7.1 Support Points

10:31 PM

Proposition 7.2 $[C, D:$ nonempty subsets of $\mathcal{H}$; $C \subseteq D]$ $C \cap \operatorname{spts} D \subseteq \operatorname{spts} C = C \cap \operatorname{spts} \bar{C}$

Proof:

$x \in C \cap \operatorname{spts} D \Leftrightarrow x \in C \wedge x \in \operatorname{spts} D = \{\tilde{x} \in D \mid \exists_{u_{\tilde{x}}: \text{normal vector} \in \mathcal{H} \setminus \{0\}} \langle \tilde{x} \mid u_{\tilde{x}} \rangle \geq \sup \langle D \mid u_{\tilde{x}} \rangle\}$

$\Updownarrow$

$\exists_{u \in \mathcal{H} \setminus \{0\}} \langle x \mid u \rangle \geq \sup \langle D \mid u \rangle$ /* $\sup \langle D \mid u \rangle = \sup_{x \in D} \langle x \mid u \rangle \geq \sup_{x \in C} \langle x \mid u \rangle = \sup \langle C \mid u \rangle$ */

$\geq \sup \langle C \mid u \rangle$ /* $D \supseteq C$ */

$\Rightarrow \exists_{u \in \mathcal{H} \setminus \{0\}} \langle x \mid u \rangle \geq \sup \langle C \mid u \rangle$ /* now: $\operatorname{spts} C = \{y \in C \mid \exists_{v_y \in \mathcal{H} \setminus \{0\}} \langle y \mid v_y \rangle \geq \sup \langle C \mid u \rangle\}$ */

$\Leftrightarrow x \in \operatorname{spts} C$

$\therefore C \cap \operatorname{spts} D \subseteq \operatorname{spts} C$

/* required info: $\sup_{x \in C} f(x) = \sup_{x \in \bar{C}} f(x)$ */

now let's show: $\operatorname{spts} C = C \cap \operatorname{spts} \bar{C}$

$\forall_x$ $x \in \operatorname{spts} C = \{\tilde{x} \in C \mid \exists_{u_{\tilde{x}} \in \mathcal{H} \setminus \{0\}} \langle \tilde{x} \mid u_{\tilde{x}} \rangle \geq \sup \langle C \mid u_{\tilde{x}} \rangle\}$

$\Leftrightarrow x \in C, \exists_{u \in \mathcal{H} \setminus \{0\}} \langle x \mid u \rangle \geq \sup \langle C \mid u \rangle = \sup \langle \bar{C} \mid u \rangle$

$\Leftrightarrow x \in C, \exists_{u \in \mathcal{H} \setminus \{0\}} \langle x \mid u \rangle \geq \sup \langle \bar{C} \mid u \rangle$

$\Leftrightarrow x \in C \cap \operatorname{spts} \bar{C}$ /* as $\operatorname{spts} \bar{C} = \{\tilde{x} \in \bar{C} \mid \exists_{u_{\tilde{x}} \in \mathcal{H} \setminus \{0\}} \langle \tilde{x} \mid u_{\tilde{x}} \rangle \geq \sup \langle \bar{C} \mid u_{\tilde{x}} \rangle\}$

$x \in C \Rightarrow x \in \bar{C}$

$\therefore \operatorname{spts} C = C \cap \operatorname{spts} \bar{C}$ ∎

Proposition 7.3.

$[C:$ nonempty convex, $\subseteq \mathcal{H}]$ $\operatorname{spts} C = \{x \in \mathcal{H} \mid N_C x \setminus \{0\}\} = N_C^{-1}(\mathcal{H} \setminus \{0\})$

Proof: $\forall_x$ /* $\operatorname{spts} C = \{y \in C \mid \exists_{u_y \in \mathcal{H} \setminus \{0\}} \langle y \mid u_y \rangle \geq \sup \langle C \mid u_y \rangle\}$

$x \in \operatorname{spts} C \Leftrightarrow \exists_{u \in \mathcal{H} \setminus \{0\}} (\langle x \mid u \rangle \geq \sup \langle C \mid u \rangle \geq \langle z \mid u \rangle \; \forall_{z \in C})$

$\Leftrightarrow \exists_{u \in \mathcal{H} \setminus \{0\}} \forall_{z \in C} \langle z \mid u \rangle \leq \langle x \mid u \rangle \Leftrightarrow \exists_{u \in \mathcal{H} \setminus \{0\}} \sup \langle C - x \mid u \rangle \leq 0$ /* recall, $[C:$ nonempty convex, $\subseteq \mathcal{H}$; $x \in \mathcal{H}]$ $N_C x = \begin{cases} (C - x)^{\ominus} = \{u \in \mathcal{H} \mid \sup \langle C - x \mid u \rangle \leq 0\}, & \text{if } x \in C \\ \emptyset, & \text{else} \end{cases}$ */

