6.1. Convex Cones

8:15 PM

Convex cones : • lies between linear subspace and convex set.

• arises in the study of convex sets via tangent cone and normal cone operators.

6.1. Convex Cones.

$\underset{\subseteq \mathcal{H}}{C}$: cone $\overset{\text{def}}{\Leftrightarrow}$ $C = \mathbb{R}_{++} C$

so $\mathcal{H}$ itself is a cone

cone C : conical hull of C $\overset{\text{def}}{\leftrightarrow}$ smallest cone in $\mathcal{H}$ containing C

$\overline{\text{cone}}\, C$: closed conical hull of C $\overset{\text{def}}{\leftrightarrow}$ smallest closed cone in $\mathcal{H}$ containing C

• Proposition 6.2.

[C : subset of $\mathcal{H}$] $\Rightarrow$

(i) cone $C = \mathbb{R}_{++} C$

(ii) $\overline{\text{cone}\, C} = \overline{\text{cone}}\, C$

(iii) cone(conv C) = conv (cone C) : smallest convex cone containing C.

(iv) $\overline{\text{cone}}$(conv C) = $\overline{\text{conv}}$ (cone C) : smallest closed convex cone containing C.

Proof :

(i) $D = \mathbb{R}_{++} C = \{\lambda x \mid x \in C, \lambda > 0\}$ /* $y \in D \Leftrightarrow \exists_{\lambda > 0} \exists_{x \in C}\ \lambda x = y$ */

first show: cone $C \subseteq \mathbb{R}_{++} C$ . . . (1)

$\Rightarrow$ D : cone $\wedge$ $C = \{1x \mid x \in C, \lambda = 1\} \subseteq D$

$\Rightarrow$ cone$(C) \subseteq$ cone$(D) = D$

/* cone C : • $C \subseteq$ cone C
• $\mathbb{R}_{++}$ cone C = cone C
• $\forall_{E : C \subseteq E,\ E : \text{cone}}$ cone $C \subseteq E$ /* smallest cone containing C */

now D : cone and $C \subseteq D$ $\Rightarrow$ cone$(C) \subseteq D$ */

now show $D \subseteq$ cone(C)

$\Leftrightarrow \forall_{y \in D}$ $y \in$ cone(C)

$\forall_y$

$y \in D \Leftrightarrow \exists_{\lambda > 0} \exists_{x \in C}$ $y = \lambda x$

$\Rightarrow x \in C \subseteq$ cone C $\Rightarrow$ $x \in$ cone C $\Rightarrow$ $\underbrace{\lambda x}_{= y} \in$ cone C $\therefore y \in$ cone C

so, $\forall_{y \in D}$ $y \in$ cone(C) $\Leftrightarrow$ $D \subseteq$ cone C (2)

(1)+(2) $\Rightarrow$ cone $C = \mathbb{R}_{++} C$. ☺

(ii) $\overline{\text{cone}}\, C$: closed cone $\Rightarrow$ cone($\overline{\text{cone}}\, C$) : the smallest cone containing $\overline{\text{cone}}\, C$ = $\overline{\text{cone}}\, C$ $\Rightarrow$ $\overline{\text{cone}(\overline{\text{cone}}\, C)} = \overline{\text{cone}}\, C$

$C \subseteq$ cone $C \subseteq \overline{\text{cone}}\, C \Rightarrow$ /* By definition of conic hull

now, cone C : smallest cone containing C,

and cone($\overline{\text{cone}}\, C$) is another cone containing C

cone $C \subseteq$ cone($\overline{\text{cone}}\, C$)

$\Rightarrow$ $\overline{\text{cone}\, C} \subseteq \overline{\text{cone}(\overline{\text{cone}}\, C)} = \overline{\text{cone}}\, C$ /* $\because A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$ */ . . . (3)

Again, $\overline{\text{cone}}\, C$ = smallest closed cone containing C

cone C = " " " C

$\Rightarrow$ $\overline{\text{cone}\, C}$ = closure of C : a closed set, contains C

$\Rightarrow$ $\overline{\text{cone}}\, C \subseteq \overline{\text{cone}\, C}$ . . . (4)

$\therefore$ (3)+(4) $\Rightarrow$ $\overline{\text{cone}\, C} = \overline{\text{cone}}\, C$.

• Proposition 6.3.

[C : subset of $\mathcal{H}$]

(i) C : cone $\Rightarrow$ (C : convex $\Leftrightarrow$ $C + C \subseteq C$)

(ii) C : convex, $0 \in C$ $\Rightarrow$ (C : cone $\Leftrightarrow$ $C + C \subseteq C$)

• Proposition 6.4.

[C : nonempty convex subset of $\mathcal{H}$]

(i) span C = cone C − cone C = cone C + cone $(-C)$

(ii) $C = -C \Rightarrow$ span C = cone C

• Definition 6.5

[K : convex cone in $\mathcal{H}$]

K : pointed $\overset{\text{def}}{\Leftrightarrow}$ $K \cap (-K) \subseteq \{0\}$

K : solid $\overset{\text{def}}{\Leftrightarrow}$ int $K \neq \emptyset$

Proposition 6.12.

