Part 1

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Theorem 9.1.

[$f : \mathcal{H} \to]-\infty, \infty]$, convex]

(i) f: weakly sequentially lower semicontinuous $\Leftrightarrow$

(ii) f: sequentially lower semi continuous $\Leftrightarrow$

(iii) f: lower semicontinuous $\Leftrightarrow$

(iv) f: weakly lower semicontinuous.

Proof:

$f : \mathcal{H} \to]-\infty, +\infty]$, convex

$\Leftrightarrow$ epi f : convex [By definition]

Now recall:

Lemma 1.24.

[$\mathcal{X}$: Hausdorff space,

$f : \mathcal{X} \to [-\infty, \infty]$]

f : lower semicontinuous $\Leftrightarrow$ epi f : closed in $\mathcal{X} \times \mathbb{R}$ $\Leftrightarrow$ $\forall_{\xi \in \mathbb{R}}$ $\mathrm{lev}_{\leq \xi} f$: closed in $\mathcal{X}$

/* this lemma tells us why in CCP function, f has to be closed, this essentially means that f : lower semicontinuous */

*Lemma 1.35.

[$\mathcal{X}$: Hausdorff space

$f : \mathcal{X} \to [-\infty, +\infty]$]

/* i.e., sequentially lower semicontinuous at every point at $\mathcal{X}$ */

(i) f: sequentially lower semicontinuous $\Leftrightarrow$

(ii) epi f: sequentially closed $\Leftrightarrow$

(iii) $\forall_{\xi \in \mathbb{R}}$ $\mathrm{lev}_{\leq \xi} f$: sequentially closed in $\mathcal{X}$

Theorem 3.32. /* this is a very important theorem which says that for a convex set all the different types of closedness coincides */

[C : convex subset of $\mathcal{H}$]

C: weakly sequentially closed $\Leftrightarrow$ C: sequentially closed $\Leftrightarrow$ C: closed $\Leftrightarrow$ C: weakly closed

/* C: weakly sequentially closed $\overset{\text{def}}{\Leftrightarrow}$ weak limit of every weakly convergent sequence in C is also in C

C: sequentially closed $\overset{\text{def}}{\Leftrightarrow}$ strong " " " strongly " " " C " " " C

C: closed $\overset{\text{def}}{\Leftrightarrow}$ strong " " " " " net " C " " " C

C: weakly closed $\overset{\text{def}}{\Leftrightarrow}$ weak " " " weakly " net " C " " " C */

f : lower semicontinuous $\Leftrightarrow$ epi f : closed, as epi f: convex we have

epi f: sequentially closed $\Leftrightarrow$ f: sequentially lower semicontinuous /* using Lemma 1.35 */

$\Updownarrow$

epi f: closed $\Leftrightarrow$ f : lower semicontinuous /* using Lemma 1.24 */

$\Updownarrow$

epi f : weakly closed $\Leftrightarrow$ f: weakly lower semicontinuous /* using Lemma 1.24 */

$\Updownarrow$

epi f : weakly sequentially closed $\Leftrightarrow$ f: weakly sequentially lower semicontinuous /* using Lemma 1.35 */

$\blacksquare$

Proposition 9.3.

$[\ (f_i)_{i\in I}$: family in $\Gamma(\mathcal{H})\ ]$

$\sup_{i\in I} f_i \in \Gamma(\mathcal{H})$

Proof:

Recall:

/*

* Lemma 1.26.

$[\![\ \mathcal{X}$: Hausdorff space, $(f_i)_{i\in I}$: family of lower semicontinuous functions from $\mathcal{X}$ to $[-\infty,+\infty]\ ]\!]$

$\Rightarrow$

• $\sup_{i\in I} f_i$: lower semicontinuous

* Proposition 8.14. $(f_i)_{i\in I}$: family of convex functions from $\mathcal{H}$ to $[-\infty,+\infty] \Rightarrow \sup_{i\in I} f_i$: convex */

from them we have $\sup_{i\in I} f_i \in \Gamma(\mathcal{H})$ $\blacksquare$

* Proposition 9.8.

$[\ f:\mathcal{H}\to[-\infty,+\infty]\ ] \Rightarrow$

(i) $\breve{f}$: largest lower semicontinuous convex function majorized by f. // $\breve{f} = \sup\{g\in\Gamma(\mathcal{H}) \mid g\leq f\}$

(ii) $\forall_{x\in\mathcal{H}}\ \breve{f} = \underline{\lim}_{y\to x} \breve{f}(y)$

(iii) $\operatorname{epi}\breve{f}$: closed, convex

(iv) $\operatorname{conv}\operatorname{dom} f \subseteq \operatorname{dom}\breve{f} \subseteq \overline{\operatorname{conv}}\operatorname{dom} f$

$f:\mathcal{X}\to[-\infty,+\infty]$

a data structure containing all the values of f on V

$\underline{\lim}_{y\to x} f(y) = \sup_{V\in\mathcal{V}(x)} \inf f(V) = \left(V\in\mathcal{V}(x) \,/\!/\, f(\cdot) \,/\!/ \inf \boxed{\cdot}\right) /\!/ \sup_{V\in\mathcal{V}(x)} (\cdot)$

Proof:

(i) $\breve{f} = \sup\{g\in\Gamma(\mathcal{H}) \mid g\leq f\}$

$= \sup_{g\in\Gamma(\mathcal{H}):g\leq f} g \in \Gamma(\mathcal{H})$ /* Proposition 9.3. /* $\Gamma(\mathcal{H})$: closed under sup */ $[\ (f_i)_{i\in I}$: family in $\Gamma(\mathcal{H})\ ]$ $\sup_{i\in I} f_i\in\Gamma(\mathcal{H})$ */

☺

(ii) From (i) we have

$\breve{f}$: largest lower semicontinuous convex function

* Lemma 1.31.

