* Proposition 8.2.

$f: \mathcal{H} \to [-\infty, +\infty]$: convex $\Rightarrow$ $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\}$: convex

Proof: $L: \mathcal{H} \times \mathbb{R} \to \mathcal{H} : (x, \xi) \mapsto x$

$$\left.\begin{aligned} L(x,\xi) &= x \\ L(y,\xi) &= y \end{aligned}\right\} \quad L(\alpha x + \beta y, \xi) = \alpha x + \beta y = \alpha L(x,\xi) + \beta L(y,\xi)$$

$\therefore$ L : linear operator

$$L(\operatorname{epi} f) = L(\{(x,\xi) \in \mathcal{H} \times \mathbb{R} \mid f(x) \leq \xi\})$$
$$= \{x \in \mathcal{H} \mid f(x) \leq \xi, \xi \in \mathbb{R}\} \quad /\!* \text{ recall: } [-\infty, \infty] = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} \ */$$
$$= \{x \in \mathcal{H} \mid f(x) < +\infty\}$$

By definition epi f : convex $\Rightarrow$ $\underbrace{L}_{\text{linear hence affine operator}}(\operatorname{epi} f)$: convex /* using :

*Proposition 3.5. (Preservation of convexity of a set under affine transformation)
[$\mathcal{K}$: real Hilbert space
$T: \mathcal{H} \to \mathcal{K}$, affine operator
C : convex subset of $\mathcal{H}$
D : convex subset of $\mathcal{K}$] $T(C)$: convex subset of $\mathcal{K}$
$T^{-1}(D)$: convex subset of $\mathcal{H}$ */

$\therefore$ dom f : convex.

* Proposition 8.12. (Convexity conditions for function on real line)

[$\varphi: \mathbb{R} \to]-\infty, +\infty]$, proper function, differentiable on $\underbrace{I}_{\text{nonempty open interval in dom } \varphi}$]

(i) φ' : increasing on $I \Rightarrow \varphi + \iota_I$: convex

(ii) φ' : strictly increasing on $I \Rightarrow \varphi$: strictly convex on I.

Proof:

$\forall_{x, y \in I} \ \forall_{\alpha \in]0,1[}$

$\psi: \mathbb{R} \to]-\infty, +\infty] : z \mapsto \alpha\varphi(x) + (1-\alpha)\varphi(z) - \varphi(\alpha x + (1-\alpha) z)$

$\psi(\underbrace{z}_{\in I}) = \alpha\varphi(x) + (1-\alpha)\varphi(z) - \varphi(\alpha x + (1-\alpha)z)$

then $\underbrace{\psi'(z)}_{\text{will exist by given}} = 0 + (1-\alpha)\varphi'(z) - \underbrace{\dfrac{d\,\varphi(\alpha x + (1-\alpha)z)}{d(\alpha x + (1-\alpha)z)}}_{\varphi'(\alpha x + (1-\alpha)z)} \cdot \underbrace{\dfrac{d(\alpha x + (1-\alpha)z)}{dz}}_{(1-\alpha)}$

$$= (1-\alpha)\varphi'(z) - (1-\alpha)\varphi'(\alpha x + (1-\alpha)z)$$
$$= (1-\alpha)\left(\varphi'(z) - \varphi'(\underbrace{\alpha x + (1-\alpha)z}_{\alpha(x-z)+z})\right) \quad \ldots (8.10)$$

now $\psi'(x) = \psi'(z)\big|_{z:=x} = (1-\alpha)\left(\varphi'(x) - \varphi'(\cancel{\alpha x} + x - \cancel{\alpha x})\right)$
$$= (1-\alpha)(\varphi'(x) - \varphi'(x)) = 0$$

(i) go through the alternative proof:

if $z < x$, then $\psi'(z) = (1-\alpha)\left(\varphi'(z) - \varphi'(\underbrace{\alpha(x-z)}_{>0} + z)\right)$

$$= (1-\alpha)\underbrace{(\varphi'(z) - \varphi'(z_+))}_{\leq 0} \leq 0 \quad [\because \text{given, } \varphi' : \text{increasing} \Leftrightarrow \varphi'(z_+) - \varphi'(z) \geq 0]$$

if $z > x$ then $\psi'(z) = (1-\alpha)\left(\varphi'(z) - \varphi'(\underbrace{\alpha(x-z)}_{<0} + z)\right)$

$$= (1-\alpha)\underbrace{(\varphi'(z) - \varphi'(z_-))}_{\geq 0} \geq 0$$

$\forall_{z_1 < x < z_2}$ $\psi'(z_1) \leq \underset{\psi'(x)}{0} \leq \psi'(z_2) \Rightarrow \psi'(z_1) \leq 0 \leq \psi'(z_2)$ $\therefore \forall_{z_1 < x < z_2}$ $\psi'(z_1) \leq 0 \leq \psi'(z_2)$ . . . (eq.F)