[C : convex, $\subseteq \mathcal{H}$]

(int $C \neq \emptyset$ $\vee$ C : closed $\vee$ $\mathcal{H}$: finite-dimensional) $\Rightarrow$ int C = core C

Proof: $\operatorname{int} C \subseteq \operatorname{core} C$ always, so it suffices to show $\operatorname{core} C \subseteq \operatorname{int} C$

$\Updownarrow$

$\forall_{x \in \operatorname{core} C}\ x \in \operatorname{int} C \Leftrightarrow \exists_{\rho \in \mathbb{R}_{++}} B(0;\rho) \subseteq C - x$

$\{\tilde{x} \in C \mid \exists_{\rho \in \mathbb{R}_{++}} B(0;\rho) \subseteq C - \tilde{x}\}$

$\operatorname{core} C = \{\tilde{x} \in C \mid \operatorname{cone}(C - \tilde{x}) = \mathcal{H}\}$

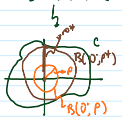
$\forall_{x \in C : x \in \operatorname{core} C}$ $\Updownarrow$ $\operatorname{cone}(C - x) = \mathcal{H}$

We can transform the coordinate system so that x becomes origin 0

in the new coordinate system C : convex, $0 \in C$, $\operatorname{cone}(C) = \mathbb{R}_{++} C = \mathcal{H}$

Want to prove $\exists_{\rho \in \mathbb{R}_{++}} B(0;\rho) \subseteq C - 0 = C$

Without loss of generality we assume $C = -C$ i.e. w.r.t 0, C is symmetric as a closed ball, as if $B(0;\rho) \subseteq C \Leftrightarrow B(0;\rho) \subseteq \underbrace{B(0;\rho^*)}_{\text{(symmetric)}}$ where ρ^* is the closest point of C to the origin



So, our modified goal is: ⟦ $C = -C$, C : convex, $\subseteq \mathcal{H}$, other antecedents ⟧ $0 \in \operatorname{core} C \Rightarrow 0 \in \operatorname{int} C$

(i) $\operatorname{int} C \neq \emptyset$, as $C = -C$, $\operatorname{int} C = \operatorname{int}(-C)$

if $y \in \operatorname{int} C \subseteq C \Rightarrow -y \in \operatorname{int}(-C) = \operatorname{int} C$

now C : convex $\Rightarrow \operatorname{int} C$: convex $\overset{\text{def}}{\Leftrightarrow} \forall_{\tilde{x}, \tilde{y} \in \operatorname{int} C}\ \forall_{\lambda \in]0,1[}\ \lambda \tilde{x} + (1-\lambda)\tilde{y} \in \operatorname{int} C$

set, $\tilde{x} = y$, $\tilde{y} := -y$, $\lambda = \frac{1}{2}$

$\Rightarrow \frac{1}{2} y + \frac{1}{2}(-y) = 0 \in \operatorname{int} C$ ⫝̸

(ii)

First note that $\bigcup_{n \in \mathbb{N}} \underbrace{nC}_{C_n : \text{ closed}} = \mathcal{H}$, /* e.g. take C to be a closed ball, then $2C$ is a closed ball with twice the radius and so on, thus $\bigcup_{n \in \mathbb{N}} nC = \mathcal{H}$ */

Lemma 1.43 (Ursescu) ⟦ X : complete metric space ⟧
(i) ($(C_n)_{n \in \mathbb{N}}$: sequence of closed subsets of X) $\Rightarrow \overline{\bigcup_{n \in \mathbb{N}} \operatorname{int} C_n} = \operatorname{int} \overline{\bigcup_{n \in \mathbb{N}} C_n}$
(ii) ($(C_n)_{n \in \mathbb{N}}$: sequence of open subsets of X) $\Rightarrow \operatorname{int} \bigcap_{n \in \mathbb{N}} \overline{C_n} = \operatorname{int} \overline{\bigcap_{n \in \mathbb{N}} C_n}$

$\overline{\bigcup_{n \in \mathbb{N}} \operatorname{int} \underbrace{C_n}_{nC}} = \operatorname{int} \overline{\bigcup_{n \in \mathbb{N}} \underbrace{C_n}_{nC}} = \operatorname{int} \overline{\mathcal{H}} = \overline{\mathcal{H}} = \mathcal{H}$ /* $\overline{X} = \mathcal{H} \Leftrightarrow X = \mathcal{H}$ */

$\Rightarrow \bigcup_{n \in \mathbb{N}} \operatorname{int} nC = \mathcal{H}$

$\Rightarrow \operatorname{int} C \neq \emptyset$ /* else, $\operatorname{int} nC = \emptyset \Rightarrow \bigcup_{n \in \mathbb{N}} \operatorname{int} nC = \emptyset$ */

$\overset{\text{using (i)}}{\Rightarrow} 0 \in \operatorname{int} C$ ⫝̸

(iii) $\mathcal{H}$: finite dimensional

let $(e_i)_{i \in I}$: orthonormal basis of $\mathcal{H}$, now as C : convex, $C = -C$, a scaled version of e_i, say $\varepsilon e_i \in C\ \forall i \in I$, /* think about $B(0;\rho^+)$ analogy, take $\underbrace{\|\varepsilon e_i\|}_{\varepsilon \|e_i\| = \varepsilon} \leq \rho^+$ will belong to C */

$\overset{\because C = -C}{\Rightarrow} -\varepsilon e_i \in C\ \forall i \in I$

$\therefore \{\varepsilon e_i, -\varepsilon e_i\}_{i \in I} \subseteq C \Rightarrow D = \operatorname{conv}\{\varepsilon e_i, -\varepsilon e_i\}_{i \in I} \subseteq C$ // as C is convex itself

Fact 6.9.1.

But $\underbrace{B(0;\varepsilon/\sqrt{\dim \mathcal{H}})}_{\text{closed}} \subseteq D \subseteq C$ /* now recall that: $A \subseteq B \Rightarrow \operatorname{int} A \subseteq \operatorname{int} B$

$\Rightarrow 0 \in \underbrace{\operatorname{int}\left(B(0;\varepsilon/\sqrt{\dim \mathcal{H}})\right)}_{\text{open ball}} \subseteq \operatorname{int} C \Rightarrow 0 \in \operatorname{int} C$.

∎

* Corollary 6.15.