$\langle z - x \mid u \rangle \leq 0$

$\sup \langle C - x \mid u \rangle \leq 0$

$\Leftrightarrow \exists_{u \in \mathcal{H} \setminus \{0\}} u \in N_C x$

$\Leftrightarrow \exists_{u \in \mathcal{H} \setminus \{0\}} N_C^{-1} u \ni x$

$\therefore \forall_x (x \in \operatorname{spts} C \Leftrightarrow \exists_{u \in \mathcal{H} \setminus \{0\}} x \in N_C^{-1} u) \Leftrightarrow \operatorname{spts} C = N_C^{-1}(\mathcal{H} \setminus \{0\})$ ∎

Theorem 7.4. (Bishop-Phelps)

$[C:$ nonempty closed convex, $\subseteq \mathcal{H}] \Rightarrow$ $\operatorname{spts} C = P_C(\mathcal{H} \setminus C)$; $\overline{\operatorname{spts} C} = \operatorname{bdry} C$

Proof: $C = \mathcal{H} \Rightarrow$ trivial

So, take $C \neq \mathcal{H}$ — First we prove $\operatorname{spts} C \subseteq P_C(\mathcal{H} \setminus C)$ — [Procedure 1]

$\forall \varepsilon \in \mathbb{R}_{++}$, $x \in \operatorname{spts} C$ /* $\operatorname{spts} C = \{y \in C \mid \exists_{u_y \in \mathcal{H} \setminus \{0\}} \langle y \mid u_y \rangle \geq \sup \langle C \mid u_y \rangle\}$

$\Updownarrow$

$\exists_{u \in \mathcal{H} \setminus \{0\}} \langle x \mid u \rangle \geq \sup \langle C \mid u \rangle \geq \langle y \mid u \rangle \; \forall y \in C$

$\langle y - x \mid u \rangle \leq 0 \; \forall y \in C \Leftrightarrow \varepsilon \langle y - x \mid u \rangle \leq 0 \; \forall y \in C$

$\Updownarrow$

$\langle y - x \mid \varepsilon u \rangle \leq 0 \; \forall y \in C$

$\Updownarrow$

$\langle y - x \mid (x + \varepsilon u) - x \rangle \leq 0 \; \forall y \in C$ /* how

/* Charecterization of projection on closed convex nonempty set */ Theorem 3.14 ★★★
$(C:$ nonempty closed convex subset of $\mathcal{H}) \Rightarrow$ • C: Chebyshev set, i.e., every point in $\mathcal{H}$ has exactly one projection on C ; • $\forall_{x \in \mathcal{H}} (p = P_C(x) \Leftrightarrow (p \in C, \forall_{y \in C} \langle y - p \mid x - p \rangle \leq 0))$ */

$x \in \operatorname{spts} C \subseteq C$

$\therefore P_C(x + \varepsilon u) = x$

now take $x + \varepsilon u \in C \Rightarrow P_C(x + \varepsilon u) = x + \varepsilon u \Rightarrow \varepsilon = 0 \vee u = 0 \Rightarrow$ contradiction

$\therefore \forall \varepsilon > 0$ $x + \varepsilon u \notin C \Rightarrow P_C(x + \varepsilon u) = x \in P_C(\mathcal{H} \setminus C)$

$\therefore \forall_{x \in \operatorname{spts} C}$ $P_C(x + \varepsilon u) = x \in P_C(\mathcal{H} \setminus C)$

$\Leftrightarrow$ $\operatorname{spts} C \subseteq P_C(\mathcal{H} \setminus C)$

also $\forall_{\varepsilon > 0}$ $x + \varepsilon u \notin C$

Now let us show: $P_C(\mathcal{H} \setminus C) \subseteq \operatorname{spts} C$

$\Updownarrow$

$\Leftrightarrow$ $\boxed{\operatorname{spts} C \subseteq P_C(\mathcal{H}\setminus C)}$

also $\forall_{\varepsilon>0}\ x+\varepsilon u \notin C$

Now let us show: $P_C(\mathcal{H}\setminus C) \subseteq \operatorname{spts} C$

$\Updownarrow$

$\forall_{y\in\mathcal{H}\setminus C}\ P_C y \in \operatorname{spts} C$

$y \in \mathcal{H}\setminus C,\ x := P_C y$ /* now * Proposition 6.46: [C: nonempty closed convex subset of $\mathcal{H}$, $x, p \in \mathcal{H}$] $p = P_C x \Leftrightarrow x - p \in N_C p$ */

$\Updownarrow$

$y - x \in N_C x \Leftrightarrow N_C^{-1}(\underbrace{y-x}_{\neq 0 \text{ as } y\notin C,\ x\in C}) \ni x$ /* now: Proposition 7.3: [C: nonempty convex set $\subseteq \mathcal{H}$] $\operatorname{spts} C = N_C^{-1}(\mathcal{H}\setminus\{0\})$ */

$\Leftrightarrow x \in N_C^{-1}(y-x) \subseteq N_C^{-1}(\mathcal{H}\setminus\{0\}) = \operatorname{spts} C$