When $z = x$, then $\psi'(z)_{z:=x} = 0$: local minimum achieved at $z = x$, the minimum value is $\psi(x) = \alpha\varphi(x) + (1-\alpha)\varphi(x) - \varphi(\alpha x + (1-\alpha)x) = \varphi(x) - \varphi(\alpha x + x - \alpha x) = 0$

so, from basic calculus, ψ achieves its global minimum of 0 on I at x

$\Leftrightarrow \forall_{z \in I, x \in I}$ $\psi(x) \leq \psi(z)$ /* $\psi(z) = \alpha\varphi(x) + (1-\alpha)\varphi(z) - \varphi(\alpha x + (1-\alpha)z)$
$\psi(x) = \alpha\varphi(x) + (1-\alpha)\varphi(x) - \varphi(\underbrace{\alpha x + (1-\alpha)x}_{\cancel{\alpha x} + x - \cancel{\alpha x} = x})$
$= \alpha\varphi(x) + \varphi(x) - \alpha\varphi(x) - \varphi(x) = 0$ */

* alternative proof:
take $x_1, x_2, x_3 \in I$: $x_1 < x_2 < x_3$
By mean value theorem: $\exists_{\xi \in (x_1, x_2)}$ $\varphi'(\xi) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$
$\exists_{\eta \in (x_2, x_3)}$ $\varphi'(\eta) = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$ $\Rightarrow \xi < \eta$
now φ' : increasing $\Rightarrow \varphi'(\xi) \leq \varphi'(\eta)$
$\Rightarrow \dfrac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \dfrac{\varphi(x_3) - \varphi(x_2)}{x_3 - x_2}$
↳ this satisfies the definition of a convex function on $I \subseteq \mathbb{R}$

f is convex on I if and only if:
$$\forall x_1, x_2, x_3 \in I : x_1 < x_2 < x_3 : \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

$\therefore \varphi$: convex on $I \Rightarrow \varphi + \iota_I$: convex */

$\Leftrightarrow \forall_{x, z \in I}$ $0 \leq \alpha\varphi(x) + (1-\alpha)\varphi(z) - \varphi(\alpha x + (1-\alpha)z)$

$\Leftrightarrow \forall_{x, z \in I}$ $\varphi(\underbrace{\alpha}_{\alpha \in]0,1[} x + (1-\alpha)z) \leq \alpha\varphi(x) + (1-\alpha)\varphi(z)$

$\Leftrightarrow$ φ : convex on I

$\Leftrightarrow$ $\varphi + \iota_I$: convex

(ii) just like (i)

$z < x \Rightarrow \psi'(z) < 0$

$z > x \Rightarrow \psi'(z) > 0$

$z := x \Rightarrow \psi'(x) = 0$

$\Rightarrow$ ψ : achieves strict minimum on I at x

$\forall_{z, x \in I}$ $\underset{0}{\psi(x)} < \psi(z)$

$\Leftrightarrow \forall_{x, z \in I}$ $\varphi(\alpha x + (1-\alpha)z) < \alpha\varphi(x) + (1-\alpha)\varphi(z)$

$\Leftrightarrow \varphi$: strictly convex on I

$\Leftrightarrow \varphi + \iota_I$: strictly convex ∎

* Proposition 8.14.

$(f_i)_{i\in I}$: family of convex functions from $\mathcal{H}$ to $[-\infty,+\infty]$ $\Rightarrow \sup\limits_{i\in I} f_i$: convex.

Proof:

/* recall

Lemma 1.6. ⟦ $(f_i)_{i\in I}$: family of functions from $\mathcal{X} \to [-\infty,+\infty]$ ⟧ (i) $\operatorname{epi}\left(\sup\limits_{i\in I} f_i\right) = \bigcap\limits_{i\in I} \operatorname{epi} f_i$.

