3, 7, 7, 6).$$

Concluding Remarks

- Methodology is numerically efficient when $m < \frac{n}{2}$.
- Calculating Groebner basis can be numerically challenging for very large system.
- Based on structure of the network it may happen that $\mathscr{F}$ is empty $\Rightarrow$ penalty based approach.

Thank You!
Questions?

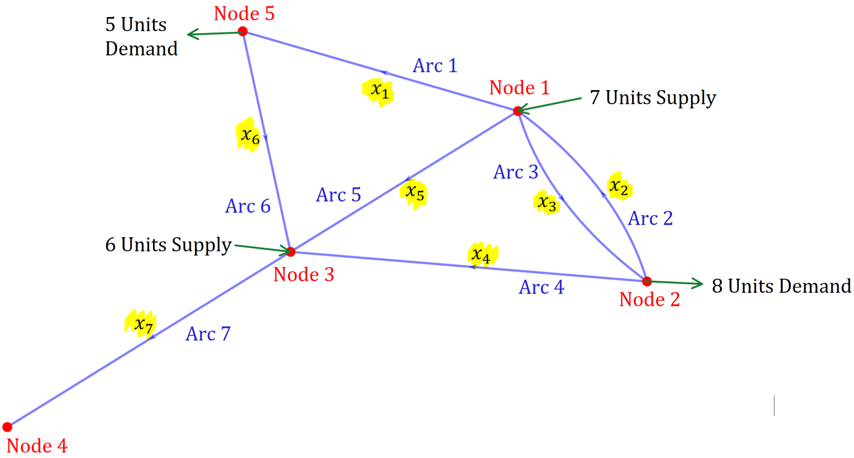
Cost Functions

Player	Cost function
1	$-\frac{x_1^4}{30}-\frac{13x_1^3}{15}+\frac{259x_1^2}{30}-\frac{263x_1}{15}+1$
2	$\frac{77x_2^5}{120}-\frac{247x_2^4}{24}+\frac{471x_2^3}{8}-\frac{3365x_2^2}{24}+\frac{6779x_2}{60}+1$
3	$\frac{47x_3^4}{24}-\frac{133x_3^3}{4}+\frac{4897x_3^2}{24}-\frac{2123x_3}{4}+485$
4	$\frac{323x_4^5}{3360}-\frac{2179x_4^4}{1120}+\frac{47393x_4^3}{3360}-\frac{48709x_4^2}{1120}+\frac{7885x_4}{168}+5$
5	$(x_5-1)^2$
6	$-\frac{x_6^4}{8}+\frac{25x_6^3}{12}-\frac{71x_6^2}{8}+\frac{95x_6}{12}+10$
7	$\lvert x_7-5\rvert$
8	$\frac{11x_8^7}{1260}-\frac{7x_8^6}{36}+\frac{119x_8^5}{72}-\frac{479x_8^4}{72}+\frac{4609x_8^3}{360}-\frac{803x_8^2}{72}+\frac{155x_8}{28}+1$
9	$-\frac{15}{16}x_9^3+\frac{365x_9^2}{16}-\frac{2865x_9}{16}+\frac{7315}{16}$
10	$(x_{10}-10)^2$
11	$\frac{5x_{11}^4}{6}-\frac{35x_{11}^3}{3}+\frac{355x_{11}^2}{6}-\frac{370x_{11}}{3}+90$
12	$\frac{5x_{12}^4}{6}-\frac{25x_{12}^3}{3}+\frac{175x_{12}^2}{6}-\frac{110x_{12}}{3}+15$
13	$\frac{5x_{13}^4}{6}-15x_{13}^3+\frac{595x_{13}^2}{6}-280x_{13}+285$
14	$\frac{5x_{14}^4}{6}-\frac{85x_{14}^3}{3}+\frac{2155x_{14}^2}{6}-\frac{6020x_{14}}{3}+4165$

Cost Functions

Player	Cost function
15	$\lvert x_{15} - 7 \rvert$
16	$\begin{cases} x_{16} + 1, & \text{if } 0 \leq x_{16} \leq 3 \\ 0, & \text{if } 4 \leq x_{16} \leq 6 \\ (x_{16} + 1)^3, & \text{if } 7 \leq x_{16} \leq 9 \\ -\frac{x_{16}^3}{6} + \frac{13x_{16}^2}{2} - \frac{244x_{16}}{3} + 330, & \text{else} \end{cases}$

A Generic Network



The Minimum Cost Flow Problem

- There is a *directed connected graph* that represents the network.
- There is *flow* of some commodity along the arcs of the graphs.
- Each arc incurs a *cost* depending on the amount of flow.
- The flow is often taken to be *integral*.
- The goal is to *minimize the total cost* of all flows subject to the *network constraints*.