⟦ $\mathcal{H}$: finite-dimensional

$\mathcal{K}$: finite dimensional real Hilbert space

$L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, C, D : nonempty convex subsets of $\mathcal{H}$ ⟧ $\Rightarrow$

(i) $\operatorname{ri} LC = L(\operatorname{ri} C)$

(ii) $\operatorname{ri}(C - D) = (\operatorname{ri} C) - (\operatorname{ri} D)$

Proof: (i)

/* * Fact 6.14. ⟦ C : nonempty convex subset of $\mathcal{H}$ ⟧

(i) $\mathcal{H}$: finite dimensional $\Rightarrow \begin{cases} \operatorname{ri} C : \text{interior of } C \text{ relative to aff } C, \operatorname{ri} C \neq \emptyset \\ \overline{\operatorname{ri} C} = \overline{C},\ \operatorname{ri} \overline{C} = \operatorname{ri} C,\ \operatorname{sri} C = \operatorname{ri} C = \operatorname{qri} C \end{cases}$ */

$\operatorname{sri} C = \operatorname{ri} C = \operatorname{qri} C \neq \emptyset$

C : nonempty convex subset of $\mathcal{H}$

/* (iv) ($\mathcal{K}$: finite dimensional real Hilbert space, $L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\operatorname{qri} C \neq \emptyset$) $\Rightarrow \operatorname{ri} L(C) = L(\operatorname{qri} C)$ */

$\therefore \operatorname{ri} LC = L \operatorname{qri} C$ ☺

(ii) $L : \mathcal{H} \times \mathcal{H} \to \mathcal{H} : (x,y) \mapsto x - y$ i.e., $L(x,y) = x - y = [1 \mid -1] \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow L \equiv [1 \mid -1]$

$\therefore \operatorname{ri}(C - D) = \operatorname{ri} L(C \times D) = L(\operatorname{ri}(C \times D))$ /* from (i) */ $= L(\operatorname{ri} C \times \operatorname{ri} D) = (\operatorname{ri} C) - (\operatorname{ri} D)$

$\{x - y \mid x \in C, y \in D\} = \{L(x,y) \mid (x,y) \in C \times D\} = L(C \times D)$

(ii) $L: \mathcal{H}\times\mathcal{H} \to \mathcal{H} : (x,y) \mapsto x-y$ i.e., $L(x,y) = x-y = [1|-1]\begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow L \equiv [1|-1]$

$\therefore \operatorname{ri}(C-D) = \operatorname{ri} L(C\times D) = L(\operatorname{ri}(C\times D))$ /* from (i) */ $= L(\operatorname{ri} C \times \operatorname{ri} D) = (\operatorname{ri} C) - (\operatorname{ri} D)$

$\{x-y \mid x\in C, y\in D\} = \{\underset{\underset{[1-1]}{\parallel}}{L}(x,y) \mid (x,y)\in C\times D\} = L(C\times D)$

$\{(x,y) \mid (x,y)\in \operatorname{ri} C\times \operatorname{ri} D\} /\!/ L = \{(x-y) \mid x\in \operatorname{ri} C, y\in \operatorname{ri} D\} = (\operatorname{ri} C) - (\operatorname{ri} D)$

/* $\operatorname{ri}(C\times D) = \operatorname{ri} C \times \operatorname{ri} D$ as the sets C, D are on different spaces connected by their cross product */ ∎

* Proposition 6.17.

[C: convex subset of $\mathcal{H}$

$\operatorname{int} C \neq \emptyset$

$0 \in C$]

(i) $0 \in \operatorname{int} C$ ⟺

(ii) $\operatorname{cone}(\operatorname{int} C) = \mathcal{H}$ ⟺

(iii) $\operatorname{cone} C = \mathcal{H}$ ⟺

(iv) $\overline{\operatorname{cone}}\, C = \mathcal{H}$ ⟺

Proof: (i)⇒(ii) $0 \in \operatorname{int} C = \{x\in C \mid \exists_{\rho\in\mathbb{R}_{++}} B(0;\rho) \subseteq C - x\}$

now, $\underset{\text{convex}}{\operatorname{int} C \neq \emptyset} \Rightarrow \operatorname{int} C = \operatorname{core} C = \operatorname{sri} C = \operatorname{ri} C = \operatorname{qri} C$ // as for a convex set with nonempty interior, the generalized interiors collapse.

$\operatorname{core} C = \{x\in C \mid \operatorname{cone}(C-x) = \mathcal{H}\}$

$0 \in \operatorname{core} C \Leftrightarrow \operatorname{cone}(C-0) = \operatorname{cone}(C) = \mathcal{H}$

/* proposition 6.16. [C: convex subset of $\mathcal{H}$, $\operatorname{int} C\neq\emptyset$, $0\in C$] ⇒ $\operatorname{int}\operatorname{cone} C = \operatorname{cone}\operatorname{int} C$ */

⇒ $\operatorname{int}\operatorname{cone} C = \operatorname{cone}\operatorname{int} C = \operatorname{int}\mathcal{H} = \mathcal{H}$

$\therefore \operatorname{cone}\operatorname{int} C = \mathcal{H}$

(ii)⇒(iii): $\operatorname{int}\operatorname{cone} C = \operatorname{cone}\operatorname{int} C = \mathcal{H} \Rightarrow \operatorname{closure}(\operatorname{int}\operatorname{cone} C) = \operatorname{cone} C = \operatorname{closure}\mathcal{H} = \mathcal{H}$.