$[\ \mathcal{X}$: Hausdorff space,

$f:\mathcal{X}\to[-\infty,+\infty]\ ]\!] \Rightarrow$

(i) $\bar{f}$: largest lower semicontinuous function majorized by f

(ii) $\operatorname{epi}\bar{f}$: closed

(iii) $\operatorname{dom} f\subseteq\operatorname{dom}\bar{f}\subseteq\overline{\operatorname{dom}} f$, (iv) $\forall_{x\in\mathcal{X}}\ \bar{f}(x) = \underline{\lim}_{y\to x} f(y)$

(v) $x\in\mathcal{X} \Rightarrow (f$: lower semicontinuous at $f \Leftrightarrow \bar{f}(x) = f(x))$

(vi) $\operatorname{epi}\bar{f} = \overline{\operatorname{epi}} f$.

$\forall_{x\in\mathcal{H}}\ \breve{f}(x) = \underline{\lim}_{y\to x} \breve{f}(y)$

(iii) $\breve{f}$: convex $\Rightarrow$ $\operatorname{epi}\breve{f}$: convex

$\breve{f}$: lower semicontinuous $\Rightarrow$ $\operatorname{epi}\breve{f}$: closed

$\Bigg\}\Rightarrow \operatorname{epi}\breve{f}$: closed, convex.

Lemma 1.24.

$[\ \mathcal{X}$: Hausdorff space,

$f:\mathcal{X}\to[-\infty,\infty]\ ]\!]$

f : lower semicontinuous $\Leftrightarrow$ $\operatorname{epi} f$: closed in $\mathcal{X}\times\mathbb{R}$ $\Leftrightarrow \forall_{\xi\in\mathbb{R}}\ \operatorname{lev}_{\leq\xi} f$: closed in $\mathcal{X}$

Lemma 1.24

[$\mathcal{X}$: Hausdorff space,
$f:\mathcal{X}\to[-\infty,\infty]$]

f : lower semicontinuous $\Leftrightarrow$ epi f : closed in $\mathcal{X}\times\mathbb{R}$ $\Leftrightarrow$ $\forall_{\xi\in\mathbb{R}}$ $\text{lev}_{\leq\xi} f$: closed in $\mathcal{X}$

/* this lemma tells us why in CCP function, f has to be closed, this essentially means that f : lower semicontinuous */

(iv) By definition,

$\breve{f}=\sup\{g\in\Gamma(\mathcal{H})\mid g\leq f\}=$ $\sup\ g$ s.t. $g\in\Gamma(\mathcal{H})$, $g\leq f$

so, $\breve{f}\leq f$ [$\because \breve{f}$ is a feasible solution]

also $\breve{f}\in\Gamma(\mathcal{H})\Rightarrow \breve{f}$: convex $\Rightarrow$ dom $\breve{f}$: convex /* A convex function f has convex domain dom $f=\{x\in\mathcal{H}\mid f(x)<+\infty\}$ */

$\therefore$ conv dom $\breve{f}$ = dom $\breve{f}$

suppose, $x\in$ dom $f \Leftrightarrow f(x)<+\infty$

$\Rightarrow \breve{f}(x)\leq f(x)<+\infty$

$\Leftrightarrow x\in$ dom $\breve{f}$

$\Rightarrow$ dom $f\subseteq$ dom $\breve{f}$

/* Fact 1.1.2 (set extension operator preserves inclusion)
[$\mathfrak{s}$: some set extension operator (e.g. span, $\overline{\text{span}}$, cone, $\overline{\text{cone}}$, conv, $\overline{\text{conv}}$ etc) that is the smallest set based on some property containing the operand set]
$A\subseteq B\Rightarrow \mathfrak{s}(A)\subseteq\mathfrak{s}(B)$ */

$\Rightarrow$ conv dom $f\subseteq$ conv dom $\breve{f}$ $=$ dom $\breve{f}$

now set $C=\overline{\text{conv}}$ dom f

$g:\mathcal{H}\to[-\infty,+\infty]: x\mapsto\begin{cases}\breve{f}(x), & \text{if } x\in C\\ +\infty, & x\notin C\end{cases}$

$g(x)=\begin{cases}\breve{f}(x), & \text{if } x\in C\\ +\infty, & \text{if } x\notin C\end{cases}=\breve{f}(x)+\iota_C(x)$

epi $\breve{f}=\{(x,t)\in\mathcal{H}\times\mathbb{R}\mid \breve{f}(x)\leq t\}$: convex /* recall that in epi f $(x,t)\in\mathcal{H}\times\mathbb{R}$, so $t\neq\pm\infty$ */

epi $g=\{(x,t)\in\mathcal{H}\times\mathbb{R}\mid g(x)=\breve{f}(x)+\iota_C(x)\leq t\}$