So, $\forall_{y\in\mathcal{H}\setminus C}\ x = P_C y \in \operatorname{spts} C \Leftrightarrow \boxed{P_C(\mathcal{H}\setminus C) \subseteq \operatorname{spts} C}$

$P_C(\mathcal{H}\setminus C) = \operatorname{spts} C$

Now let us show $\operatorname{bdry} C \subseteq \overline{\operatorname{spts}}\, C$

$\forall z$

// From Procedure 2

$z \in \operatorname{bdry} C \Rightarrow \forall_{\varepsilon>0} \exists_{y\in\mathcal{H}\setminus C} \|z-y\| < \varepsilon,\ z \in C$

now take $p := P_C y \subseteq P_C(\mathcal{H}\setminus C) = \operatorname{spts} C \Leftrightarrow p \in \operatorname{spts} C$

/* now Proposition 4.8: [C: nonempty closed convex set of $\mathcal{H}$] $\Rightarrow P_C$: firmly nonexpansive */ $\Rightarrow P_C$: expansive

$\therefore \|\underbrace{P_C y}_{p} - \underbrace{P_C z}_{z}\| = \|p - z\| \leq \|y - z\| < \varepsilon$ $\therefore \|p - z\| < \varepsilon$

/* $z \in \operatorname{bdry} C \subseteq C \therefore P_C z = z$ */

$\forall_{\varepsilon>0}\ p \in \operatorname{spts} C,\ \|p - z\| < \varepsilon$ ($z \in \operatorname{bdry} C$) $\Leftrightarrow p \in \operatorname{spts} C,\ p \in V_\varepsilon(z)\ \forall_{\varepsilon>0} \Rightarrow \forall_{\varepsilon>0}\ V_\varepsilon(z) \cap \operatorname{spts} C \ni p$ /* recall: $z \in \bar{S} \Leftrightarrow \forall_{\varepsilon>0}\ V_\varepsilon(z) = \{y \mid \|y - z\| < \varepsilon\} \cap S \neq \emptyset$ */

$\Updownarrow$

$\forall_{\varepsilon>0}\ V_\varepsilon(z) \cap \operatorname{spts} C \neq \emptyset$

$\Updownarrow$

$z \in \overline{\operatorname{spts}}\, C$ So, $\forall z \in \operatorname{bdry} C\ \ z \in \overline{\operatorname{spts} C}$

$\therefore \boxed{\operatorname{bdry} C \subseteq \overline{\operatorname{spts}}\, C}$

Now let us show: $\overline{\operatorname{spts}}\, C \subseteq \underbrace{\operatorname{bdry} C}_{C/\operatorname{int} C}$: lets simplify them

$\Leftrightarrow \forall_{z\in\overline{\operatorname{spts}}\, C}\ z \in \operatorname{bdry} C$

now: by def: $z \in \bar{S} \Leftrightarrow \forall_{\varepsilon>0}\ V_\varepsilon(z) \cap S \neq \emptyset$ $\therefore z \in \overline{\operatorname{spts}}\, C \Leftrightarrow \forall_{\varepsilon>0}\ V_\varepsilon(z) \cap \underbrace{\operatorname{spts} C}_{\{\tilde{q}\in C \mid \exists_{u_{\tilde{q}}\neq 0} \langle \tilde{q} \mid u_{\tilde{q}}\rangle \geq \sup\langle C \mid u_{\tilde{q}}\rangle\}} \neq \emptyset \Leftrightarrow \forall_{\frac{\varepsilon}{2}>0} \exists_{q\in C} \|q - z\| < \frac{\varepsilon}{2},\ \exists_{u\neq 0} \langle q \mid u\rangle \geq \sup\langle C \mid u\rangle$

$\Downarrow$ similar to [Procedure 1]

$\exists_{u\neq 0} \sup\langle C - q \mid (q + \frac{\varepsilon}{2}u) - q\rangle \leq 0$

$\Downarrow$

$\underbrace{q}_{\in C} = P_C(\underbrace{q + \frac{\varepsilon}{2}u}_{\notin C})$

Procedure 2

and $z \in \operatorname{bdry} C = C \setminus \operatorname{int} C \Leftrightarrow z \in C,\ z \notin \operatorname{int} C \Leftrightarrow z \in C,\ \forall_{\varepsilon>0} \exists_{y: \|y - z\| < \varepsilon}\ y \notin C$: this is what we want to show

$\Updownarrow$

/* $\neg(\exists_{\varepsilon>0}\ V_\varepsilon(z) \subseteq C) \Leftrightarrow \forall_{\varepsilon>0}\ \neg(\forall_{y: \|z - y\| < \varepsilon}\ y \in C) \Leftrightarrow \forall_{\varepsilon>0} \exists_{y: \|z - y\| < \varepsilon}\ y \notin C$ */

$\Updownarrow$

$\big(\forall_{y\in V_\varepsilon(z)} y \in C\big)$, $y \in V_\varepsilon(z) \Leftrightarrow \|z - y\| < \varepsilon$

Proving this:

$\forall_{\varepsilon>0}$ set $y = q + \frac{\varepsilon}{2}u \notin C$, $\|y - z\| = \|q + \frac{\varepsilon}{2}u - z\| \leq \underbrace{\|q - z\|}_{<\frac{\varepsilon}{2}} + \underbrace{\|\frac{\varepsilon}{2}u\|}_{\frac{\varepsilon}{2}\|u\| = \frac{\varepsilon}{2}} < \varepsilon$ ☑

$\tilde{y} := q \in C$

now $\|P_C z - z\| \leq \|\tilde{y} - z\|\ \forall \tilde{y} \in C \Rightarrow \forall_{\varepsilon/2>0} \exists_{q\in C} \left(\|P_C z - z\| \leq \|q - z\| < \frac{\varepsilon}{2}\right)$

$\Rightarrow \forall_{\varepsilon>0}\ \|P_C z - z\| < \varepsilon$ /* Result 1.2.2 (Equality of two numbers, and two vectors) $[a, b \in \mathbb{R}]\ (a = b) \Leftrightarrow \forall_{\varepsilon>0} |a - b| < \varepsilon$; $[x, y \in \mathcal{H}]\ x = y \Leftrightarrow \forall_{\varepsilon>0} \|x - y\| < \varepsilon$ */

$\Leftrightarrow P_C z = z$

$\Leftrightarrow \boxed{z \in C}$ ☑

$\therefore z \in \operatorname{bdry} C$

So, $\forall_{z\in\overline{\operatorname{spts}}\, C}\ z \in \operatorname{bdry} C \Leftrightarrow \boxed{\overline{\operatorname{spts}}\, C \subseteq \operatorname{bdry} C}$

$\Rightarrow \overline{\operatorname{spts}}\, C = \operatorname{bdry} C$ ■

Corollary 7.6.

⟦ C: nonempty closed convex subset of $\mathcal{H}$

one of the following holds (i) $\operatorname{int} C \neq \emptyset$,

(ii) C: closed affine subspace

(iii) $\mathcal{H}$: finite dimensional ⟧

$\Rightarrow$

$\operatorname{spts} C = \operatorname{bdry} C$

Proof:

(i)

/*

* Theorem 7.4. (Bishop-Phelps) ⟦ C: nonempty closed convex subset of $\mathcal{H}$ ⟧ $\operatorname{spts} C = P_C(\mathcal{H} \setminus C)$

$\overline{\operatorname{spts} C} = \operatorname{bdry} C$

Proposition 7.5. ⟦ C: convex subset of $\mathcal{H}$, $\operatorname{int} C \neq \emptyset$ ⟧ $\operatorname{bdry} C \subseteq \operatorname{spts} \overline{C}$

$C \cap \operatorname{bdry} C \subseteq \operatorname{spts} C$

*/

$\overline{\operatorname{spts} C} = \operatorname{bdry} C \subseteq \operatorname{spts} \overline{C} = \operatorname{spts} C$ [$\because C$: closed]

now we want to show that, $\operatorname{spts} C \subseteq \operatorname{bdry} C$

as $\operatorname{spts} C = P_C(\mathcal{H} \setminus C)$, so any point outside C will be projected on the boundary, i.e. $\operatorname{bdry} C$

$\therefore \operatorname{spts} C = P_C(\mathcal{H} \setminus C) \subseteq \operatorname{bdry} C$

$\therefore \operatorname{spts} C = \operatorname{bdry} C$.

(ii) C: closed affine subspace

if $C = \mathcal{H}$ then obviously $\operatorname{spts} C = \operatorname{bdry} C$

if $C \neq \mathcal{H}$

$V = C - C$: closed linear subspace parallel to C [page 1, Bauschke]

$x \in \operatorname{bdry} C$, $u \in V^{\perp} \setminus \{0\}$

$\forall_{\tilde{x} \in C}$ $C = \tilde{x} + V$ [page 1, Bauschke] $\Rightarrow$ $C = \underset{\in \operatorname{bdry} C \subseteq C}{x} + V$

/* Proposition 3.17. $P_{\underset{\in \mathcal{H}}{y} + \underset{\text{nonempty closed convex subset of } \mathcal{H}}{C}} \underset{\in \mathcal{H}}{x} = y + P_C(x - y)$ */ $\therefore P_C(x+u) = P_{x+V}(x+u) = x + P_V(x+u-x) = x + P_V u$

Now recall : /* (Corollary 3.12)

* Projection onto a closed linear subspaces.

⟦ V: closed linear subspace of $\mathcal{H}$, $x \in \mathcal{H}$ ⟧

(i) $P_V x$: $P_V x \in V$, $x - P_V x \perp V$

(ii) $\|P_V x\|^2 = \langle P_V x, x \rangle$

(iii) ($P_V \in \mathcal{B}(\mathcal{H})$, $\|P_V\| = 1$) $\Leftrightarrow$ $V \neq \{0\}$

$\|P_V\| = 0 \Leftrightarrow V = \{0\}$

(iv) $V^{\perp\perp} = V$

(v) $P_{V^\perp} = \operatorname{Id} - P_V$

(vi) $P_V^* = P_V$

(vii) $\|x\|^2 = \|P_V x\|^2 + \|P_{V^\perp} x\|^2$ ...