(ii) I : finite $\Rightarrow \operatorname{epi}\left(\min\limits_{i\in I} f_i\right) = \bigcup\limits_{i\in I} \operatorname{epi} f_i$. */

$\operatorname{epi}\left(\sup\limits_{i\in I} f_i\right) = \bigcap\limits_{i\in I} \underbrace{\operatorname{epi} f_i}_{\text{convex by given}}$: convex /* intersection of convex sets is convex */

$\Rightarrow \sup\limits_{i\in I} f_i$: convex. ∎

* Proposition 8.19.

⟦ $f : \mathcal{H} \to]-\infty,+\infty]$: convex

$\varphi : \mathbb{R} \to]-\infty,+\infty]$: convex

$C = \operatorname{conv}(\mathbb{R} \cap \operatorname{ran} f)$ /* $\operatorname{ran} f = f(\mathcal{H})$, as $\pm\infty \notin \mathbb{R}$, so $C \not\ni \pm\infty$ */

$\tilde{\varphi}$: extension of φ, : $\underbrace{]-\infty,+\infty]}_{=\mathbb{R}\cup\{+\infty\}} \to]-\infty,+\infty]$, $\tilde{\varphi}(+\infty) = +\infty$

⟧

($C \subseteq \operatorname{dom}\varphi$, /* $\operatorname{dom}\varphi = \{x \in]-\infty,+\infty] \mid \varphi(x) < +\infty\}$ */

φ : increasing on C)

$\Rightarrow$

$\tilde{\varphi} \circ f$: convex

Proof:

$\operatorname{dom}(\tilde{\varphi}\circ f) = \{x \in \mathcal{H} \mid (\tilde{\varphi}\circ f)(x) < +\infty\} = \{x \in \mathcal{H} \mid (\tilde{\varphi}\circ f)(x) < +\infty,\ \underset{\Updownarrow}{\overset{x\in\operatorname{dom} f}{\overset{\Updownarrow}{f(x) \neq \infty}}}\} = \{x \in \operatorname{dom} f \mid (\tilde{\varphi}\circ f)(x) < +\infty\} \subseteq \operatorname{dom} f$

$\tilde{\varphi}(f(x)) < +\infty \Rightarrow f(x) \neq \infty$ [as $\tilde{\varphi}(+\infty) = +\infty$]

on $\operatorname{dom}(\tilde{\varphi}\circ f)$, $\tilde{\varphi}\circ f$ coincides with $\varphi\circ f$

as $x \in \operatorname{dom}(\tilde{\varphi}\circ f) \Leftrightarrow x \in \operatorname{dom} f,\ (\tilde{\varphi}\circ f)(x) < +\infty$

$\Leftrightarrow \underbrace{-\infty < f(x) < +\infty}_{/\ast\ \because f : \mathcal{H} \to]-\infty,+\infty]\ \ast/}$, $\tilde{\varphi}(\underbrace{f(x)}_{\in]-\infty,+\infty[= \mathbb{R}}) = \varphi(f(x))$

$\therefore$ on $\operatorname{dom}(\tilde{\varphi}\circ f)$, $\tilde{\varphi}\circ f$ and $\varphi\circ f$ are same.

$\forall_{x,y \in \operatorname{dom}(\tilde{\varphi}\circ f)}\ \forall_{\alpha \in]0,1[}$

$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$: Clearly both L.H.S and R.H.S will belong to $\operatorname{conv}(\mathbb{R} \cap \operatorname{ran} f)$

now φ : convex, increasing on $C \subseteq \operatorname{dom}\varphi$

$\Updownarrow$

$x \geq y \Rightarrow \varphi(x) \geq \varphi(y)$

so,

$\varphi(f(\alpha x + (1-\alpha)y)) \leq \varphi(\alpha f(x) + (1-\alpha) f(y)) \leq \alpha\, \varphi(f(x)) + (1-\alpha)\, \varphi(f(y))$

[$\because \varphi$: convex on $\operatorname{dom}\varphi$ and $C \subseteq \operatorname{dom}\varphi$ by given

$\forall_{\tilde{x},\tilde{y} \in C}\ \forall_{\tilde{\alpha} \in]0,1[}\ \varphi(\tilde{\alpha}\tilde{x} + (1-\tilde{\alpha})y) \leq \tilde{\alpha}\varphi(\tilde{x}) + (1-\tilde{\alpha})\varphi(\tilde{y})$ */