$=\underbrace{\{(x,t)\in\mathcal{H}\times\mathbb{R}\mid \breve{f}(x)\leq t\}}_{\text{epi }\breve{f}}\cap(C\times\mathbb{R})$ /* otherwise $g(x)=\infty$ */

$=\underbrace{\text{epi}\breve{f}}_{\text{convex, closed}}\cap\underbrace{(C\times\mathbb{R})}_{\text{closed, convex}}$: closed, convex

/* lower semicontinuous function has closed epigraph */

epi g : closed $\Leftrightarrow$ g : lower semicontinuous /* " */

epi g : convex $\Leftrightarrow$ g : convex /* by definition */

$\therefore g\in\Gamma(\mathcal{H})$

$\forall_x$

$x\in C\Rightarrow g(x)=\breve{f}(x)\leq f(x)$, now if $x\notin C\Rightarrow g(x)=\breve{f}(x)+\iota_C(x)=+\infty$, also, dom $f\subseteq\overline{\text{conv}}$ dom $f=C$, so $x\notin C\Rightarrow x\notin$ dom $f\Leftrightarrow f(x)=+\infty$

$\therefore x\notin C\Rightarrow g(x)=f(x)=+\infty$

$\therefore \forall_{x\in\mathcal{H}}\ g(x)\leq f(x)$

$\Leftrightarrow g\leq f$

and as $g\in\Gamma(\mathcal{H})$, $\breve{f}=\sup\{\tilde{g}\in\Gamma(\mathcal{H})\mid\tilde{g}\leq f\}$ $\breve{f}\geq g$

but $\breve{f}\leq f$ $\quad \forall_{\tilde{g}\in\Gamma(\mathcal{H}):\tilde{g}\leq f}$ $\breve{f}\geq\tilde{g}$ set $\tilde{g}:=g$

$\therefore g\leq\breve{f}$

$\Rightarrow$ dom $g\supseteq$ dom $\breve{f}$ /* as, $\forall_{x\in\text{dom}\breve{f}}\Leftrightarrow\breve{f}(x)<+\infty$ $\ g(x)\leq f(x)<+\infty\Rightarrow g(x)<+\infty$ $\therefore$ dom $\breve{f}\subseteq$ dom g */

again $\forall_{x\in\text{dom}\,g}$ $g(x)<+\infty\Leftrightarrow f(x)+\iota_C(x)<\infty\Rightarrow x\in C$ /* otherwise $\iota_C(x)=+\infty$ */

$\Rightarrow \operatorname{dom} g \supseteq \operatorname{dom} \check{f}$ /* as, $\forall_{x \in \operatorname{dom} \check{f}} \Leftrightarrow \check{f}(x) < +\infty$ $\quad g(x) \leq f(x) < +\infty \Rightarrow g(x) < +\infty$ $\quad \therefore \operatorname{dom} \check{f} \subseteq \operatorname{dom} g$ */

again $\forall_{x \in \operatorname{dom} g}$ $\underset{f(x) + \iota_C(x)}{\underset{\parallel}{g(x)}} < +\infty \Leftrightarrow f(x) + \iota_C(x) < \infty \Rightarrow x \in C$ /* otherwise $\iota_C(x) = +\infty$ */

$\therefore \operatorname{dom} g \subseteq C$

$\operatorname{dom} \check{f} \subseteq \operatorname{dom} g \subseteq C = \overline{\operatorname{conv}} \operatorname{dom} f$

$\therefore \operatorname{conv} \operatorname{dom} f \subseteq \operatorname{dom} \check{f} \subseteq \overline{\operatorname{conv}} \operatorname{dom} f.$

$\blacksquare$

Proposition 9.14.

$[f\in\Gamma_0(\mathcal{H}),\ x\in\mathcal{H},\ y\in \operatorname{dom} f\ ;$

$\forall_{\alpha\in]0,1[}\ x_\alpha := (1-\alpha)x+\alpha y\,]$

$\lim_{\alpha\downarrow 0} f(x_\alpha) = f(x)$

Proof:

* Definition 1.21.

$[\ X$: Hausdorff space, $f: X\to[-\infty,+\infty],\ x\in X\ ]$

f: lower semicontinuous at $x \overset{\text{def}}{\Leftrightarrow} \forall_{(x_a)_{a\in A}:\ \text{net in } X,\ x_a\to x}\ \underline{\lim}\, f(x_a)\geq f(x)$ \# definition in terms of a net

$\overset{\text{def}}{\Leftrightarrow} \forall_{\xi\in]-\infty,f(x)[}\ \exists_{V\in\mathcal{V}(x)}\ f(V)\subseteq]\xi,+\infty]$

$\overset{\text{def}}{\Leftrightarrow} \forall_{\xi:\ \xi<f(x)}\ \exists_{V\in\mathcal{V}(x)}\ \forall_{f(y)\in f(V)}\ \xi<f(y)$ (see the pic)