*/

so, $\|u\|^2 = \|P_V u\|^2 + \|\underbrace{P_{V^\perp} u}_{u \ [\because u \in V^\perp]}\|^2 = \|P_V u\|^2 + \|u\|^2$

$\Rightarrow \|P_V u\|^2 = 0$

$\Leftrightarrow P_V u = 0$

$P_C(x+u) = x$, $\therefore \forall_{x \in \operatorname{bdry} C}$ $P_C(x+u) = x$

now, $x + u \in \mathcal{H}$, but $x + u \notin C$ /* if $\underset{\operatorname{bdry} C}{x} + \underset{\in V^\perp \setminus \{0\}}{u} \in C = x + V \Rightarrow u \in V \Rightarrow$ contradiction */

$\Rightarrow x + u \in \mathcal{H} \setminus C$

$\Rightarrow \underbrace{P_C(x+u)}_{= x} \in P_C(\mathcal{H} \setminus C) = \operatorname{spts} C$ /* using Recall Bishop-Phelps theorem ⟦ C: nonempty closed convex subset of $\mathcal{H}$ ⟧ $\operatorname{spts} C = P_C(\mathcal{H} \setminus C)$

$\overline{\operatorname{spts} C} = \operatorname{bdry} C$ */

$\Rightarrow \forall_{x \in \operatorname{bdry} C}$ $x \in \operatorname{spts} C$ $\therefore \operatorname{bdry} C \subseteq \operatorname{spts} C$,

and, $\operatorname{spts} C \subseteq \operatorname{bdry} C$

$\therefore \operatorname{bdry} C = \operatorname{spts} C$. ☺

(iii)

$\mathcal{H}$: finite dimensional

if $\operatorname{int} C \neq \emptyset \Rightarrow \operatorname{spts} C = \operatorname{bdry} C$ [proved in (i)]

now consider $\operatorname{int} C = \emptyset$

$\begin{cases} D = \operatorname{aff} C \\ x \in \operatorname{bdry} C \end{cases}$

/*

* Proposition 6.12.

⟦ C: convex subset of $\mathcal{H}$ ⟧

($\operatorname{int} C \neq \emptyset$ $\vee$ C: closed $\vee$ $\mathcal{H}$: finite-dimensional) $\Rightarrow$ $\operatorname{int} C = \operatorname{core} C$ */

$\therefore \operatorname{int} C = \operatorname{core} C = \{x \in C \mid \operatorname{cone}(C - x) = \mathcal{H}\} = \emptyset$

$\neg \left(\exists_{x \in C} \ \operatorname{cone}(C-x) = \mathcal{H} \right)$

$\leftrightarrow \forall_{x \in C} \ \operatorname{cone}(C-x) \neq \mathcal{H}$

$D = \operatorname{aff} C \neq \mathcal{H}$ /* explanation needed */

/* Affine hull of a set in a finite-dimensional space is closed */

$\Rightarrow$ D: proper closed affine subspace of $\mathcal{H}$

$\Rightarrow$ $x \in \operatorname{bdry} D$

using (ii) : and Proposition 7.2.: $x \in C \cap \operatorname{spts} D \subseteq \operatorname{spts} C$. /* explanation needed */

∎

7.2 Support Functions, 7.3 Polar Sets

3:08 PM

Proposition 7.9.

$[C \subseteq \mathcal{H},\ \forall_{u\in\mathcal{H}}\ H_u = \{x\in\mathcal{H} \mid \langle x|u\rangle \leq \sigma_C(u)\}]\quad \overline{\text{conv}}\, C = \bigcap_{u\in\mathcal{H}} H_u$

Proof:

- $C=\emptyset \Rightarrow$ obvious
- $C\neq\emptyset$.

$D := \bigcap_{u\in\mathcal{H}} H_u$

$H_u = \{x\in\mathcal{H} \mid \langle x|u\rangle \leq \sigma_C(u)\}$

(intersection of closed sets) (half-spaces are convex)

By definition H_u is a closed half-space containing C, i.e. $\forall_{u\in\mathcal{H}}\ C\subseteq H_u \Rightarrow D$: closed, convex. $\overline{\text{conv}}\, C \subseteq D$

smallest closed convex set containing C

/* $\sigma_C(u) = \sup_{\tilde{x}\in C}\langle \tilde{x}|u\rangle \geq \langle x|u\rangle\ \ \forall_{x\in C} \rightarrow \forall_{x\in C}\ \langle x|u\rangle \leq \sigma_C(u)$ $\Leftrightarrow x\in H_u$

