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An Optimization Model to Utilize Regenerative Braking Energy in a Railway Network

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Abstract— **In this paper, we study the railway timetabling problem to utilize regenerative braking energy produced by trains in a railway network. An electric train produces regenerative energy while braking, which is often lost in present technology. A positive overlapping time between braking and accelerating phases of a suitable train pair makes it possible to save electrical energy by transferring the regenerative energy of the braking train to the accelerating one. We propose a novel optimization model to determine a timetable that saves energy by maximizing the total overlapping time of all suitable train pairs. We apply our optimization model to different instances of a railway network for a time horizon spanning six hours. For each instance, our model finds an optimal or near-optimal timetable within an acceptable running time. We observe significant increase in the final overlapping time compared to the existing timetable for every instance, thus making it possible to save the associated electrical energy.**

I. Introduction

A. Background and Motivation

In recent years, much emphasis has been placed on efficient energy management of electric vehicles using mathematical optimization techniques [1]–[4]. Among all vehicles, trains are preferred by many people for being safer and cheaper. Nowadays in most railway networks, trains use electricity as their primary source of energy. In this regard, a relevant problem is how to efficiently utilize electrical energy in a railway network.

In modern railway networks, trains are generally equipped with regenerative braking technology. When a train makes a trip from an origin platform to a destination platform, its optimal speed profile consists of four phases: accelerating, speed holding, coasting and braking [5]. A train consumes most of its required energy during the accelerating phase, and it produces electrical energy, called regenerative braking energy, during the braking phase [6] as shown in Figure 1. Naturally, proper utilization of regenerative braking energy of trains can lead to significant energy saving. However, transferring the regenerative braking energy back to the electrical grid requires specialized technology such as reversible electrical substations [7, page 30], and storing it using current technology, e.g., via super-capacitors [8], is very expensive [9, page 66]. A better strategy to utilize the regenerative braking energy of a train would be to synchronize its braking phase with the accelerating phase of another nearby train operating under the same electrical substation. A positive overlapping time arising from such a synchronization process would make it possible to transfer the regenerative braking energy of the first train to the latter via the overhead contact line, thus saving the electrical energy which would be lost otherwise. Our objective is to design an energy efficient railway timetable, that contains the arrival and departure time of every train to and from all the platforms it visits, such that the duration of the **s**ynchronization **p**rocesses between **s**uitable **t**rain **p**airs (**SPSTP**s) is maximized subject to the different constraints present.

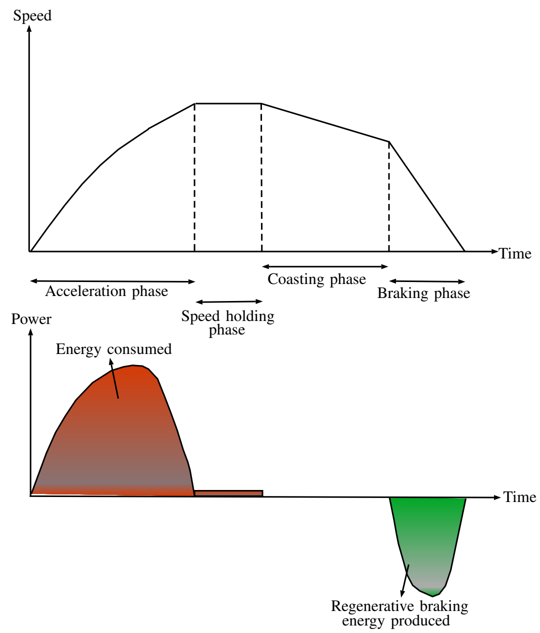

Fig. 1. The speed profile of a train (top), and corresponding energy consumed and produced in accelerating phase and braking phase respectively (bottom).

B. Related Works

Though the general timetabling problem in a railway network has been studied extensively over the past three decades [10], very few works exist which attempt to determine a timetable to utilize regenerative braking energy of trains. The work by Peña-Alcaraz et al. [11] proposes a Mixed Integer Programming (**MIP**) model, where the objective is to maximize the total duration of all possible synchronization processes between all train pairs and applies it successfully to line 3 of the Madrid underground system in Spain.