(iii)⇒(iv): $\operatorname{cone} C = \mathcal{H} \Rightarrow \overline{\operatorname{cone} C} = \mathcal{H}$ /* now, Proposition 6.2. [$C \subseteq \mathcal{H}$] $\overline{\operatorname{cone} C} = \overline{\operatorname{cone}}\, C$ */ $\therefore \overline{\operatorname{cone}}\, C = \mathcal{H}$

(iv)⇒(ii):

$\overline{\operatorname{cone}}\, C = \overline{\operatorname{cone}\, C} = \mathcal{H}$

convex

$\therefore \operatorname{conv} C = C$

Proposition 6.2.

[C: subset of $\mathcal{H}$] ⇒

(i) $\operatorname{cone} C = \mathbb{R}_{++} C$ // most intuitive way creating cone C out of C

(ii) $\overline{\operatorname{cone} C} = \overline{\operatorname{cone}}\, C$

(iii) $\operatorname{cone}(\operatorname{conv} C) = \operatorname{conv}(\operatorname{cone} C)$: smallest convex cone containing C

(iv) $\overline{\operatorname{cone}}(\operatorname{conv} C) = \overline{\operatorname{conv}}(\operatorname{cone} C)$: smallest closed convex cone containing C.

now: $\operatorname{cone}(\underbrace{\operatorname{conv} C}_{C}) = \underbrace{\operatorname{conv}(\operatorname{cone} C)}_{\text{convex by structure}}$

$\therefore \operatorname{cone} C$: convex

/* * Proposition 6.16 [C: convex subset of $\mathcal{H}$, $\operatorname{int} C \neq \emptyset$, $0 \in C$] ⇒ $\operatorname{int}(\operatorname{cone} C) = \operatorname{cone}(\operatorname{int} C)$ */

now $\mathcal{H} = \operatorname{int}\mathcal{H} = \operatorname{int}(\overline{\operatorname{cone} C}) = \operatorname{int}\operatorname{cone} C$

$= \operatorname{cone}\operatorname{int} C$

$\therefore \operatorname{cone}\operatorname{int} C = \mathcal{H}$

Proposition 3.36.

C: convex subset of $\mathcal{H}$ ⇒ (i) $\overline{C}$: convex (ii) $\operatorname{int} C$: convex (iii) $\operatorname{int} C \neq \emptyset \Rightarrow \begin{cases} \operatorname{int} C = \operatorname{int}\overline{C} \\ \overline{C} = \overline{\operatorname{int} C} \end{cases}$

(ii)⇒(i)

$\operatorname{cone}\operatorname{int} C = \mathcal{H} \Rightarrow 0 \in \mathcal{H} = \operatorname{cone}\operatorname{int} C$ /* $\operatorname{cone} C = \mathbb{R}_{++} C$ */

$= \mathbb{R}_{++}\operatorname{int} C$

$= \bigcup_{\lambda\in\mathbb{R}_{++}} \lambda \operatorname{int} C$

$\Rightarrow \exists_{\lambda\in\mathbb{R}_{++}}\ 0 \in \lambda \operatorname{int} C$

$\Rightarrow 0 \in \operatorname{int} C$. ∎

* Proposition 6.20.

[m: integer, ≥ 2,

$I = \{1,\dots,m\}$

$(C_i)_{i\in I}$: convex subsets of $\mathcal{H}$

one of the following holds:

(i) $\forall_{i\in\{2,\dots,m\}} (C_i - \bigcap_{j=1}^{i-1} C_j)$: closed linear subspace

(ii) $(C_i)_{i\in I}$: linear subspaces, $\forall_{i\in\{2,\dots,m\}} (C_i + \bigcap_{j=1}^{i-1} C_j)$: closed

(iii) $C_m \cap (\bigcap_{i=1}^{m} \operatorname{int} C_i) \neq \emptyset$

(iv) $\mathcal{H}$: finite-dimensional, $\bigcap_{i\in I} \operatorname{ri} C_i \neq \emptyset$] ⇒

$0 \in \bigcap_{i=2}^{m} \operatorname{sri}(C_i - \bigcap_{j=1}^{i-1} C_j)$

Proof: Apply proposition 6.19.

* Proposition 6.19. [$\cancel{C}$: convex subset of $\mathcal{H}$, $\cancel{\mathcal{K}}$: real Hilbert space, $\cancel{L} \in \cancel{\mathcal{B}(\mathcal{H},\mathcal{K})}$, $\cancel{D}$: convex subset of $\mathcal{K}$] ($\forall_{i\in\{1,\dots,m\}}$ C_i; $\mathcal{H}$; Id; $\mathcal{B}(\mathcal{H},\mathcal{H})$; $\bigcap_{j=1}^{i-1} C_j$)

Suppose one of the following holds:

✓(i) $\cancel{D} - \cancel{L}(C)$ ($C_i - \bigcap_{j=1}^{i-1} C_j$) : closed linear subspace ✓ $C_i + \mathrm{Id}(\bigcap_{j=1}^{i-1} C_j) = C_i + \bigcap_{j=1}^{i-1} C_j$: closed

✓(ii) $\cancel{C}, \cancel{D}$ (C_i, $\bigcap_{j=1}^{i-1} C_j$) : linear subspaces and (• $\cancel{D} + \cancel{L}(C)$: closed ✓

• D : closed, $L(C)$: finite dimensional or finite co-dimensional ✓

• D: finite-dimensional or finite-codimensional and $L(C)$: closed)

(iii) D : cone, $D - L(C)$: closed linear subspace

(iv) $D = L(C)$, span D : closed

(v) $0 \in \operatorname{core}(D - L(C))$

(vi) $0 \in \operatorname{int}(D - L(C))$ $\bigcap_{i=1}^{m-1} C_i$ $\therefore C_m \cap \operatorname{int} \mathrm{Id}(\bigcap_{i=1}^{m-1} C_i)$

✓(vii) $\cancel{D}$ (C_m) $\cap \operatorname{int} \cancel{L(C)} \neq \emptyset$ ✓