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first note that $(x_\alpha)_{\alpha\in]0,1[}$: net, now if we partially order them as: $x_{1--}, \ldots, x_{0++}$, i.e., $\lim x_\alpha = \lim_{\alpha\downarrow 0}((1-\alpha)x+\alpha y) = x$

then f: lower semicontinuous at $x \overset{\text{def}}{\Leftrightarrow} f(x)\leq \underline{\lim}\, f(x_\alpha)$ ↪ as $\alpha\downarrow 0$ $= \underline{\lim}_{\alpha\downarrow 0} f(x_\alpha)$

$= \underline{\lim}_{\alpha\downarrow 0} f((1-\alpha)x+\alpha y)$ /* now f: convex */

$\leq \underline{\lim}_{\alpha\downarrow 0} (1-\alpha)f(x)+\alpha f(y)$ /* now note that $\lim_{\alpha\downarrow 0}(1-\alpha)f(x) = f(x)\lim_{\alpha\downarrow 0}(1-\alpha) = f(x) = \underline{\lim}_{\alpha\downarrow 0}(1-\alpha)f(x)$

$\lim_{\alpha\downarrow 0} \alpha \underbrace{f(y)}_{\text{finite}} = 0 = \underline{\lim}_{\alpha\downarrow 0}\alpha f(y)$

finite /* as $y\in\operatorname{dom} f$, f: proper $\Leftrightarrow -\infty\notin f(\mathcal{H})$

$\Downarrow$

$f(y)<+\infty$

$f(y)$ = finite */

$\therefore \lim_{\alpha\downarrow 0}(1-\alpha)f(x)+\lim_{\alpha\downarrow 0}\alpha f(y)$

$= \lim_{\alpha\downarrow 0}(1-\alpha)f(x)+\alpha f(y) = f(x)$

$\therefore \underline{\lim}_{\alpha\downarrow 0}(1-\alpha)f(x)+\alpha f(y) = \lim_{\alpha\downarrow 0}(1-\alpha)f(x)+\alpha f(y) = f(x)$ */

$= f(x)$

so, the inbetween terms would collapse.

$\therefore \lim_{\alpha\downarrow 0} f(x_\alpha) = f(x)$

■

* Proposition 9.17.

$[\ f\in\Gamma_0(\mathcal{H}),\ (x,\xi)\in\mathcal{H}\times\mathbb{R},\ (p,\pi)\in\mathcal{H}\times\mathbb{R}\ ]$

$$(p,\pi) = P_{\operatorname{epi} f}(x,\xi) \Leftrightarrow \begin{cases} \bullet\ \max\{\xi, f(p)\}\leq\pi \\ \bullet\ \forall_{y\in\operatorname{dom} f}\ \langle y-p\mid x-p\rangle + (f(y)-\pi)(\xi-\pi)\leq 0 \end{cases}$$

Proof:

$$f\in\Gamma_0(\mathcal{H}) \Rightarrow \begin{cases} f:\text{convex} \Rightarrow \operatorname{epi} f:\text{convex} & /\!*\text{ By def }*/ \\ f:\text{proper} \Rightarrow \operatorname{epi} f:\text{nonempty} & /\!*\ f:\text{proper} \overset{\text{def}}{\Leftrightarrow} -\infty\notin f(\mathcal{H}),\ \operatorname{dom} f=\{x\in\mathcal{H}\mid f(x)<+\infty\}\neq\emptyset,\ \operatorname{epi} f=\{(x,t)\in\mathcal{H}\times\mathbb{R}\mid f(x)\leq t\}\neq\emptyset\ */ \\ f:\text{lower semicontinuous} \Rightarrow \operatorname{epi} f:\text{closed} & /\!*\text{Lemma 1.24}*/ \end{cases}$$

$\exists_{(x,t)\in\mathcal{H}\times\mathbb{R}}\ f(x)\leq t$

$\Rightarrow \operatorname{epi} f$: convex, nonempty, closed

$(p,\pi) := P_{\operatorname{epi} f}(x,\xi)$ well defined

/* Charecterization of projection on closed convex nonempty set */ Theorem 3.14. ★★★

(C: nonempty closed convex subset of $\mathcal{H}$) $\Rightarrow$
$\begin{cases} \bullet\ C: \text{Chebyshev set, i.e., every point in } \mathcal{H} \text{ has exactly one projection on } C \\ \bullet\ \forall_{x\in\mathcal{H}}\ (p=P_C(x) \Leftrightarrow (p\in C,\ \forall_{y\in C}\ \langle y-p\mid x-p\rangle\leq 0)) \end{cases}$

now recall

/*

/* Charecterization of projection on closed convex nonempty set */ Theorem 3.14. ★★★