$\Leftrightarrow (\tilde{\varphi}\circ f)(\alpha x + (1-\alpha)y) \leq \alpha(\tilde{\varphi}\circ f)(x) + (1-\alpha)(\tilde{\varphi}\circ f)(y)$ [$\because$ on $\operatorname{dom}(\tilde{\varphi}\circ f)$, $\tilde{\varphi}\circ f = \varphi\circ f$]

so $\forall_{x,y \in \operatorname{dom}(\tilde{\varphi}\circ f)}\ \forall_{\alpha\in]0,1[}\ (\tilde{\varphi}\circ f)(\alpha x + (1-\alpha)y) \leq \alpha(\tilde{\varphi}\circ f)(x) + (1-\alpha)(\tilde{\varphi}\circ f)(y)$

$\Leftrightarrow (\tilde{\varphi}\circ f)$: convex. ∎

* Proposition 8.23.

⟦ $\varphi : \mathcal{H} \to]-\infty,+\infty]$: convex

$f : \mathbb{R}\times\mathcal{H} \to]-\infty,+\infty] : (\xi, x) \mapsto \begin{cases} \xi\varphi(x/\xi), & \text{if } \xi > 0 \\ +\infty, & \text{otherwise} \end{cases}$

⟧

Proof:

$C = \{1\} \times \operatorname{epi}\varphi$

$\varphi(\bar{x}) \leq \underbrace{\bar{\xi}}_{<+\infty} \Leftrightarrow (\bar{x},\bar{\xi}) \in \operatorname{epi}\varphi$

φ : convex $\Rightarrow \operatorname{epi}\varphi = \{(\bar{x},\bar{\xi}) \in \mathcal{H}\times\mathbb{R} \mid \varphi(\bar{x}) \leq \bar{\xi}\}$: convex set /* By definition */

$\Rightarrow C := \{1\}\times\operatorname{epi}\varphi = \{(1,\bar{x},\bar{\xi}) \in \mathbb{R}\times\mathcal{H}\times\mathbb{R} \mid \varphi(\bar{x}) \leq \bar{\xi}\}$: convex set on $\mathbb{R}\times\mathcal{H}\times\mathbb{R}$

$= \{(1,\bar{x},\bar{\xi}) \in \mathbb{R}\times\mathcal{H}\times\mathbb{R} \mid (\bar{x},\bar{\xi}) \in \operatorname{epi}\varphi\}$

/* By using Proposition 3.6. (i)

$(C_i)_{i\in I}$: totally ordered finite family of m convex subsets of $\mathcal{H}$

$\Rightarrow$

$\underset{i\in I}{\times} C_i$: convex set */

$f(\xi, x) = \begin{cases} \xi\varphi(x/\xi), & \text{if } \xi > 0 \\ +\infty, & \text{otherwise} \end{cases}$

now,

$\operatorname{epi} f = \{((\xi,x),\zeta) \in \mathbb{R}\times\mathcal{H}\times\mathbb{R} \mid \xi > 0,\ f(\xi,x) \leq \zeta\}$

/* because else $f(\xi,x) = +\infty$, and by definition epi f only contains points <u>not</u> associated with $+\infty$ */

$= \{(\xi,x,\zeta) \in \mathbb{R}_{++}\times\mathcal{H}\times\mathbb{R} \mid \underbrace{\xi\varphi(x/\xi) \leq \zeta}\} = \{(\xi,x,\zeta) \in \mathbb{R}_{++}\times\mathcal{H}\times\mathbb{R} \mid \left(\frac{x}{\xi}, \frac{\zeta}{\xi}\right) \in \operatorname{epi}\varphi\}$

/* $\varphi\left(\frac{x}{\xi}\right) \leq \underbrace{\frac{\zeta}{\xi}}_{<+\infty} \Leftrightarrow \left(\frac{x}{\xi}, \frac{\zeta}{\xi}\right) \in \operatorname{epi}\varphi$ */

$= \{(\xi,x,\zeta) \in \mathbb{R}_{++}\times\mathcal{H}\times\mathbb{R} \mid \left(\frac{x}{\xi}, \frac{\zeta}{\xi}\right) \in \operatorname{epi}\varphi\}$ /* let $\frac{x}{\xi} = y$, $\frac{\zeta}{\xi} = \eta$, then, $(\overset{\in\mathbb{R}_{++}}{\xi}, x, \zeta) = \xi\left(1, \frac{x}{\xi}, \frac{\zeta}{\xi}\right) = \xi(1, y, \eta)$ */