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However, for many railway networks, considering all the train pairs in the objective is not always realistic, because energy transfer between trains not close to each other suffers from drastic transmission loss. From a computational point of view, including all train pairs is not very efficient, as the search space for the optimization problem can become quite large. Also, the model is only applicable to a single train-line because of the absence of connection and turn-around constraints. The works by Yang et al. [12] and Le et al. [13] apply local search techniques such as genetic algorithm and simulated annealing respectively to utilize regenerative braking energy with application to Chinese railway network. Their models only consider the energy transfer between train pairs on the same train-line, which can also lead to significant transmission loss. Moreover, being local search models, their final timetables do not have any guarantee of optimality.

C. Contributions

Our contributions in this paper can be summarized as follows:

- Using mathematical programming techniques, we propose a novel MIP model to utilize the regenerative braking energy of trains in a railway network. In comparison with the existing works, our model can be applied to any railway network.
- For most of the existing railway networks, the railway management has a feasible timetable, which we exploit to devise an optimization strategy. This strategy produces a smaller search space and is computationally more tractable in comparison with the related works. We prove that all possible cases arising from an SPSTP are modelled accurately by our model.
- We apply our optimization model to different instances of a real-life railway network and find that our model gives optimal or near-optimal timetables in an acceptable running time with a significant increase in the final overlapping time.

D. Organization

This paper is organized as follows. In Section II we describe the notation and notions to be used in this paper. In Section III we define an SPSTP mathematically, propose our optimization strategy and formulate a mixed integer linear objective function. Then in Section IV we model and justify various constraints present in the railway network. We propose the full optimization model in Section V. In Section VI we apply our model to realistic instances of different size. Section VII presents the conclusion.

II. Notation and Notions

All the sets described in this paper are strictly ordered and finite unless otherwise specified. The cardinality and the ith element of such a set S is denoted by $|S|$ and $S(i)$ respectively. Consider a railway network where the set of all stations is denoted by $\mathcal{S}$. The set of all platforms in the railway network is indicated by $\mathcal{N}$. A directed arc between two distinct and non-opposite platforms is called a track. The set of all tracks is represented by $\mathcal{A}$. The directed graph of the railway network is expressed by $\mathcal{G} = (\mathcal{N}, \mathcal{A})$. A train-line or line is a directed path with the set of nodes representing non-opposite platforms and the set of arcs representing non-opposite tracks. A crossing-over is a special type of directed arc that connects two train-lines. If after arriving at the terminal platform of a train-line, a train turns around by traversing the crossing-over and starts travelling through another train-line, then the same physical train is treated and labelled functionally as two different trains by the railway management [14, page 41]. The set of all trains to be considered in our problem is denoted by $\mathcal{T}$. The sets of all platforms and all tracks visited by a train t in chronological order are denoted by $\mathcal{N}^t \subseteq \mathcal{N}$ and $\mathcal{A}^t \subseteq \mathcal{A}$ respectively. The train-path of a train is the directed path containing all platforms and tracks visited by it in chronological order.

The decision variables to be determined are the arrival and departure times of trains to and from the associated platforms respectively. Let a_i^t be the arrival time of train $t \in \mathcal{T}$ at platform $i \in \mathcal{N}^t$ and d_j^t be the departure time of train t from platform $j \in \mathcal{N}^t$. Our objective is to maximize the duration of total overlapping time of all SPSTPs subject to the constraints, which would make it possible to save significant amount of electrical energy produced by the braking trains by transferring it to the accelerating trains.

III. Modelling the Objective

At first, we need to characterize the train pairs and the associated platform pairs necessary to describe the SPSTPs. The platform pairs to consider are those opposite to each other and powered by the same electrical substations, because the transmission loss in transferring electrical energy between them is negligible. The set containing all such platform pairs is denoted by Ω. Consider any such platform pair $(i, j) \in \Omega$, and let $\mathcal{T}_i \subseteq \mathcal{T}$ be the set of all the trains which arrive at, dwell and then depart from platform i. Suppose, $t \in \mathcal{T}_i$. Now, we are interested in finding another train $\tilde{t}$ on platform j, i.e., $\tilde{t} \in \mathcal{T}_j$, which along with t would form a suitable pair for the transfer of regenerative braking energy. To achieve this, we use the fact that for most of the existing railway networks, an initial feasible timetable is available to the railway management, where too much deviation from it is not desired. For every train t, this timetable provides a feasible arrival time $\bar{a}_i^t$ and a feasible departure time $\bar{d}_i^t$ to and from every platform $i \in \mathcal{N}^t$ respectively. Intuitively, among all the trains going through platform j, the one which is temporally closest to t in the initial timetable would be the best candidate to form a pair with t. The temporal proximity can be of two types with respect to t, which results in the following definitions.