$\Rightarrow \forall_{x\in C}\ x\in H_u \Leftrightarrow C\subseteq H_u$ */

/* as $C\subseteq H_u\ \forall_{u\in\mathcal{H}}$ now each H_u: closed, convex

H_u is a closed convex set containing C

$\Rightarrow \overline{\text{conv}}\, C \subseteq \overline{\text{conv}}\, H_u$ /* $A\subseteq B \Rightarrow \overline{\text{conv}}\, A \subseteq \overline{\text{conv}}\, B$

$= H_u$

$\Rightarrow \overline{\text{conv}}\, C \subseteq H_u\ \ \forall_{u\in\mathcal{H}} \Rightarrow \overline{\text{conv}}\, C \subseteq \bigcap_{u\in\mathcal{H}} H_u = D$ */

$\therefore \overline{\text{conv}}\, C \subseteq D = \bigcap_{u\in\mathcal{H}} H_u$

let's prove $D\subseteq \overline{\text{conv}}\, C$

let. $x\in D$, $p = P_{\overline{\text{conv}}\, C}\, x$. $\tilde{C} = \overline{\text{conv}}\, C$

/*

/* Characterization of projection on closed convex nonempty set */ Theorem 3.14 ★★★

(C: nonempty closed convex subset of $\mathcal{H}$) $\Rightarrow$ { • C: Chebyshev set, i.e. every point in $\mathcal{H}$ has exactly one projection on C
• $\forall_{x\in\mathcal{H}}\ (p = P_C(x) \Leftrightarrow (p\in C,\ \forall_{y\in C}\ \langle y-p|x-p\rangle \leq 0))$ } */

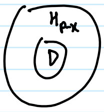
$\forall_{y\in\tilde{C}}\ \langle y-p|x-p\rangle = \langle y|x-p\rangle - \langle p|x-p\rangle \leq 0$

$\Leftrightarrow \forall_{y\in\tilde{C}}\ \langle y|x-p\rangle \leq \langle p|x-p\rangle$

$\Leftrightarrow \sup_{y\in\tilde{C}}\langle y|x-p\rangle = \sigma_{\tilde{C}}(x-p) \leq \langle p|x-p\rangle$

$\Leftrightarrow \sigma_{\overline{\text{conv}}\, C}(x-p) \leq \langle p|x-p\rangle$

now, $x\in D = \bigcap_{u\in\mathcal{H}} H_u \subseteq H_{x-p}$ [$\because p-x\in\mathcal{H}$]



$H_{x-p} = \{\square\in\mathcal{H} \mid \langle\square|x-p\rangle \leq \sigma_C(x-p)\}$

$\Rightarrow x\in H_{p-x} \Leftrightarrow \langle x|x-p\rangle \leq \sigma_C(x-p)$

So, $\|x-p\|^2 = \langle x-p|x-p\rangle = \underbrace{\langle x|x-p\rangle}_{\leq\sigma_C(x-p)} - \underbrace{\langle p|x-p\rangle}_{\leq -\sigma_{\overline{\text{conv}}\, C}(x-p)} \leq \sigma_C(x-p) - \sigma_{\overline{\text{conv}}\, C}(x-p)$

$\Leftrightarrow \|x-p\|^2 \leq \sigma_C(x-p) - \sigma_{\overline{\text{conv}}\, C}(x-p)$

now, $C\subseteq \overline{\text{conv}}\, C$

so, $\underbrace{\sup_{\tilde{x}\in\overline{\text{conv}}\, C}\langle\tilde{x}|x-p\rangle}_{=\sigma_{\overline{\text{conv}}\, C}(x-p)}$ is a relaxation of $\underbrace{\sup_{\tilde{x}\in C}\langle\tilde{x}|x-p\rangle}_{=\sigma_C(x-p)} \Rightarrow \sigma_{\overline{\text{conv}}\, C}(x-p) \geq \sigma_C(x-p)$

/* relaxation always have better objective value */

$\Rightarrow 0\leq\|x-p\|^2\leq 0 \Leftrightarrow \|x-p\|^2 = 0 \Leftrightarrow x = p = P_{\overline{\text{conv}}\, C}\, x \in \overline{\text{conv}}\, C$

$\therefore \forall_{x\in\bigcap_{u\in\mathcal{H}} H_u}\ x\in\overline{\text{conv}}\, C \Leftrightarrow \bigcap_{u\in\mathcal{H}} H_u \subseteq \overline{\text{conv}}\, C$ ($\bigcap_{u\in\mathcal{H}} H_u = D$)

$\overline{\text{conv}}\, C = \bigcap_{u\in\mathcal{H}} H_u$ ■

$\overline{\text{conv}}\, C = \bigcap_{u \in \mathcal{H}} H_u$ ∎

Theorem 7-16 [$C \subseteq \mathcal{H}$] $C^{\ominus\ominus} = \overline{\text{conv}}(C \cup \{0\})$

$\sigma_C(u) = \sup\langle C | u \rangle$

Proof:

/ recall: [$C \subseteq \mathcal{H}$] $C^{\ominus} = \text{lev}_{\leq 1}\, \sigma_C = \{u \in \mathcal{H} \mid \sigma_C(u) \leq 1\}$