$<+\infty$

$= \{(\xi, x, \zeta) \in \mathbb{R}_{++} \times \mathcal{H} \times \mathbb{R} \mid (\frac{x}{\xi}, \frac{\zeta}{\xi}) \in \operatorname{epi} \phi\}$ /* let $\frac{x}{\xi} = y$, $\frac{\zeta}{\xi} = \eta$, then, $(\underbrace{\xi}_{\in \mathbb{R}_{++}}, x, \zeta) = \xi(1, \frac{x}{\xi}, \frac{\zeta}{\xi}) = \xi(1, y, \eta)$ */

$= \{\xi(1, y, \eta) \in \mathbb{R} \times \mathcal{H} \times \mathbb{R} \mid \xi \in \mathbb{R}_{++}, (y, \eta) \in \operatorname{epi} \phi\}$ $\{\frac{x}{\xi} = y, \frac{\zeta}{\xi} = \eta\}$

$= \underbrace{\xi}_{\in \mathbb{R}_{++}} \underbrace{\{(1, y, \eta) \in \mathbb{R} \times \mathcal{H} \times \mathbb{R} \mid (y, \eta) \in \operatorname{epi} \phi\}}_{= C}$ $\{\frac{x}{\xi} = y, \frac{\zeta}{\xi} = \eta\}$

$= \mathbb{R}_{++} C$ { Recall : C : convex }

$= \operatorname{cone} C$ /* By Proposition 6.3. [[C : subset of $\mathcal{H}$]] $\Rightarrow$ $\operatorname{cone} C = \mathbb{R}_{++} C$ */

/* Recall Proposition 6.2. (iii)

$[C \subseteq \mathcal{H}]$ $\operatorname{cone}(\operatorname{conv} C) = \operatorname{conv}(\operatorname{cone} C)$

had a consequence C : convex $\Rightarrow$ $\operatorname{cone} C$: convex */

$\therefore$ $\operatorname{epi} f = \operatorname{cone} C$: convex

$\leftrightarrow$ f : convex function.

■

*** Proposition 8.26.**

[[$\mathcal{K}$: real Hilbert space

$F : \mathcal{H} \times \mathcal{K} \to]-\infty, +\infty]$: convex]]

$\underbrace{f : \mathcal{H} \to [-\infty, +\infty] : x \mapsto \inf F(x, \mathcal{K})}_{\text{marginal function}}$: convex

Proof :

$\forall_{x_1, x_2 \in \operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\}}$

$\forall_{\alpha \in]0,1[}$

$\begin{cases} f(x_1) = \inf F(x_1, \mathcal{K}) \leq F(x_1, \tilde{y}_1) & \forall_{\tilde{y}_1 \in \mathcal{K}} \\ f(x_2) = \inf F(x_2, \mathcal{K}) \leq F(x_2, \tilde{y}_2) & \forall_{\tilde{y}_2 \in \mathcal{K}} \end{cases}$ as $x_1 \in \operatorname{dom} f_1$, $f_1(x_1) = \inf F(x_1, \mathcal{K}) < +\infty$, also, as by given $\forall_{\tilde{y}_1 \in \mathcal{K}}$ $F(x_1, \tilde{y}_1) > -\infty$

so, $f(x_1) =$ finite $\therefore$ $f(x_1) = \min F(x_1, \mathcal{K})$ $\hookrightarrow$ $\exists_{y_1 \in \mathcal{K}}$ $y_1 = \operatorname{argmin}_{\tilde{y}_1 \in \mathcal{K}} F(x_1, \tilde{y}_1) = \operatorname{argmin} F(x_1, \mathcal{K})$ $\therefore$ $f(x_1) = F(x_1, y_1)$

similarly, $\exists_{y_2 \in \mathcal{K}}$ $y_2 = \operatorname{argmin} F(x_2, \mathcal{K})$ $\therefore$ $f(x_2) = F(x_2, y_2)$

now,

$f(\alpha x_1 + (1-\alpha) x_2)$

$= \inf F(\alpha x_1 + (1-\alpha) x_2, \mathcal{K})$ $\mathcal{K}$; real Hilbert space

we can treat this as a column vector

$\leq F(\alpha x_1 + (1-\alpha) x_2, \alpha y_1 + (1-\alpha) y_2)$ /* $\because$ $\alpha y_1 + (1-\alpha) y_2 \in \mathcal{K}$ */