Definition 1: Consider any $(i, j) \in \Omega$. For every train $t \in \mathcal{T}_i$, the train $\overset{\rightharpoonup}{t} \in \mathcal{T}_j$ is called the **temporally closest train to the right of** t if

$$\overset{\rightharpoonup}{t} = \operatorname*{argmin}_{t' \in \{x \in \mathcal{T}_j : 0 \leq \frac{\bar{a}_j^x + \bar{d}_j^x}{2} - \frac{\bar{a}_i^t + \bar{d}_i^t}{2} \leq r\}} \left\{ \left| \frac{\bar{a}_i^t + \bar{d}_i^t}{2} - \frac{\bar{a}_j^{t'} + \bar{d}_j^{t'}}{2} \right| \right\}, \tag{1}$$

where r is an empirical parameter determined by the timetable designer and is much smaller than the time horizon of the entire timetable.

Definition 2: Consider any $(i,j) \in \Omega$. For every train $t \in \mathcal{T}_i$, the train $\overleftarrow{t} \in \mathcal{T}_j$ is called the **temporally closest train to the left of** t if

$$\overleftarrow{t} = \underset{t' \in \{x \in \mathcal{T}_j : 0 < \frac{\bar{a}_i^t + \bar{d}_i^t}{2} - \frac{\bar{a}_j^x + \bar{d}_j^x}{2} \le r\}}{\operatorname{argmin}} \left\{ \left| \frac{\bar{a}_i^t + \bar{d}_i^t}{2} - \frac{\bar{a}_j^{t'} + \bar{d}_j^{t'}}{2} \right| \right\}. \tag{2}$$

Definition 3: Consider any $(i,j) \in \Omega$. For every train $t \in \mathcal{T}_i$, the train $\tilde{t} \in \mathcal{T}_j$ is called the **temporally closest train to** t if

$$\tilde{t} = \underset{t' \in \{\overrightarrow{t}, \overleftarrow{t}\}}{\operatorname{argmin}} \left\{ \left| \frac{\bar{a}_i^t + \bar{d}_i^t}{2} - \frac{\bar{a}_j^{t'} + \bar{d}_j^{t'}}{2} \right| \right\}. \tag{3}$$

If both $\overrightarrow{t}$ and $\overleftarrow{t}$ are temporally equidistant from t, we pick one of them arbitrarily.

Any SPSTP can be described by specifying the corresponding i, j, t and $\tilde{t}$ by using the definitions above. We construct a set of all the SPSTPs, which we denote by $\mathcal{E}$. Each element of this set is a tuple of the form $(i,j,t,\tilde{t})$. Because $\tilde{t}$ is unique for any t in each element of $\mathcal{E}$, we can partition $\mathcal{E}$ into two sets denoted by $\overrightarrow{\mathcal{E}}$ and $\overleftarrow{\mathcal{E}}$ containing elements of the form $(i,j,t,\overrightarrow{t})$ and $(i,j,t,\overleftarrow{t})$ respectively. For every $(i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}}$ our strategy is to synchronize the accelerating phase of t with the braking phase of $\overrightarrow{t}$. On the other hand, for every $\overleftarrow{t} \in \overleftarrow{\mathcal{E}}$, then it would be convenient to synchronize the accelerating phase of $\overleftarrow{t}$ with the braking phase of t. For every $(i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}}$, the corresponding overlapping time is denoted by $\sigma_{ij}^{t\overrightarrow{t}}$, and for every $(i,j,t,\overleftarrow{t}) \in \overleftarrow{\mathcal{E}}$, the corresponding overlapping time is denoted by $\sigma_{ij}^{t\overleftarrow{t}}$. Our objective is to maximize the sum of overlapping times over all the elements of $\overrightarrow{\mathcal{E}}$ and $\overleftarrow{\mathcal{E}}$, i.e.,

$$\begin{aligned} &\text{maximize} && \sum_{(i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}}} \sigma_{ij}^{t\overrightarrow{t}} + \sum_{(i,j,t,\overleftarrow{t}) \in \overleftarrow{\mathcal{E}}} \sigma_{ij}^{t\overleftarrow{t}} \\ &\text{subject to} && \text{the constraints present in the system.} \end{aligned} \tag{4}$$