(iv) $D = L(C)$, span D : closed

(v) $0 \in \operatorname{core}(D - L(C))$

(vi) $0 \in \operatorname{int}(D - L(C))$ $\quad \therefore\ C_m \cap \operatorname{int} \operatorname{Id}\left(\bigcap_{i=1}^{m-1} C_i\right)$ 

(vii) $\cancel{D}^{\,C_m} \cap \operatorname{int} L(C)^{\,\bigcap_{i=1}^{m-1} C_i} \neq \emptyset$ $\quad = C_m \cap \operatorname{int}\left(\bigcap_{i=1}^{m-1} C_i\right) = C_m \cap \left(\bigcap_{i=1}^{m-1} \operatorname{int} C_i\right) \neq \emptyset$

$L(C) \cap \operatorname{int} D \neq \emptyset$

(viii) K: finite-dimensional and $(\operatorname{ri} \cancel{D}^{\,C_i}) \cap (\operatorname{ri} L(C)^{\,\bigcap_{j \in I \setminus \{i\}} C_j}) \neq \emptyset$ $\quad \operatorname{ri} C_i \cap \left(\operatorname{ri} \operatorname{Id}\left(\bigcap_{j \in I \setminus \{i\}} C_j\right)\right)$

(ix) K: finite-dimensional and $(\operatorname{ri} D) \cap L(\operatorname{qri} C) \neq \emptyset$ $\quad = \operatorname{ri} \underbrace{C_i}_{\text{convex}} \cap \left(\operatorname{ri} \underbrace{\bigcap_{j \in I \setminus \{i\}} C_j}_{\text{convex}}\right)$

(x) $(\mathcal{H}, K)$: finite dimensional and $(\operatorname{ri} D) \cap L(\operatorname{ri} C) \neq \emptyset$

Then $0 \in \operatorname{sri}(D - L(C))$.

$= \bigcap_{j \in I} \operatorname{ri} C_i \neq \emptyset$

Fact 6.14

/* now by (v) [$\mathcal{H}$: finite dimensional, D: convex subset of $\mathcal{H}$, $\operatorname{ri} \underset{}{\overset{\text{convex}}{C}} \cap \operatorname{ri} D \neq \emptyset$] $\Rightarrow \operatorname{ri}(C \cap D) = \operatorname{ri} C \cap \operatorname{ri} D$

so, one of (i), (ii), (vii), (viii) are given

so, $0 \in \operatorname{sri}(\underbrace{D}_{C_i} - \underbrace{L(C)}_{\operatorname{Id}\left(\bigcap_{j=1}^{i-1} C_j\right) = \bigcap_{j=1}^{i-1} C_j}) = \operatorname{sri}\left(C_i - \bigcap_{j=1}^{i-1} C_j\right)$ ∎

6.3 Polar and Dual Cone, 6.4 Tangent and Normal Cone

9:31 PM

Proposition 6.26.

$[K_1, K_2 : \text{nonempty cones in } \mathcal{H}] \Rightarrow$

$(K_1 + K_2)^\ominus = K_1^\ominus \cap K_2^\ominus$

in particular: $[K_1, K_2 : \text{linear subspaces}] \Rightarrow (K_1+K_2)^\perp = K_1^\perp \cap K_2^\perp$

Proof: $x_1 \in K_1$, $x_2 \in K_2$

want to prove, $(K_1+K_2)^\ominus \subseteq K_1^\ominus \cap K_2^\ominus$

$\forall_u \; u \in (K_1+K_2)^\ominus = (\{\tilde{x}+\tilde{y} : \tilde{x} \in K_1, \tilde{y} \in K_2\})^\ominus$

$= \{\tilde{u} \in \mathcal{H} \mid \sup \langle K_1+K_2 \mid \tilde{u} \rangle \leq 0\}$

$= \{\tilde{u} \in \mathcal{H} \mid \forall_{\tilde{x}+\tilde{y} \in K_1+K_2} \langle \tilde{x}+\tilde{y} \mid \tilde{u} \rangle \leq 0\}$ $\therefore u \in (K_1+K_2)^\ominus \Leftrightarrow \forall_{\tilde{x}+\tilde{y} \in K_1+K_2} \langle \tilde{x}+\tilde{y} \mid u \rangle \leq 0$ ... (eq 0)

now K_1, K_2: cones $\Leftrightarrow K_1 = \mathbb{R}_{++} K_1$, $K_2 = \mathbb{R}_{++} K_2 \Rightarrow K_1+K_2 = \mathbb{R}_{++}(K_1+K_2)$

$\therefore \forall_{\lambda_1 \in \mathbb{R}_{++}, \lambda_2 \in \mathbb{R}_{++}} \lambda_1 x_1 \in K_1, \lambda_2 x_2 \in K_2 \Rightarrow \lambda_1 x_1 + \lambda_2 x_2 \in \lambda_1 K_1 + \lambda_2 K_2 = K_1 + K_2$

$\forall_{\lambda_1 \in \mathbb{R}_{++}, \lambda_2 \in \mathbb{R}_{++}} \langle \lambda_1 x_1 + \lambda_2 x_2 \mid u \rangle \leq 0$

set $\lambda_2 = 1$, $\lambda_1 := (\lambda_{1,n})_{n \in \mathbb{N}} \subseteq \mathbb{R}_{++}, \lambda_{1,n} \to 0$

$\Rightarrow \langle \lambda_{1,n} x_1 + x_2 \mid u \rangle \leq 0$

$\therefore \forall_{x_2 \in K_2} \langle x_2 \mid u \rangle \leq 0$

*Fact 1.3.1.
(comes handy in dealing with sequences)