$\leq \alpha F(x_1, y_1) + (1-\alpha) F(x_2, y_2)$ /* $\because$ $\alpha(x_1, y_1) + (1-\alpha)(x_2, y_2) = (\alpha x_1, \alpha y_1) + ((1-\alpha) x_2, (1-\alpha) y_2)$

$\leq \alpha f(x_1) + (1-\alpha) f(x_2)$ $= (\alpha x_1 + (1-\alpha) x_2, \alpha y_1 + (1-\alpha) y_2)$ and F : convex */

now by letting $\xi_1 \downarrow f(x_1)$, $\xi_2 \downarrow f(x_2)$ we have,

$f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$ $\overset{\text{def}}{\longleftrightarrow}$ f : convex ■

*** Theorem 8.29.**

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper, convex

$x_0 \in \operatorname{dom} f$]]

(i) f : locally Lipschitz continuous near x_0 $\Leftrightarrow$

(ii) f : continuous at x_0 $\Leftrightarrow$

(iii) f : bounded on a neighborhood of x_0 $\Leftrightarrow$

(iv) f : bounded above on a neighborhood of x_0.

Moreover, if one of these conditions holds, then f : locally Lipschitz continuous on $\operatorname{int} \operatorname{dom} f$.

Proof : (i) means $\exists_{\rho > 0}$ $\exists_{L > 0}$ $\forall_{x, y \in B(x_0; \rho)}$ $\|f(x) - f(y)\| \leq L \|x - y\|$

(i) $\to$ (ii) $\to$ (iii) $\to$ (iv) : clear

(iv) $\to$ (ii) : take $\rho \in \mathbb{R}_{++}$; $\eta = \sup f(B(x_0, \rho)) < +\infty$ /* this is allowed as f : bounded above on a neighborhood of x_0

$\alpha \in]0,1[$ $\Rightarrow$ $\exists_{V_\varepsilon(x_0)}$ $\sup(f(V_\varepsilon(x_0))) < +\infty$, so any $B(x_0; \rho) \subseteq V_\varepsilon(x_0)$ will

$x \in B(x_0; \alpha\rho)$ satisfy $\sup(f(B(x_0, \rho))) < +\infty$ */

/*

Proposition 8.28.

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper, convex

$x_0 \in \operatorname{dom} f$

$\rho \in \mathbb{R}_{++}$]

(i) [$\eta = \sup f(B(x_0; \rho)) < +\infty$, $\alpha \in]0,1[$] $\Rightarrow$ $\forall_{x \in B(x_0; \alpha\rho)}$ $|f(x) - f(x_0)| \leq \alpha(\eta - f(x_0))$

define maximum value f achieves over a ball of radius ρ centered at x_0 */

(ii) $\delta = \operatorname{diam} f(B(x_0; \rho)) < +\infty$ $\Rightarrow$ f : Lipschitz continuous relative to $B(x_0; \rho)$ with constant $\frac{\delta}{\rho}$. */

$\forall_{x \in B(x_0; \alpha\rho)}$ $|f(x) - f(x_0)| \leq \alpha(\eta - f(x_0))$

/* recall, f : continuous at x $\overset{\text{def}}{\leftrightarrow}$ $x_a \to x \Rightarrow f(x_a) \to f(x)$ */

$x \to x_0 \Leftrightarrow \alpha \to 0$, $x \in B(x_0; \alpha\rho)$

$\Rightarrow$ $|f(x) - f(x_0)| \leq \alpha(\underset{\substack{\| \\ \sup f(B(x_0; \rho)) < +\infty}}{\eta} - f(x_0)) \to 0$ $<+\infty$, as $x_0 \in B(x_0; \rho)$!

$\Rightarrow$ $|f(x) - f(x_0)| \to 0$ $\Rightarrow$ $f(x) \to f(x_0)$ $\therefore$ f is continuous at x_0.

(iii) $\to$ (i) : follows from Proposition 8.28. (ii)

Now we have shown (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i)

so, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) : note that this infact establishes the following equivalence:

Now we have shown (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i)

so,

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) : note that this infact establishes the following equivalence:

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)

$\Rightarrow$ (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)

Now let us prove that : f: locally Lipschitz continuous on int dom f. if one of (i)-(iv) holds
(as they are equivalent)

Assume (iv) holds,

$$\exists_{\rho \in \mathbb{R}_{++}} \ \eta = \sup f(B(x_0;\rho)) < +\infty$$

then from (iv) $\Rightarrow$ (i): f: locally Lipschitz continuous near x_0

take $\forall\ x \in \operatorname{int} \operatorname{dom} f \setminus \{x_0\}$, $\gamma \in \mathbb{R}_{++}$: $B(x;\gamma) \subseteq \operatorname{dom} f$.