We model $\sigma_{ij}^{t\overrightarrow{t}}$ for all $(i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}}$ and $\sigma_{ij}^{t\overleftarrow{t}}$ for all $(i,j,t,\overleftarrow{t}) \in \overleftarrow{\mathcal{E}}$ in terms of the arrival and departure times of trains. Consider the case, when $(i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}}$. We need to ensure that after applying the optimization strategy $\overrightarrow{t}$ still stays the temporally closest train to the right of t. Otherwise, the only way to achieve a positive overlapping time is to synchronize the braking phase of t with the accelerating phase of $\overrightarrow{t}$, which might result in a large deviation of event times compared to the original timetable, especially when there is no or very little overlapping to begin with. We write this constraint as follows:

$$\frac{\left(a_j^{\overrightarrow{t}} + d_j^{\overrightarrow{t}} - a_i^t - d_i^t\right)}{\left(\bar{a}_j^{\overrightarrow{t}} + \bar{d}_j^{\overrightarrow{t}} - \bar{a}_i^t - \bar{d}_i^t + \epsilon\right)} \ge 0. \tag{5}$$

Here ϵ is a very small positive number to prevent division by zero.

Let us denote the start of the braking phase of train t before arriving at platform i by a_i^{t-} and the end of its accelerating phase after departing from the same platform by d_i^{t+}. For all $t \in \mathcal{T}$ and for all $i \in \mathcal{N}^t$, the durations of the associated braking phase $\beta_i^t = a_i^t - a_i^{t-}$ and the associated accelerating phase $\alpha_i^t = d_i^{t+} - d_i^t$ are assumed to be known parameters, as they can be calculated from driving profiles determined by existing simulation tools [15, page 3]. To model the overlapping time $\sigma_{ij}^{t\overrightarrow{t}}$ for all $(i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}}$, we propose the following lemma.

Lemma 1: For all $(i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}}$, the overlapping time $\sigma_{ij}^{t\overrightarrow{t}}$ between the accelerating phase of t on platform i and the braking phase of $\overrightarrow{t}$ on platform j, where $(i,j) \in \Omega$, can be modelled by

$$a_j^{\overrightarrow{t}-} - d_i^{t+} + \epsilon \le M(1 - \lambda_{ij}^{t\overrightarrow{t}}), \tag{6}$$
$$d_i^t - a_j^{\overrightarrow{t}} + \epsilon \le M(1 - \lambda_{ij}^{t\overrightarrow{t}}), \tag{7}$$
$$\sigma_{ij}^{t\overrightarrow{t}} \ge 0, \tag{8}$$
$$\sigma_{ij}^{t\overrightarrow{t}} \le \alpha_i^t \lambda_{ij}^{t\overrightarrow{t}}, \tag{9}$$
$$\sigma_{ij}^{t\overrightarrow{t}} \le \beta_j^{\overrightarrow{t}} \lambda_{ij}^{t\overrightarrow{t}}, \tag{10}$$
$$\sigma_{ij}^{t\overrightarrow{t}} \le d_i^{t+} - a_j^{\overrightarrow{t}-} + M(1 - \lambda_{ij}^{t\overrightarrow{t}}), \tag{11}$$
$$\sigma_{ij}^{t\overrightarrow{t}} \le a_j^{\overrightarrow{t}} - d_i^t + M(1 - \lambda_{ij}^{t\overrightarrow{t}}), \tag{12}$$

where M is a large positive number, ϵ is a small positive number smaller than time granularity considered and $\lambda_{ij}^{t\overrightarrow{t}}$ is a binary variable which is one if and only if $\sigma_{ij}^{t\overrightarrow{t}}$ is positive.

Proof: We omit the proof due to space constraint. A short proof sketch is as follows. We use the hypograph approach to model the overlapping time $\sigma_{ij}^{t\overrightarrow{t}}$ in terms of the associated event times [16, page 75, 134]. Next we use interval algebra [17] to show that there can be thirteen different kinds of overlapping possible between the accelerating phase of train t and the braking phase of train $\overrightarrow{t}$. We arrive at the goal by implementing integer programming modelling rules [18, pages 166, 172-174, 183-184]. ■