- $\forall_{n \in \mathbb{N}} \; a_n \leq a \Rightarrow \underline{\lim}\, a_n \leq a$
- $\forall_{n \in \mathbb{N}} \; b \leq b_n \Rightarrow b \leq \overline{\lim}\, b_n$
- $\forall_{n \in \mathbb{N}} \; a_1 \leq a_n \leq a_2 \Rightarrow$ $a_1 \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq a_2$; $\underline{\lim}\, a_n$ exists in $[a_1, a_2]$; $\overline{\lim}\, a_n$ exists in $[a_1, a_2]$

now recall:

$u \in K_2^\ominus = \{\tilde{u} \in \mathcal{H} \mid \langle K_2 \mid u \rangle \leq 0\} \Leftrightarrow \forall_{x \in K_2} \langle x_2 \mid u \rangle \leq 0$

$\therefore u \in K_2^\ominus$, similarly we can show: $u \in K_1^\ominus$

$\Leftrightarrow u \in K_1^\ominus + K_2^\ominus$

$\therefore (K_1+K_2)^\ominus \subseteq K_1^\ominus + K_2^\ominus$

Now, let us show

$K_1^\ominus \cap K_2^\ominus \subseteq (K_1+K_2)^\ominus$

$\Leftrightarrow \forall_{u \in K_1^\ominus \cap K_2^\ominus} u \in (K_1+K_2)^\ominus$

$u \in K_1^\ominus \cap K_2^\ominus \Leftrightarrow u \in K_1^\ominus, u \in K_2^\ominus$

$\Leftrightarrow \sup \langle K_1 \mid u \rangle \leq 0$, $\sup \langle K_2 \mid u \rangle \leq 0$

$\Leftrightarrow \forall_{x_1 \in K_1} \langle x_1 \mid u \rangle \leq 0$, $\forall_{x_2 \in K_2} \langle x_2 \mid u \rangle \leq 0$

adding

$\Rightarrow \forall_{x_1 \in K_1, x_2 \in K_2} \langle x_1 \mid u \rangle + \langle x_2 \mid u \rangle = \langle x_1 + x_2 \mid u \rangle \leq 0 \Leftrightarrow u \in (K_1+K_2)^\ominus$ $\therefore K_1^\ominus \cap K_2^\ominus \subseteq (K_1+K_2)^\ominus$

$\therefore \boxed{(K_1+K_2)^\ominus = K_1^\ominus \cap K_2^\ominus}$

if K_1, K_2: linear subspaces $\Rightarrow K_1^\ominus = K_1^\perp$, $K_2^\ominus = K_2^\perp$ /* using *Proposition 6.22. [C: linear subspace of $\mathcal{H}$] $\Rightarrow C^\ominus = C^\perp$ */

$\Rightarrow K_1 + K_2$: linear subspace $\Rightarrow (K_1+K_2)^\ominus = (K_1+K_2)^\perp$

$(K_1+K_2)^\perp = K_1^\perp \cap K_2^\perp$ ∎

*Proposition 6.27.

[K: nonempty closed convex cone in $\mathcal{H}$, $x, p \in \mathcal{H}$]

$p = P_K(x) \Leftrightarrow (p \in K,\; x - p \perp p,\; x - p \in K^\ominus)$

Proof: ($\Rightarrow$)

/* Recall Theorem 3.14.

(C: nonempty closed convex subset of $\mathcal{H}$) $\Rightarrow$
- C: $\underbrace{\text{Chebyshev}}$ set — every point in $\mathcal{H}$ has exactly one projection on C.
- $\forall_x \left(p = P_C x \Leftrightarrow \left(p \in C, \forall_{y \in C} \langle y - p \mid x - p \rangle \leq 0 \right) \right)$ */

$C := K$, then $p = P_K(x) \Leftrightarrow \begin{cases} p \in K \\ \forall_{y \in K} \langle y - p \mid x - p \rangle \leq 0 \end{cases}$

$p \in K \Rightarrow \forall_{\alpha \in \mathbb{R}_{++}} \alpha p \in K$ — set $y := \alpha p$

$\langle \alpha p - p \mid x - p \rangle = \langle p(\alpha - 1) \mid x - p \rangle = (\alpha - 1)\langle p \mid x - p \rangle \leq 0 \quad \forall \alpha \in \mathbb{R}_{++}$

as $(\alpha - 1)$ can have either sign (based on what $\alpha \in \mathbb{R}_{++}$ we pick) the only possibility is $\langle p \mid x - p \rangle = 0 \Leftrightarrow p \perp x - p$

$\forall_{y \in K} \langle y \mid x - p \rangle = \langle y - p + p \mid x - p \rangle = \underbrace{\langle y - p \mid x - p \rangle}_{\leq 0} + \underbrace{\langle p \mid x - p \rangle}_{= 0}$ /* By definition, $\square \in C^\ominus \Leftrightarrow \forall_{\square \in C} \langle \square \mid \square \rangle \leq 0$

$= \langle y - p \mid x - p \rangle \leq 0$

$\Leftrightarrow \forall_{y \in K} \langle y \mid x - p \rangle \leq 0$

/* now $K^\ominus = \{u \in \mathcal{H} : \forall_{y \in K} \langle y \mid u \rangle \leq 0\}$, so $u \in K^\ominus \Leftrightarrow \forall_{y \in K} \langle y \mid x - p \rangle \leq 0$ */

so, $(x - p) \in K^\ominus$

($\Leftarrow$:)

so. $(x-p)\in K^{\ominus}$

$(\Leftarrow:)$

given, $p\in K$, $\underbrace{x-p\perp p}$ and $\underbrace{x-p\in K^{\ominus}}$

$\langle x-p|p\rangle=0 \quad \forall_{y\in K}\ \langle x-p|y\rangle\le 0$

$\forall_{y\in K}\ -\langle x-p|p\rangle+\langle x-p|y\rangle\le 0$

$\leftrightarrow \forall_{y\in K}\ \langle x-p|y-p\rangle\le 0$

$\therefore\ p\in K \wedge \forall_{y\in K}\langle x-p|y-p\rangle\le 0.$

$\Leftrightarrow \quad p=P_K(x)$ ■

* Theorem 6.29. (Moreau)