($\cancel{C}$ $\operatorname{epi} f$: nonempty closed convex subset of $\mathcal{H}$) $\Rightarrow$
$\begin{cases} \bullet\ \cancel{C}\ \operatorname{epi} f: \text{Chebyshev set, i.e., every point in } \mathcal{H} \text{ has exactly one projection on } C \\ \bullet\ \forall_{\cancel{x\in\mathcal{H}}\ (x,\xi)\in\mathcal{H}\times\mathbb{R}}\ (\cancel{p}\,(p,\pi) = P_{\operatorname{epi} f}\,\cancel{(x)}\,(x,\xi) \Leftrightarrow (\cancel{p\in C}\ (p,\pi)\in\operatorname{epi} f,\ \cancel{\forall_{y\in C}\ \langle y-p\mid x-p\rangle\leq 0})\ \forall_{(y,\eta)\in\operatorname{epi} f}\ \langle (y,\eta)-(p,\pi)\mid(x,\xi)-(p,\pi)\rangle\leq 0) \end{cases}$

*/

$(p,\pi)\in\operatorname{epi} f \Leftrightarrow f(p)\leq\pi$

$\langle (y,\eta)-(p,\pi)\mid(x,\xi)-(p,\pi)\rangle = \langle (y-p,\eta-\pi)\mid(x-p,\xi-\pi)\rangle = \langle y-p\mid x-p\rangle + (\eta-\pi)(\xi-\pi)\leq 0$ [$\because x^{T}y = \sum_{i=1}^{n} x_i y_i = (x_1,\ldots,x_{n-1})^{T}(y_1,\ldots,y_{n-1}) + x_n y_n$ */

/* now,
$\forall_y\forall_\eta\ (y,\eta)\in\operatorname{epi} f \Leftrightarrow \forall_{y\in\operatorname{dom} f}\ \forall_{\eta\in\mathbb{R}}\ f(y)\leq\eta$ /* $\forall_\eta\ \exists_{\lambda\in\mathbb{R}_+}\ \eta = f(y)+\lambda$ */ $\Leftrightarrow \forall_{y\in\operatorname{dom} f}\ \forall_{\lambda\in\mathbb{R}_+}\ (y, f(y)+\lambda)\in\operatorname{epi} f$ */

so, put $\eta = f(y)+\lambda$

$\Leftrightarrow \forall_{y\in\operatorname{dom} f}\ \forall_{\lambda\in\mathbb{R}_+}\ \langle y-p\mid x-p\rangle + (f(y)+\lambda-\pi)(\xi-\pi)\leq 0$

so if $\lambda\uparrow+\infty$ then we must have $\xi\leq\pi$ (otherwise nothing can stop the L.H.S >0)

so if $\lambda \uparrow +\infty$ then we have $\xi \leq \pi$ (otherwise nothing can stop the L.H.S >0)

$\boxed{\max\{\xi, f(p)\} \leq \pi}$

$\lambda := 0$ yields:

$\boxed{\forall_{y \in \operatorname{dom} f} \quad \langle y-p | x-p \rangle + (f(y)-\pi)(\xi-\pi) \leq 0}$

∎

✱ Theorem 9.19.

$[\![f \in \Gamma_0(\mathcal{H})]\!]$

f: possesses a continuous affine minorant.

Proof: Set,

$x \in \operatorname{dom} f \leftrightarrow f(x) < +\infty$

$\xi \in]-\infty, f(x)[\subseteq \mathbb{R} \Rightarrow \xi < f(x) \Rightarrow (x, \xi) \notin \operatorname{epi} f = \{(\tilde{x}, \tilde{\xi}) \in \mathcal{H} \times \mathbb{R} \mid \tilde{\xi} \geq f(\tilde{x})\}$

$(p, \pi) = P_{\operatorname{epi} f}(x, \xi)$, as $(p, \pi) \in \operatorname{epi} f = \{(\tilde{x}, \tilde{\xi}) \in \mathcal{H} \times \mathbb{R} \mid \tilde{\xi} \geq f(\tilde{x})\} \Rightarrow p \in \mathcal{H},\ \pi \in \mathbb{R},\ f(p) \leq \pi$

/✱ recall:

✱ Proposition 9.18.

$[\![f \in \Gamma_0(\mathcal{H}),\ x \in \operatorname{dom} f,\ \xi \in]-\infty, f(x)[,\ (p, \pi) \in \mathcal{H} \times \mathbb{R}]\!]$

$(p, \pi) = P_{\operatorname{epi} f}(x, \xi) \Leftrightarrow \begin{cases} \bullet\ \xi < f(p) = \pi \\ \bullet\ \forall_{y \in \operatorname{dom} f}\ \langle y-p | x-p \rangle \leq (f(y)-f(p)) \underbrace{(f(p)-\xi)}_{>0} \end{cases}$ ✱/

$f(p) = \pi \in \mathbb{R} \rightarrow p \in \operatorname{dom} f$

$\langle y-p \mid \underbrace{\frac{x-p}{f(p)-\xi}}_{=u} \rangle \leq f(y)-f(p) \Leftrightarrow \langle y-p | u \rangle + f(p) \leq f(y) \Leftrightarrow f(p) - \langle p | u \rangle \leq f(y) - \langle y | u \rangle$

/✱ u is a constant w.r.t variable y ✱/

as, $f(p) = \pi$: finite
and $\langle p | u \rangle$: finite $\Big\}$ $\underbrace{f(p) - \langle p | u \rangle}_{\zeta} = \text{finite} > -\infty$

$\therefore \forall_{y \in \operatorname{dom} f} \quad -\infty < \zeta \leq f(y) - \langle y | u \rangle$

and also, $\forall_{y \in \mathcal{H} \setminus \operatorname{dom} f} \quad \underset{+\infty}{\underset{\shortparallel}{f(y)}} - \underbrace{\langle y | u \rangle}_{\text{finite}} = +\infty > -\infty$

$\therefore \forall_{y \in \mathcal{H}} \quad f(y) - \langle y | u \rangle > -\infty$

$\leftrightarrow \quad f - \langle \cdot | u \rangle$: bounded below

$\overset{\text{def}}{\leftrightarrow} \quad f$: has a continuous affine minorant

∎

Corollary 9.20.