Now consider the case when $(i,j,t,\overleftarrow{t}) \in \overleftarrow{\mathcal{E}}$. Like the previous case, in this case as well, we need to ensure that after applying the optimization strategy, $\overleftarrow{t}$ still stays the temporally closest train to the left of t. Analogous constraint to that of Equation (5) can be easily found by replacing t

and $\overrightarrow{t}$ in Equation (5) with $\overleftarrow{t}$ and t respectively as follows:

$$\frac{\left(a_i^t + d_i^t - a_j^{\overleftarrow{t}} - d_j^{\overleftarrow{t}}\right)}{\left(\bar{a}_i^t + \bar{d}_i^t - \bar{a}_j^{\overleftarrow{t}} - \bar{d}_j^{\overleftarrow{t}}\right)} \geq 0. \tag{13}$$

Note that the denominator can never be zero on the left hand side of the equation above because of the definition of $\overleftarrow{t}$ in Equation (2). To model the overlapping time $\sigma_{ij}^{t\overleftarrow{t}}$ for all $(i, j, t, \overleftarrow{t}) \in \overleftarrow{\mathcal{E}}$, we propose the following lemma.

Lemma 2: For all $(i, j, t, \overleftarrow{t}) \in \overleftarrow{\mathcal{E}}$, the overlapping time $\sigma_{ij}^{t\overleftarrow{t}}$ between the braking phase of t on platform i and the accelerating phase of $\overleftarrow{t}$ on platform j, where $(i, j) \in \Omega$, can be modelled by the equations:

$$a_i^{t-} - d_j^{\overleftarrow{t}+} + \epsilon \leq M(1 - \lambda_{ij}^{t\overleftarrow{t}}), \tag{14}$$
$$d_j^{\overleftarrow{t}} - a_i^t + \epsilon \leq M(1 - \lambda_{ij}^{t\overleftarrow{t}}), \tag{15}$$
$$\sigma_{ij}^{t\overleftarrow{t}} \geq 0, \tag{16}$$
$$\sigma_{ij}^{t\overleftarrow{t}} \leq \alpha_j^{\overleftarrow{t}} \lambda_{ij}^{t\overleftarrow{t}}, \tag{17}$$
$$\sigma_{ij}^{t\overleftarrow{t}} \leq \beta_i^t \lambda_{ij}^{t\overleftarrow{t}}, \tag{18}$$
$$\sigma_{ij}^{t\overleftarrow{t}} \leq d_j^{\overleftarrow{t}+} - a_i^{t-} + M(1 - \lambda_{ij}^{t\overleftarrow{t}}), \tag{19}$$
$$\sigma_{ij}^{t\overleftarrow{t}} \leq a_i^t - d_j^{\overleftarrow{t}} + M(1 - \lambda_{ij}^{t\overleftarrow{t}}), \tag{20}$$

where M is a large positive number, ϵ is a small positive number smaller than time granularity considered and $\lambda_{ij}^{t\overleftarrow{t}}$ is a binary variable which is one if and only if $\sigma_{ij}^{t\overleftarrow{t}}$ is positive.

Proof: The lemma can be easily proved by replacing t and $\overrightarrow{t}$ with $\overleftarrow{t}$ and t respectively in Lemma 1. ■

IV. Modelling the Constraints

The constraints in the railway network show how the events are related. In this section, we describe, model and justify the constraints.

A. Trip Time Constraint

Consider the trip of any train $t \in \mathcal{T}$ from platform i to platform j along the track $(i, j) \in \mathcal{A}^t$. The train t departs from platform i at time d_i^t and arrives at platform j at time a_j^t and the train can have a trip time between $\underline{\tau}_{ij}^t$ and $\overline{\tau}_{ij}^t$. The trip time constraint can be written as follows:

$$\forall t \in \mathcal{T}\ \forall (i, j) \in \mathcal{A}^t \quad \left(\underline{\tau}_{ij}^t \leq a_j^t - d_i^t \leq \overline{\tau}_{ij}^t\right). \tag{21}$$

B. Dwell Time Constraint

When any train $t \in \mathcal{T}$ arrives at a platform $i \in \mathcal{N}^t$, it dwells there for a certain time interval denoted by $[\underline{\delta}_i^t, \overline{\delta}_i^t]$ so that the passengers can get off and get on the train. After the dwelling time is over, the train departs from the station. The difference between the departure time d_i^t and arrival time a_i^t corresponding to the dwelling mentioned lies between $\underline{\delta}_i^t$ and $\overline{\delta}_i^t$. The dwell time constraint can be written as follows:

$$\forall t \in \mathcal{T}\ \forall i \in \mathcal{N}^t \quad (\underline{\delta}_i^t \leq d_i^t - a_i^t \leq \overline{\delta}_i^t). \tag{22}$$

Every train $t \in \mathcal{T}$ arrives at the first platform $\mathcal{N}^t(1)$ in its train-path either from the depot or by turning around from some other line, and departs from the final platform $\mathcal{N}^t(|\mathcal{N}^t|)$ in order to either return to the depot or start as a new train on another line by turning around. So, the train t dwells at all the platforms in $\mathcal{N}^t$. This is the reason why in Equation (22) the platform index i is varied over all the elements of the set $\mathcal{N}^t$.