* Proposition 7.14. [C, D: subsets of $\mathcal{H}$]
  - (i) $C^{\perp} \subseteq C^{\ominus} \subseteq C^{\ominus}$
  - (ii) $0 \in C^{\ominus}$, $C^{\ominus}$: closed and convex ✓
  - (iii) $C \cup \{0\} \subseteq C^{\ominus\ominus}$ ✓
  - (iv) $C \subseteq D \Rightarrow D^{\ominus} \subseteq C^{\ominus}$, $C^{\ominus\ominus} \subseteq D^{\ominus\ominus}$
    - inclusion flips by polar set operator
    - double polar set operator maintains the inclusion
  - (v) $C^{\ominus\ominus\ominus} = C^{\ominus}$
  - (vi) $(\overline{\text{conv}}\, C)^{\ominus} = C^{\ominus}$

$C \cup \{0\} \subseteq C^{\ominus\ominus}$ /* as $A \subseteq B \Rightarrow \overline{\text{conv}}\, A \subseteq \overline{\text{conv}}\, B$

$\Rightarrow \overline{\text{conv}}(C \cup \{0\}) \subseteq \overline{\text{conv}}(C^{\ominus\ominus})$ // now $C^{\ominus}$: closed, convex

$= C^{\ominus\ominus}$ // $\Rightarrow C^{\ominus\ominus}$: closed, convex

$\therefore \overline{\text{conv}}(C \cup \{0\}) \subseteq C^{\ominus\ominus}$

lets prove $C^{\ominus\ominus} \subseteq \overline{\text{conv}}(C \cup \{0\})$

Per absurdum assume $C^{\ominus\ominus} \not\subseteq \overline{\text{conv}}(C \cup \{0\})$

$\Leftrightarrow \neg\left(\forall_{x \in C^{\ominus\ominus}}\ x \in \overline{\text{conv}}(C \cup \{0\})\right)$

$\Leftrightarrow \exists_{x \in C^{\ominus\ominus}}\ x \notin \overline{\text{conv}}(C \cup \{0\})$

$\Leftrightarrow \exists_x\ x \in C^{\ominus\ominus} \setminus \overline{\text{conv}}(C \cup \{0\})$

now $0 \in C^{\ominus\ominus}$ and $0 \in \overline{\text{conv}}(C \cup \{0\})$, $x \neq 0$

so more precisely:

$\exists_{x \in \mathcal{H} \setminus \{0\}}\ x \in C^{\ominus\ominus} \setminus \overline{\text{conv}}(C \cup \{0\})$

/*
Theorem 3.38.
[$\tilde{C}$: nonempty closed convex subset of $\mathcal{H}$
$x \in \mathcal{H} \setminus \tilde{C}$] $\Rightarrow$ x: strongly seperated from $\tilde{C} \Leftrightarrow \exists_{v \in \mathcal{H} \setminus \{0\}}\ \sup\langle \tilde{C} | v \rangle < \langle x | v \rangle$ */

$\tilde{C} := \overline{\text{conv}}(C \cup \{0\})$

$\Rightarrow \exists_{v \in \mathcal{H} \setminus \{0\}}\ \langle x | v \rangle > \sup\langle \tilde{C} | v \rangle = \sigma_{\tilde{C}}(v)$ // By definition: $\sigma_C(u) = \sup\langle C | u \rangle$

$= \sigma_{\overline{\text{conv}}(C \cup \{0\})}(v)$

now: $\sigma_{\overline{\text{conv}}(C \cup \{0\})}(v) = \sup\langle \overline{\text{conv}}(C \cup \{0\}) | v \rangle \geq \max\{\sigma_C(v), 0\}$

/* using $\tilde{C} = \overline{\text{conv}}(C \cup \{0\}) \supseteq C \Rightarrow \forall_z\ \sup\langle \tilde{C} | z \rangle \geq \sup\langle C | z \rangle = \sigma_C(z)$ (sup over a larger set is larger set)

$\tilde{C} = \overline{\text{conv}}(C \cup \{0\}) \supseteq 0 \Rightarrow \forall_z\ \sup\langle \tilde{C} | z \rangle \geq \sup\langle 0 | z \rangle = 0$

$\Rightarrow \forall_z\ \sup\langle \tilde{C} | z \rangle \geq \max\{\sigma_C(z), 0\}$ */

$\therefore \langle x | v \rangle > \sigma_{\overline{\text{conv}}(C \cup \{0\})}(v) \geq \max\{\sigma_C(v), 0\} \geq 0$

$\beta > 0$ // say $\langle x | v \rangle$ is strictly positive

$\Rightarrow \langle x | v \rangle > \max\{\sigma_C(v), 0\} \geq 0$ /* $C^{\ominus} = \text{lev}_{\leq 1}\, \sigma_C = \{\tilde{v} \in \mathcal{H} \mid \sigma_C(\tilde{v}) \leq 1\}$ */

after scaling v if necessary: setting $v = \alpha u$: $\alpha > 0$

$\Rightarrow$

$\alpha \langle x | u \rangle = \beta > \max\{\sigma_C(\alpha u), 0\} = \alpha \max\{\sigma_C(u), 0\}$