set,

$$y = x_0 + \frac{1}{1-\alpha}(x - x_0), \quad \alpha = \frac{\gamma}{\gamma + \|x - x_0\|} \in\ ]0,1[$$

then

$$y - x = x_0 + \frac{1}{1 - \frac{\gamma}{\gamma + \|x - x_0\|}}(x - x_0) - x$$

$$= \frac{\gamma + \|x - x_0\|}{\|x - x_0\|}(x - x_0) - (x - x_0)$$

$$= \left[\frac{\gamma + \|x - x_0\|}{\|x - x_0\|} - 1\right](x - x_0)$$

$$= \frac{\gamma}{\|x - x_0\|}(x - x_0) = \gamma \underbrace{\frac{x - x_0}{\|x - x_0\|}}_{\text{a vector with unit norm}}$$

$$\Rightarrow \|y - x\| = \gamma \underbrace{\left\|\frac{x - x_0}{\|x - x_0\|}\right\|}_{=1} = \gamma \Rightarrow y \in B(x;\gamma)$$

take, $z \in B(x;\alpha\rho)$

$$w = x_0 + \frac{z - x}{\alpha} \stackrel{?}{=} \frac{z - (1-\alpha)y}{\alpha}$$

$$\leftrightarrow x_0 + \cancel{\frac{z}{\alpha}} - \frac{x}{\alpha} \stackrel{?}{=} \cancel{\frac{z}{\alpha}} - \frac{(1-\alpha)y}{\alpha}$$

$$\leftrightarrow \frac{\alpha x_0 - x}{\cancel{\alpha}} \stackrel{?}{=} \frac{-(1-\alpha)y}{\cancel{\alpha}}$$

$$\leftrightarrow y \stackrel{?}{=} -\frac{\alpha x_0 - x}{1-\alpha} = \frac{x - \alpha x_0}{1-\alpha} = \frac{(1-\alpha)x_0 - x_0 + x}{1-\alpha} = x_0 + \frac{x - x_0}{1-\alpha} \checkmark$$

$$w - x_0 = \frac{z - x}{\alpha}$$

$$\Rightarrow \|w - x_0\| = \frac{1}{\alpha}\|z - x\| \leq \rho \ \left[\because z \in B(x;\alpha\rho) \leftrightarrow \|z - x\| \leq \alpha\rho \leftrightarrow \frac{\|z - x\|}{\alpha} \leq \rho\right]$$

$$\Rightarrow w \in B(x_0;\rho)$$

$$w = \frac{z - (1-\alpha)y}{\alpha}$$

$$\leftrightarrow z = \alpha w + (1-\alpha)y : \alpha \in\ ]0,1[$$

now:

$$f(z) = f(\alpha w + (1-\alpha)y) \leq \alpha f(w) + (1-\alpha)f(y) \ [\because f: \text{convex}]$$

now, $z \in B(x;\alpha\rho)$, $w \in B(x_0;\rho)$, $y \in B(x,\gamma)$

as $y \in B(x;\gamma) \subseteq \operatorname{dom} f \Rightarrow f(y) < +\infty$

$$\Rightarrow f(z) \leq \alpha \underbrace{\sup_{w \in B(x_0;\rho)} f(w)}_{= \sup f(B(x_0;\rho)) = \eta} + (1-\alpha)f(y) = \alpha\eta + (1-\alpha)\underbrace{f(y)}_{<+\infty}$$

$<+\infty$ [By construction]

$$\text{so } \forall_{z \in B(x;\alpha\rho)} \ f(z) < +\infty$$

$$\leftrightarrow \sup(f(B(x;\alpha\rho))) < +\infty$$

$\leftrightarrow$ f: bounded above on $B(x;\alpha\rho)$ so from (iv) $\Leftrightarrow$ (iii) we have:
$\in \operatorname{int} \operatorname{dom} f$

if one of (i)-(iv) holds, then f: locally Lipschitz continuous on int dom f.

∎

*Corollary 8.32.

[$\mathcal{H}$: finite dimensional,

$f: \mathcal{H} \to]-\infty,+\infty]$, proper, convex

C: nonempty, closed, bounded $\subseteq$ ri dom f /* ri $C = \{x \in C \mid \operatorname{cone}(C-x) = \operatorname{span}(C-x)\}$ */

] $\Rightarrow$

f: Lipschitz continuous relative to C.