C. Connection Constraint:

In many cases, a single train might not exist between the origin and desired destination of a passenger. To circumvent such issues, connecting trains are often used by the railway management at interchange stations. Let $\chi \subseteq \mathcal{N} \times \mathcal{N}$ be the set of platform pairs where passengers transfer between trains. If $(i, j) \in \chi$, then both the platforms i and j are situated at the same station, and there exist a train $t \in \mathcal{T}$ arriving at platform i and another train $t' \in \mathcal{T}$ departing from platform j such that a connection time window needs to be maintained between train t and t' for passengers to get off from the first train and get on the latter. Note that order matters here. Let $\mathcal{C}_{ij}$ be the set of connecting train pairs for a platform pair $(i, j) \in \chi$. Then the connection constraint can be written as:

$$\forall (i, j) \in \chi\ \forall (t, t') \in \mathcal{C}_{ij} \quad (\underline{\chi}_{ij}^{tt'} \leq d_j^{t'} - a_i^t \leq \overline{\chi}_{ij}^{tt'}), \tag{23}$$

where $\underline{\chi}_{ij}^{tt'}$ and $\overline{\chi}_{ij}^{tt'}$ are the lower bound and upper bound of the time window to achieve the described connection between the associated trains.

D. Turn-around Constraint:

After arriving at the terminal platform of a train-line, a train might turn around by traversing the crossing-over and start travelling through another train-line. From a railway management perspective, a time window has to be maintained between the departure of the train from the terminal platform of the first line and the arrival time of it (labelled as a different train, though it is physically the same train) on another platform of the second line. Let φ be the set of all crossing-overs, where turn-around events occur. Consider any crossing-over $(i, j) \in \varphi$, where the platforms i and j are situated on different train-lines. Let $\mathcal{B}_{ij}$ be the set of all train pairs involved in corresponding turn-around events. Let $(t, t') \in \mathcal{B}_{ij}$. Train $t \in \mathcal{T}$ turns around at platform i by travelling through the crossing-over (i, j), and beginning from platform j starts traversing a different train-line as train $t' \in \mathcal{T} \setminus \{t\}$. A time window denoted by $[\underline{\kappa}_{ij}^{tt'}, \overline{\kappa}_{ij}^{tt'}]$ has to be maintained between the mentioned events, where $\underline{\kappa}_{ij}^{tt'}$ and $\overline{\kappa}_{ij}^{tt'}$ are the lower bound and upper bound respectively. We can write this constraint as follows:

$$\forall (i, j) \in \varphi\ \forall (t, t') \in \mathcal{B}_{ij} \quad (\underline{\kappa}_{ij}^{tt'} \leq a_j^{t'} - d_i^t \leq \overline{\kappa}_{ij}^{tt'}). \tag{24}$$

E. Headway Constraint:

In any railway network, a minimum amount of time between the departures of consecutive trains is always maintained. This time is called headway time. Let $(i, j) \in \mathcal{A}$

TABLE I

RESULTS OF THE NUMERICAL STUDY (RUNNING TIME 1200 S)