$[\![f \in \Gamma_0(\mathcal{H})]\!]$

$\forall_{\substack{\text{nonempty bounded} \\ \text{subset of } \mathcal{H}}}$ f: bounded below on that set

Proof:

C: nonempty bounded subset of $\mathcal{H}$

$\beta = \sup_{x \in C} \|x\| \in \mathbb{R}_+ \leftrightarrow \forall_{x \in C} \quad \|x\| \leq \beta$

As by Theorem 9.19. $[\![f \in \Gamma_0(\mathcal{H})]\!]$ $\exists_{\substack{g:\text{ continuous} \\ \text{affine function}}}$ $g \leq f$

say $g = \langle \cdot | u \rangle + \eta$

now, $\forall_{y \in \mathcal{H}} \quad g(y) \leq f(y)$

$\Rightarrow \forall_{x \in C \subseteq \mathcal{H}} \quad f(x) \geq g(x)$

$= \langle x | u \rangle + \eta$

$\geq -|\langle x | u \rangle| + \eta$

$\geq -\|x\| \|u\| + \eta$ /✱ Cauchy-Schwarz: $|\langle x | u \rangle| \leq \|x\| \|u\|$

$\geq \underbrace{-\beta}_{\text{finite}} \underbrace{\|u\|}_{\text{finite}} + \underbrace{\eta}_{\text{finite}}$ $\quad \leftrightarrow -|\langle x | u \rangle| \geq -\|x\| \|u\|$ ✱/

$> -\infty$

$\therefore \forall_{x \in C} \quad f(x) > -\infty$

■

***Proposition 9.26.**

[$g: \mathcal{H} \to]-\infty, +\infty]$, proper, convex

dom g: open, g: continuous on dom g

$$f: \mathcal{H} \to]-\infty, +\infty]: x \mapsto \begin{cases} g(x), & x \in \operatorname{dom} g \\ \lim_{y \to x} g(y), & x \in \operatorname{bdry} \operatorname{dom} g \\ +\infty, & x \in \mathcal{H} \setminus \overline{\operatorname{dom}} g \end{cases}$$

]

$\Rightarrow$

$f = \breve{g}$, $f \in \Gamma_0(\mathcal{H})$.

Proof: Part 1: We will show that, $\forall_{x \in \operatorname{dom} g} \; f = \breve{g}$

$C = \operatorname{dom} g$

Recall:

* Proposition 9.8. ★★

[$f: \mathcal{H} \to [-\infty, +\infty]$] $\Rightarrow$

(i) $\breve{f}$: largest lower semicontinuous convex function majorized by f.

(ii) $\forall_{x \in \mathcal{H}} \; \breve{f} = \underline{\lim}_{y \to x} f(y)$ /* $\underline{\lim}_{y \to x} f(y) = (V \in \mathcal{V}(x) \,//\, f(x) \,//\, \inf \boxed{2}) \,//\, \sup_{V \in \mathcal{V}(x)} \boxed{1}$ */

(iii) epi $\breve{f}$: closed, convex

(iv) $\operatorname{conv} \operatorname{dom} f \subseteq \operatorname{dom} \breve{f} \subseteq \overline{\operatorname{conv}} \operatorname{dom} f$.

$\breve{g} = \sup\{\tilde{g} \in \Gamma(\mathcal{H}) \mid \tilde{g} \leq g\}$, $\breve{g} \in \Gamma(\mathcal{H})$

$\Rightarrow \breve{g} \leq g$ /* recall: $f \geq g \Rightarrow \operatorname{dom} f \subseteq \operatorname{dom} g$ */

$\Rightarrow \operatorname{dom} \breve{g} \supseteq \operatorname{dom} g = C$

as g: convex $\Rightarrow$ dom g: convex /* proposition 8.2 */

$\Rightarrow \operatorname{conv} \operatorname{dom} g = \operatorname{dom} g$, $\overline{\operatorname{conv}} \underbrace{\operatorname{dom} g}_{\text{convex}} = \overline{\operatorname{dom}}\, g$

$C = \operatorname{dom} g \subseteq \operatorname{dom} \breve{g} \subseteq \overline{\operatorname{dom}}\, g = \overline{C}$

Assume. $x \in C = \operatorname{dom} g = \{\tilde{x} \in \mathcal{H} \mid g(\tilde{x}) < +\infty\} \Rightarrow g(x) < +\infty$

then

$\underbrace{\breve{g}(x) \leq g(x)} < +\infty$

$(\because \breve{g} \leq g)$

/*
recall Theorem 9.9. [$f: \mathcal{H} \to [-\infty, +\infty]$] $\Rightarrow$ epi $\breve{f} = \overline{\operatorname{conv}}$ epi f. */

now, g: convex $\Rightarrow$ epi g: convex by definition.