Proof:

$z \in \operatorname{dom} f$

consider, $z - \operatorname{dom} f$ /* just z becomes the new origin in $z - \operatorname{dom} f$ */

construct span$(z - \operatorname{dom} f)$ /* span C: smallest linear subspace containing C */

consider, z-dom f /* just z becomes the new origin in z-dom f */

construct

span(z-dom f) /* span C: smallest linear subspace containing C */

$C \subseteq$ ri dom f

assume $C \subseteq$ int dom f without loss of generality /* need explanation */

// as a simple explanation, let us assume int dom f=nonempty, then ri dom f and int dom f will coincide, as dom f is convex set due to f being convex

/*

* Corollary 8.30.

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex,

one of the following holds

- f: bounded above on some neighborhood
- f: lower continuous,
- $\mathcal{H}$: finite-dimensional] $\Rightarrow$ cont f = int dom f /* cont f: domain of continuity of a function f */

*/

now $\mathcal{H}$: finite-dimensional $\Rightarrow$ cont f = int dom f $\leftrightarrow$ f: continuous on int dom f

Take any $x_0 \in int\ dom\ f \subseteq dom\ f$, then f: continuous at x_0

/*

* Theorem 8.29.

f: proper $\Leftrightarrow$ $-\infty \notin f(\mathcal{H})$, dom $f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \neq \emptyset$

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex

$x_0 \in$ dom f]

(i) f: locally Lipschitz continuous near x_0, $\Leftrightarrow$

(ii) f: continuous at x_0, $\Leftrightarrow$

(iii) f: bounded on a neighborhood of x_0. $\Leftrightarrow$

(iv) f: bounded above on a neighborhood of x_0.

Moreover, if one of these conditions holds, then f: locally Lipschitz continuous on int dom f. */

Now, as $C \subseteq$ int dom f, f: locally Lipschitz continuous on int dom f

$\Rightarrow$ f: locally Lipschitz continuous on $C \subseteq$ int dom f

C: closed, bounded $\Rightarrow$ C: compact /* on a finite dimensional space, closed and bounded sets are compact */

/* f: locally Lipschitz continuous on a compact set $\Rightarrow$ f: Lipschitz continuous on a compact set */ see proof below:

$\therefore$ f: Lipschitz continuous on C $\boxtimes$

Remark 10.10. If F is locally Lipschitz, then F is Lipschitz continuous on compact sets. Actually, for every compact set K there exists a neighborhood U of K such that F is Lipschitz on U.

Proof (by contradiction). Assume that F is locally Lipschitz, but that there is some compact set K such that F is not Lipschitz continuous on any neighborhood of K. In particular, F is continuous on X. Let $U_n = \{x \in X; d(x, K) < 1/n\}$. Let (Λ_n) be a sequence of positive numbers with $\Lambda_n \to \infty$ as $n \to \infty$. Since F is not Lipschitz on U_n, for each $n \in \mathbb{N}$ there exist $x_n, y_n \in U_n$ such that

$$\|F(x_n) - F(y_n)\| > \Lambda_n \|x_n - y_n\|, \qquad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, there exists some $w_n, z_n \in K$ with $\|w_n - x_n\| < 1/n$ and $\|z_n - y_n\| < 1/n$. Since K is compact, after choosing subsequences, $w_n \to x$ and $z_n \to z$ for some $x, z \in K$. Then $x_n \to x$ and $y_n \to z$. Since F is continuous, $F(x_n) \to F(x)$ and $F(y_n) \to F(z)$. The inequality above implies that $x_n - y_n \to 0$ and so $x_n, y_n \to x$. Since F is locally Lipschitz, there exist some $\epsilon > 0$ and $\Lambda > 0$ such that

$$\|F(z) - F(y)\| \le \Lambda \|z - y\| \qquad \text{if } z, y \in K, \|z - x\| < \epsilon, \|y - z\| < \epsilon.$$

Choosing subsequences again, we can arrange that $\|x_n - x\| < \epsilon$ and $\|y_n - x\| < \epsilon$ for all $n \in \mathbb{N}$. This implies

$$\Lambda \ge \frac{\|F(x_n) - F(y_n)\|}{\|x_n - y_n\|} \ge \Lambda_n \to \infty,$$

a contradiction. □