Headway Time (s)	Number of Trains	Number of Constraints	Binary Variables	Real Variables	Non-zero Coefficients	Explored Nodes	CPU Time (s)	Initial Overlapping Time	Final Overlapping Time	Increase in Overlapping Time	Relative Optimality Gap
257	108	4223	100	2469	9246	0	1.09	0	1000	Infinity	0%
231	121	6370	346	2999	15508	30	3.8	1887	3460	83.36%	0%
193	144	8496	542	3699	21328	349	9.54	2151	5420	151.98%	0%
178	156	7253	308	3729	16970	6	4.52	222	3080	1287.39%	0%
154	180	9619	534	4479	23510	285	11.48	379	5340	1308.97%	0%
144	192	11610	762	4971	29316	228676	1200	2887	7425	157.19%	0.97%
136	204	12047	768	5241	30238	819	25.44	2337	7680	228.63%	0%
128	216	14108	1006	5743	36264	203	17.5	3837	10060	162.18%	0%
121	228	15645	1170	6167	40650	1450	43.54	3661	11700	219.58%	0%
115	240	19400	1650	6911	52000	1672	66.71	6968	16500	136.80%	0%
110	252	17401	1308	6833	45266	1428	69.51	4752	13080	175.25%	0%
105	263	18096	1356	7121	47040	1027	56.81	5284	13560	156.62%	0%
100	275	15425	918	6947	38194	156	17.06	4318	9180	112.60%	0%
96	288	20722	1614	7927	54356	1423	122.39	5515	16140	192.66%	0%
92	300	20319	1500	8077	52638	1475	116.11	3989	15000	276.03%	0%
89	311	18746	1224	8041	47284	1224	115.8	4372	12240	179.96%	0%
82	336	27714	2388	9753	74532	16200	1200	9666	23680	144.98%	0.32%

be the track between two platform i and j, and $\mathcal{H}_{ij}$ be the set of train-pairs who move along that track successively in the order of their departures. Now, assume train t and train t' are moving along this track in same direction where $(t,t') \in \mathcal{H}_{ij}$. Let $h_i^{tt'}$ and $h_j^{tt'}$ be the associated headway times at platform i and platform j respectively. So, the headway constraint can be written as:

$$\forall (i,j) \in \mathcal{A}\ \forall (t,t') \in \mathcal{H}_{ij} \quad (h_i^{tt'} \leq d_i^{t'} - d_i^t \ \wedge\ h_j^{tt'} \leq d_j^{t'} - d_j^t). \tag{25}$$

F. Total Travel Time Constraint:

To maintain the quality of service in the railway network, it is desired that for every train $t \in \mathcal{T}$, the total travel time to traverse its train-path stays within a time window $[\underline{\tau}_{\mathcal{P}}^t, \overline{\tau}_{\mathcal{P}}^t]$, where $\underline{\tau}_{\mathcal{P}}^t$ and $\overline{\tau}_{\mathcal{P}}^t$ are the corresponding lower and upper bound respectively. We can write this constraint as follows:

$$\forall t \in \mathcal{T} \quad (\underline{\tau}_{\mathcal{P}}^t \leq a^t_{\mathcal{N}^t(|\mathcal{N}^t|)} - d^t_{\mathcal{N}^t(1)} \leq \overline{\tau}_{\mathcal{P}}^t). \tag{26}$$

Here $\mathcal{N}^t(1)$ and $\mathcal{N}^t(|\mathcal{N}^t|)$ are the first and last platform in the train-path of t.

V. FULL OPTIMIZATION MODEL

In this section, we collect the objective and all the constraints discussed in the previous two sections, and propose our optimization problem to maximize the total duration of overlapping times of the SPSTPs in order to utilize regenerative braking energy produced by trains in a railway network. The full optimization model is as follows:

$$\text{maximize} \sum_{(i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}}} \sigma_{ij}^{t\overrightarrow{t}} + \sum_{(i,j,t,\overleftarrow{t}) \in \overleftarrow{\mathcal{E}}} \sigma_{ij}^{t\overleftarrow{t}}$$

subject to

Equations (21), (22), (23), (24), (25) and (26)

$$\forall (i,j,t,\overrightarrow{t}) \in \overrightarrow{\mathcal{E}} \quad \text{Equations (5),(6)-(12)}$$

$$\forall (i,j,t,\overleftarrow{t}) \in \overleftarrow{\mathcal{E}} \quad \text{Equations (13),(14)-(20)}$$

$$\forall t \in \mathcal{T}\ \forall i \in \mathcal{N}^t \quad (d_i^{t+} = d_i^t + \alpha_i^t, a_i^t = \beta_i^t + a_i^{t-}, \; a_i^t \geq 0, d_i^t \geq 0)$$

$$\forall (i,j,t,\tilde{t}) \in \mathcal{E} \quad (\lambda_{ij}^{t\tilde{t}} \in \{0,1\}, \sigma_{ij}^{t\tilde{t}} \geq 0).$$

The decision variables are $a_i^t, d_i^t, \lambda_{ij}^{t\tilde{t}}$ and $\sigma_{ij}^{t\tilde{t}}$. As the model is a MIP with bounds, the optimization problem is $\mathcal{NP}$-hard [19, page 242]. However, in the next section we show that for the size of the railway data considered in practice, the running time is quite acceptable.