using Theorem 9.9. epi $\breve{g} = \overline{\operatorname{conv}}$ epi g = $\overline{\text{epi}}\, g$

/* $(x,t) \in \operatorname{epi} \breve{g} \leftrightarrow \breve{g}(x) \leq t \in \mathbb{R}$

$(x, \breve{g}(x)) \in \operatorname{epi} \breve{g} \Leftrightarrow \breve{g}(x) \leq \breve{g}(x) < +\infty$ */

$\therefore (x, \breve{g}(x)) \in \operatorname{epi} \breve{g} = \overline{\operatorname{epi}}\, g$

recall that:

* lemma 1.10. /* This is quite useful

[C: subset of a Hausdorff space $\mathcal{X}$,

$x \in \mathcal{X}$]

$x \in \overline{C} \Leftrightarrow \exists_{(x_a)_{a \in A}: \text{net in } C} \; x_a \to x$

Theorem 3.32. /* this is a very important theorem which says that for a convex set all the different types of closedness coincides */

[C: convex subset of $\mathcal{H}$]

C: weakly sequentially closed $\Leftrightarrow$ C: sequentially closed $\Leftrightarrow$ C: closed $\Leftrightarrow$ C: weakly closed

$\Rightarrow$ [C: convex] $x \in \overline{C} \Leftrightarrow \exists_{(x_n)_{n \in \mathbb{N}}: \text{sequence in } C} \; x_n \to x$

as, $(x, \breve{g}(x)) \in \overline{\operatorname{epi}}\, g$

$\Rightarrow$

$\exists_{(x_n, \xi_n)_{n \in \mathbb{N}}: \text{sequence in epi} g} \; (x_n, \xi_n) \to (x, \breve{g}(x))$

$\Rightarrow x_n \to x$,

$\xi_n \to \breve{g}(x)$

$\Rightarrow \breve{g}(x) = \lim \xi_n = \underline{\lim}\, \xi_n$ /* When the limit exists the lim sup, $\overline{\lim}$ and lim inf, $\underline{\lim}$ are same */

$\geq \underline{\lim}\, g(x_n)$ /* $\because (x_n, \xi_n) \in \operatorname{epi} g \;\; \forall_{n \in \mathbb{N}}$

$\Leftrightarrow g(x_n) \leq \xi_n < +\infty \;\; \forall_{n \in \mathbb{N}}$

$\Leftrightarrow \underline{\lim}\, g(x_n) \leq \underline{\lim}\, \xi_n$ */

$\geq \lim \breve{g}(x_n)$ [$\because g \geq \breve{g}$ by definition]

$\Leftrightarrow g(x_n) \leqslant \xi_n < +\infty \quad \forall_{n\in\mathbb{N}}$

$\Leftrightarrow \underline{\lim}\, g(x_n) \leqslant \underline{\lim}\, \xi_n$ */

$\geqslant \underline{\lim}\, \breve{g}(x_n)$ [$\because$ $g \geqslant \breve{g}$ by definition]

$\geqslant \breve{g}(x)$ /* $\because$ f : lower semicontinuous at x $\overset{def}{\Leftrightarrow}$ $\forall_{(x_a)_{a\in A} : \text{net in } x : x_a \to x}$ $\underline{\lim}\, f(x_a) \geqslant f(x)$

as $\breve{g}$: lower semicontinuous, and $x_n \to x$ $\Rightarrow$ $\underline{\lim}\, \breve{g}(x_n) \geqslant \breve{g}(x)$ */

So, the inequality system will collapse $\therefore \underline{\lim}\, g(x_n) = \underline{\lim}\, \breve{g}(x_n) = \breve{g}(x)$

So, for $x \in C = \operatorname{dom} g$

$\underbrace{f(x) = g(x)}_{\text{by given}} = \lim g(x_n)$ /* $x_n \to x$, g: continuous on $\operatorname{dom} g = C$ by given, by def $\therefore$ $g(x_n) = g(x)$ */

$= \underline{\lim}\, g(x_n)$

$= \breve{g}(x)$ [Part 1 proved]

Part 2: We will show,

$\forall (x \in \mathcal{H} \setminus \bar{C} = \mathcal{H} \setminus \overline{\operatorname{dom}}\, g)$ $f(x) = \breve{g}(x) = +\infty$

$x \in \mathcal{H} \setminus \bar{C}$

says:

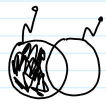
$\operatorname{dom} g \subseteq \operatorname{dom} \breve{g} \subseteq \overline{\operatorname{dom}}\, g = \bar{C}$

$\operatorname{dom} \breve{g} \subseteq \overline{\operatorname{dom}}\, g$

$\Rightarrow \mathcal{H} \setminus \operatorname{dom} \breve{g} \supseteq \mathcal{H} \setminus \overline{\operatorname{dom}}\, g = \mathcal{H} \setminus \bar{C}$

bdry dom g

dom g



$x \in \mathcal{H} \setminus \bar{C} \Rightarrow x \in \mathcal{H} \setminus \operatorname{dom} \breve{g} \Rightarrow \breve{g}(x) = +\infty$

By construction, $x \in \mathcal{H} \setminus \bar{C} \Rightarrow f(x) = +\infty$

$\Rightarrow \forall_{x \in \mathcal{H} \setminus \bar{C}}$ $f(x) = \breve{g}(x)$.