[K: nonempty convex cone in $\mathcal{H}$,
$x\in\mathcal{H}$]

(i) $x=P_Kx+P_{K^{\ominus}}x$

(ii) $P_Kx\perp P_{K^{\ominus}}x$

(iii) $\|x\|^2=d_K^2(x)+d_{K^{\ominus}}^2(x)$

/*Fact used: $K\subseteq K^{\ominus\ominus}$ */

Proof: (i)

$q=x-P_Kx$ (A)

/*

* Proposition 6.27. [K: nonempty closed convex cone in $\mathcal{H}$; $x,p\in\mathcal{H}$] $\Rightarrow$ $p=P_Kx \Leftrightarrow (p\in K,\ (x-p)\perp p,\ x-p\in K^{\ominus})$

Charecterization of projection on a closed convex cone.

/* projection on a closed convex cone is charecterized by its polar cone */

*/

[K: nonempty closed convex cone in $\mathcal{H}$; $x\in\mathcal{H}$]

$\underbrace{P_Kx}_{x-q}\in K,\ (\underbrace{x-P_Kx}_{q})\perp \underbrace{P_Kx}_{x-q},\ \underbrace{x-P_Kx}_{q}\in K^{\ominus}$

$\leftrightarrow\ x-q\in \underset{\subseteq K^{\ominus\ominus}}{K},\ q\perp x-q,\ q\in K^{\ominus}$

now, $K^{\ominus}$ is a cone too so $P_{K^{\ominus}}x$ will satisfy: $P_{K^{\ominus}}x\in K^{\ominus},\ x-P_{K^{\ominus}}x\perp P_{K^{\ominus}}x,\ x-P_{K^{\ominus}}x\in \underbrace{K^{\ominus\ominus}}_{\{u\in\mathcal{H}|\ \forall_{x\in K^{\ominus}}\ \langle x|u\rangle\le 0\}}$; so after comparison:

so, we have $q=P_{K^{\ominus}}x$, thus from A: $x=P_Kx+P_{K^{\ominus}}x$

(ii) in $q=P_{K^{\ominus}}x$

$P_{K^{\ominus}}x\perp P_Kx$

(iii)
$$\|x\|^2=\|P_Kx+P_{K^{\ominus}}x\|^2 \quad /*\ \overset{\text{from (i)}}{x=P_Kx+P_{K^{\ominus}}x}\ */$$

$$=\|P_{K^{\ominus}}x\|^2+\|P_Kx\|^2+2\underbrace{\langle P_Kx|P_{K^{\ominus}}x\rangle}_{0} \quad /*\text{from (ii) } P_Kx\perp P_{K^{\ominus}}x\ */$$

$$=\underbrace{\|x-P_Kx\|^2}_{d_K(x)}+\underbrace{\|x-P_{K^{\ominus}}x\|^2}_{d_{K^{\ominus}}x} \quad /*\text{ using (ii) }*/$$

$$=d_K^2(x)+d_{K^{\ominus}}^2(x)$$ ■

Proposition 6.31.

[K: nonempty closed convex cone in $\mathcal{H}$;
$x\in\mathcal{H}$, $\langle x|x-P_Kx\rangle\le 0$] $\Rightarrow$ $x\in K$

$0\le$
$\|x-P_Kx\|^2=\langle x-P_Kx|x-P_Kx\rangle$

$=\underbrace{\langle x|x-P_Kx\rangle}_{\le 0\,//\text{given}}-\underbrace{\langle P_Kx|x-P_Kx\rangle}_{0}$ /* now.

* Proposition 6.27. [K: nonempty closed convex cone in $\mathcal{H}$; $x,p\in\mathcal{H}$] $\Rightarrow$ $p=P_Kx \Leftrightarrow (p\in K,\ (x-p)\perp p,\ x-p\in K^{\ominus})$

Charecterization of projection on a closed convex cone.

/* projection on a closed convex cone is charecterized by its polar cone */

*/

≤ 0

$\Rightarrow\ \|x-P_Kx\|^2=0 \leftrightarrow x=P_Kx \leftrightarrow x\in K.$

■

* Proposition 6.32.

[C: nonempty convex subset of $\mathcal{H}$] $C^{\ominus\ominus}=\overline{\text{cone}}\,C$

Proof: $K:=\overline{\text{cone}}\,C$

Part I: $K\subseteq C^{\ominus\ominus}$

/* as $\overline{\text{cone}}\,C$ smallest closed cone containing C, so set inclusion will not change */

/*fact: C: convex $\rightarrow$ $C\subseteq C^{\ominus\ominus}$ */ $\Rightarrow$ $\overline{\text{cone}}\,C\subseteq\overline{\text{cone}}\,C^{\ominus\ominus}$ $\boxed{\overbrace{\overline{\text{cone}}\,C}^{K}\subseteq C^{\ominus\ominus}}$

* Proposition 6.23. [C: subset of $\mathcal{H}$] $\Rightarrow$ (i) $D\subseteq C \Rightarrow$ $\cdot\, C^{\ominus}\subseteq D^{\ominus}$ $\cdot\, C^{\ominus}\subseteq D^{\ominus}$ /* inclusion flips */

(ii) $C^{\ominus}, C^{\ominus}$: nonempty closed convex cones $\rightarrow$ $C^{\ominus\ominus}$: nonempty closed convex cone $\Rightarrow$ $\overline{\text{cone}}\,C^{\ominus\ominus}=C^{\ominus\ominus}$