VI. NUMERICAL EXPERIMENT

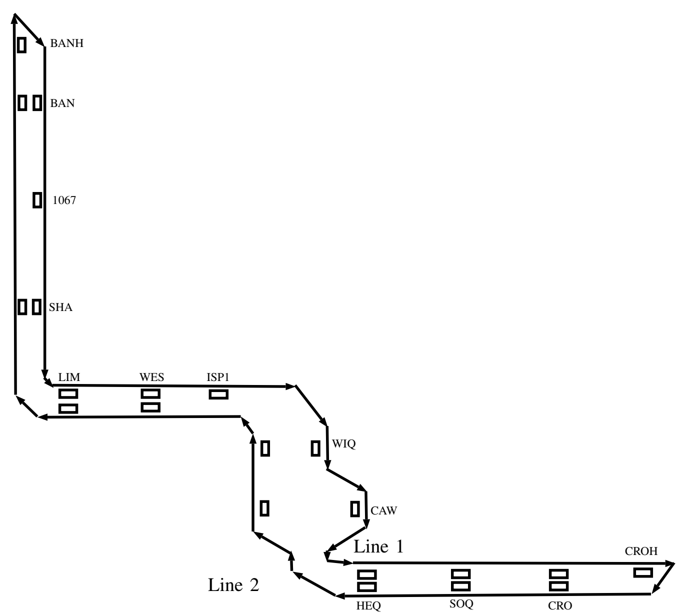

Fig. 2. Railway network considered for numerical experiment

In this section, we apply our model to different problem instances of varying size. All the experiments were executed on a Intel Core i5-3317U 1.70GHz CPU with 4096 MB of RAM running the Windows 8.1 operating system. We have used IBM ILOG CPLEX Optimization Studio 12.6 academic version with OPL as our modelling language to perform the optimization.

We have applied the proposed optimization model to a railway network that is a part of the Docklands Light Railway as shown in Figure 2. Platforms are denoted by rectangles in the figure. The railway network has two train-lines denoted by **Line 1** and **Line 2**. There are ten stations in this network denoted by capitalized words and each station has two opposite platforms, e.g., BAN is a station which has two opposite platforms: one on Line 1 and the other on Line 2. The platforms denoted by 1067 and ISP1 are intermediate stopping points on Line 1. The platforms indicated by BANH and CROH are turn-around points on Line 1 and Line 2 respectively.

For our numerical study, we have considered 18 different instances with varying headway times and number of trains. As the headway time decreases, the number of trains in the network increases. In each instance we have an initial feasible timetable with a duration of six hours. In most of the railway networks the duration of the off-peak or rush hours is smaller than six hours, so a timetable spanning six hours is sufficient for practical purpose. The feasible timetables are generated by the Timetable Compiler software which is a proprietary tool developed by Thales Canada Incorporated. We have taken $M = 1000$ and $\epsilon = 0.005$. We have applied our optimization model to find the optimal timetable that maximizes the total overlapping time of the SPSTPs. We see from Table I that for each instance, our optimization model produces an optimal or near-optimal timetable with significant increase in the total overlapping time in comparison with the initial timetable, with the minimum one being 83.36%. Such increase in the total overlapping time would make it possible to save significant amount of electrical energy produced by the braking trains by transferring it to the accelerating trains via the overhead contact lines. Optimal solutions are obtained for all the instances within the running time of 1200 s, except two instances corresponding to headway times of 144 s and 82 s. For those that did not reach optimality, the relative optimality gaps are very small, the largest being 0.97%.

VII. Conclusion

In this paper, we have presented an energy-efficient optimization model that utilizes the regenerative energy produced by the braking trains in a railway network. We have proposed an optimization strategy and devised an objective function based on the hypograph approach and interval algebra. We have modelled the different constraints present in a railway network. The final optimization model is a MIP model that can be applied to any railway network. We have applied our optimization model to different instances of a railway network for a time horizon of six hours. We have found that for each instance, our model produces an optimal or near-optimal solution within an acceptable running time, with a significant increase in the final overlapping time compared to the existing timetable. Such an increase in the total overlapping time would make it possible to save considerable amount of electrical energy by transferring the regenerative energy of the braking trains to the accelerating ones. As future work, we would like to develop solution techniques for large scale instances of the optimization problem. Additionally, modifying the current model to perform online optimization can be of considerable interest.

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