Part 3. in this part we will prove that: $\forall_{x \in \operatorname{bdry} \operatorname{dom} g}$ $f(x) = \breve{g}(x)$; now $\operatorname{bdry} \operatorname{dom} g = (\operatorname{bdry} \operatorname{dom} g \setminus \overline{\operatorname{dom}}\, g) \cup (\operatorname{bdry} \operatorname{dom} g \cap \overline{\operatorname{dom}}\, g)$:

so we will focus on them seperately

consider $x \in (\operatorname{bdry} C) \setminus (\overline{\operatorname{dom}}\, g) = (\operatorname{bdry} \operatorname{dom} g) \setminus (\overline{\operatorname{dom}}\, g)$

$\Rightarrow \breve{g}(x) = +\infty$. /* because $\operatorname{dom} g \subseteq \overline{\operatorname{dom}}\, g$, we are removing $\operatorname{dom} \breve{g}$, $g(x) = +\infty$ */

$f(x) = \underbrace{\underline{\lim}_{y \to x}\, g(y)}$ $\geqslant \underline{\lim}_{y \to x}\, \breve{g}(y)$ /* $\because$ $g \geqslant \breve{g}$ */

// by construction $\geqslant \breve{g}(x)$ /* $\breve{g}$: lower semicontinuous $\overset{def}{\Leftrightarrow}$ $y \to x \Rightarrow \underline{\lim}\, \breve{g}(y) \geqslant \breve{g}(x)$ */

$= +\infty$

$\forall_{x \in \operatorname{bdry} C \setminus \operatorname{dom} \breve{g}}$ $f(x) = \breve{g}(x) = +\infty$

Now consider,

$x \in \operatorname{bdry} C \cap \operatorname{dom} \breve{g}$

$= \operatorname{bdry} \operatorname{dom} g \cap \operatorname{dom} \breve{g}$ $\Rightarrow$ $x \in \operatorname{dom} \breve{g}$ $\Rightarrow$ $(x, \breve{g}(x)) \in \operatorname{epi} \breve{g} = \overline{\operatorname{epi}}\, g$ /* Using

says $\exists_{(x_n, \xi_n)_{n\in\mathbb{N}} : \text{sequence in } \operatorname{epi} g}$ $(x_n, \xi_n) \to (x, \breve{g}(x))$ $\leftrightarrow$ $(x, \breve{g}(x)) \in \overline{\operatorname{epi}}\, g$

Definition 1.23.0.

[$\mathcal{X}$: Hausdorff space; $f : \mathcal{X} \to [-\infty, +\infty]$] a data structure containing all the values of f in V

$\underline{\lim}_{y \to x}\, f(y) = \sup_{V \in \mathcal{V}(x)} \inf f(V) = \left(V \in \mathcal{V}(x) \,/\!/\, f(x) \,/\!/\, \inf \boxed{?} \right) /\!/ \sup_{V \in \mathcal{V}(x)} (?)$

now, using similar logic as in Lemma 1.23 we have: /* as $x_n \to x \in \operatorname{dom} \breve{g}$; $\underline{\lim}_{y \to x}\, g(y) = \min_{(x_a)_{a\in A} : x_a \to x}$ $\underline{\lim}\, g(x_a) \leqslant \underline{\lim}\, g(x_n)$ */

lower semicontinuous

Lemma 1.23.

[$\mathcal{X}$: Hausdorff space

$f : \mathcal{X} \to [-\infty, +\infty]$

$x \in \mathcal{X}$

$N(x)$: set of all nets in $\mathcal{X}$ converging to x]

$\underline{\lim}_{y \to x}\, f(y) = \min_{(x_a)_{a\in A} \in N(x)} \underline{\lim}\, f(x_a)$

$f(x) = \underline{\lim}_{y \to x}\, g(y) \geqslant \underline{\lim}_{y \to x}\, \breve{g}(y) = \breve{g}(x) = \lim \xi_n = \underline{\lim}\, \xi_n \geqslant \underline{\lim}\, g(x_n) \geqslant \underline{\lim}_{y \to x}\, g(y) = f(x)$ $\Rightarrow$ collapse of inequalities

$\Rightarrow f(x) = \breve{g}(x)$ /* $g \geqslant \breve{g}$ */ /* $(x_n, \xi_n) \in \operatorname{epi} g \leftrightarrow g(x_n) \leqslant \xi_n \Rightarrow \underline{\lim}\, g(x_n) \leqslant \underline{\lim}\, \xi_n$ */

$\therefore \forall_{x \in \operatorname{bdry} C \cap \operatorname{dom} \breve{g}}$ $f(x) = g(x)$

Combining Parts 1,2,3 we have $f = \breve{g} \in \Gamma_0(\mathcal{H})$

$\therefore$ f : lower semicontinuous, convex, proper.

■