$\Rightarrow \langle x | u \rangle = \frac{\beta}{\alpha} > \max\{\sigma_C(u), 0\}$ /* for any β we can choose an α: such that $\langle x | u \rangle = \frac{\beta}{\alpha} > 1$ */

so we can say:

$\langle x | u \rangle > 1 \geq \sigma_C(u) = \sup\langle C | u \rangle$

$\Downarrow$

$\langle x|u\rangle > 1 \geq \sigma_C(u) = \sup\langle C|u\rangle$

$\Downarrow$

$u \in C^{\ominus}$

now $\sigma_{C^{\ominus}}(x) = \sup\langle C^{\ominus}|x\rangle \geq \langle u|x\rangle$

but $x \in \text{lev}_{\leq 1}\sigma_{C^{\ominus}} \setminus \overline{\text{conv}}(C\cup\{0\})$

$\Rightarrow \sigma_{C^{\ominus}}(x) \leq 1$

$\langle u|x\rangle \leq \sigma_{C^{\ominus}}(x) \leq 1$

$1 < \langle x|u\rangle \leq \sigma_{C^{\ominus}}(x) \leq 1 \Rightarrow 1<1 \Rightarrow$ contradiction

■

Corollary 7.17.

$[\![C \subseteq \mathcal{H}]\!]$

(i) $(C$: closed, convex, $0\in C) \Leftrightarrow C^{\ominus\ominus} = C$

(ii) $(C$: nonempty, closed, convex$) \Leftrightarrow C^{\ominus\ominus} = C$

(iii) $(C$: closed linear subspace$) \Leftrightarrow C^{\perp\perp} = C$

Proof:

(i)

/* * Proposition 7.14. $[\![C, D$: subsets of $\mathcal{H}]\!]$ (i) $C^{\perp} \subseteq C^{\ominus} \subseteq C^{\odot}$

(ii) $0 \in C^{\odot}$, $C^{\odot}$: closed and convex */

($\Leftarrow$) $(C^{\odot})^{\odot} \Rightarrow$ closed, convex, $0 \in C^{\odot\odot}$

($\Rightarrow$)

C: closed, convex, $0\in C$

$\Rightarrow C = C\cup\{0\}$: nonempty, convex, closed

$\Rightarrow \overline{\text{conv}}(C\cup\{0\}) = C$ /* Theorem 7.16. $[\![C\subseteq\mathcal{H}]\!]$ $C^{\odot\odot} = \overline{\text{conv}}(C\cup\{0\})$ */ ☺

($\overline{\text{conv}}(C\cup\{0\}) = C^{\odot\odot}$)

(ii)

* Proposition 6.23. $[\![C$: subset of $\mathcal{H}]\!] \Rightarrow$ (i) $D \subseteq C \Rightarrow$ • $C^{\ominus} \subseteq D^{\ominus}$ /* inclusion flips */
• $C^{\oplus} \subseteq D^{\oplus}$

(ii) $C^{\ominus}, C^{\oplus}$: nonempty closed convex cones

(iii) $C^{\ominus} = (\text{cone}\, C)^{\ominus} = (\text{conv}\, C)^{\ominus} = \overline{C}^{\ominus}$ /* to the polar cone operator, a set, its cone, convex hull and closure all are same */

(iv) $\overline{\text{cone}}\, C = -\overline{\text{cone}}\, C \Rightarrow C^{\ominus} = C^{\oplus} = C^{\perp}$

C: nonempty closed convex cone

$\Rightarrow C^{\ominus}$: nonempty closed convex cone $= C^{\odot}$ /* $\because K$: cone in $\mathcal{H} \Rightarrow K^{\odot} = K^{\ominus}$ */

$\Rightarrow C^{\ominus\ominus}$: nonempty closed convex cone, and contains 0 /* $\forall_{K\subseteq\mathcal{H}}$ $K^{\ominus} = \{u\in\mathcal{H} \mid \sup\langle C|u\rangle \leq 0\} \Rightarrow 0\in K^{\ominus}$ as $\sup\langle C|0\rangle = 0$ */

$= (C^{\odot})^{\ominus} = (C^{\odot})^{\odot} = C$ [from (i): $C^{\odot\odot} = C$]

$\therefore C^{\ominus\ominus} = C$ ☺

(iii) C: closed linear subspace

$\Rightarrow C$: nonempty closed convex cone /* A closed linear subspace is also a nonempty closed convex cone */

$\Rightarrow C^{\ominus\ominus} = C$

$\Rightarrow C$: nonempty closed convex cone /* A closed linear subspace is also a nonempty closed convex cone */

$\Rightarrow C^{\ominus\ominus} = C$

But *Proposition 6.22. [C: linear subspace of $\mathcal{H}$] $\Rightarrow C^{\ominus} = C^{\perp}$

$\therefore C^{\perp\perp} = C$ ∎