(iii) $C^{\ominus}=(\text{cone}\,C)^{\ominus}=(\text{conv}\,C)^{\ominus}=\overline{C}^{\ominus}$ /* to the polar cone operator, a set, its cone, convex hull and

* Proposition 6.23. [C: subset of $\mathcal{H}$] $\Rightarrow$ (i) $D \subseteq C \Rightarrow$ • $C^{\ominus} \subseteq D^{\ominus}$ /* inclusion flips */
• $C^{\oplus} \subseteq D^{\oplus}$

(ii) $C^{\ominus}, C^{\oplus}$: nonempty closed convex cones $\rightarrow$ $C^{\ominus\ominus}$: nonempty closed convex cone $\Rightarrow$ $\overline{\text{cone}}\, C^{\ominus\ominus} = C^{\ominus\ominus}$

(iii) $C^{\ominus} = (\text{cone}\, C)^{\ominus} = (\text{conv}\, C)^{\ominus} = \overline{C}^{\ominus}$ /* to the polar cone operator, a set, its cone, convex hull and closure all are same */

(iv) $\overline{\text{cone}}\, C = -\overline{\text{cone}}\, C \Rightarrow C^{\ominus} = C^{\oplus} = C^{\perp}$

Part 2: $C^{\ominus\ominus} \subseteq K$.

Take, $x \in C^{\ominus\ominus}$ /* now, $K = \overline{\text{cone}}\, C = \overline{\text{cone}\, C}$ /* Proposition 6.2: $\overline{\text{cone}}\, C = \overline{\text{cone}\, C}$ */

$K^{\ominus} = (\overline{\text{cone}\, C})^{\ominus} = (\text{cone}\, C)^{\ominus} = C^{\ominus}$

$\Rightarrow K^{\ominus\ominus} = C^{\ominus\ominus}$ */

$\therefore x \in C^{\ominus\ominus} = K^{\ominus\ominus}$

$x - P_K x \in K^{\ominus}$

* Proposition 6.27. [K: nonempty closed convex cone in $\mathcal{H}$; $x, p \in \mathcal{H}$] $\Rightarrow$ $p = P_K x \Leftrightarrow (p \in K,\ (x-p) \perp p,\ x - p \in K^{\ominus})$

Charecterization of projection on a closed convex cone.

/* projection on a closed convex cone is charecterized by its polar cone */

$\langle x \mid x - P_K x\rangle \leq 0$ /* As By defn: $y \in C^{\ominus} \Leftrightarrow \forall_{x \in C} \langle x \mid y\rangle \leq 0$

$\Rightarrow \boxed{x \in K}$

$\because x \in (K^{\ominus})^{\ominus} \Leftrightarrow \forall_{\tilde{y} \in K^{\ominus}} \langle \tilde{y} \mid x\rangle \leq 0$

$\tilde{y} := x - P_K x \in K^{\ominus} \Rightarrow \langle x \mid x - P_K x\rangle \leq 0$ */

/* as
* Proposition 6.31. [K: nonempty closed convex cone in $\mathcal{H}$
$x \in \mathcal{H}$, $\langle x \mid x - P_K x\rangle \leq 0$] $\Rightarrow x \in K$ */

$\therefore \forall_{x \in C^{\ominus\ominus}}\ x \in K = \overline{\text{cone}}\, C \Leftrightarrow \boxed{C^{\ominus\ominus} \subseteq \overline{\text{cone}}\, C}$

$C^{\ominus\ominus} = \overline{\text{cone}}\, C$. ∎

Proposition 6.43. /* Relationship between tangent and normal cone */

[C: nonempty convex, $\subseteq \mathcal{H}$; $x \in C$]

(i) $T_C^{\ominus}(x) = N_C x$, $N_C^{\ominus}(x) = T_C(x)$

(ii) $x \in \text{core}\, C \Rightarrow T_C x = \mathcal{H} \Leftrightarrow N_C x = \{0\}$

Proof:

$$T_C \tilde{x} = \begin{cases} \overline{\text{cone}}\,(C - \tilde{x}), & \text{if } \tilde{x} \in C \\ \emptyset, & \text{else} \end{cases}$$

given, $x \in C$

$\Rightarrow T_C(x) = \underbrace{\overline{\text{cone}}\,(C-x)}_{\text{smallest closed cone containing } (C-x)} \supseteq C - x$

$\Rightarrow T_C(x)^{\ominus} \subseteq (C-x)^{\ominus}$

/* now
* Proposition 6.23. [C: subset of $\mathcal{H}$] $\Rightarrow$ (i) $D \subseteq C \Rightarrow$ • $C^{\ominus} \subseteq D^{\ominus}$ /* inclusion flips */
• $C^{\oplus} \subseteq D^{\oplus}$
(ii) $C^{\ominus}, C^{\oplus}$: nonempty closed convex cones
(iii) $C^{\ominus} = (\text{cone}\, C)^{\ominus} = (\text{conv}\, C)^{\ominus} = \overline{C}^{\ominus}$ /* to the polar cone operator, a set, its cone, convex hull and closure all are same */
(iv) $C^{\ominus} \cap C^{\oplus} = C^{\perp}$ /* intersection of polar and dual cone is orthogonal complement */
(v) $\overline{\text{cone}}\, C = -\overline{\text{cone}}\, C \Rightarrow C^{\ominus} = C^{\perp} = C^{\oplus}$ /* sufficient condition for the collapse of polar, dual cone and orthogonal complement */
*/

now,
$$N_C \tilde{x} = \begin{cases} (C - \tilde{x})^{\ominus}, & \text{if } \tilde{x} \in C \\ \emptyset, & \text{else} \end{cases}$$

as $x \in C \Rightarrow (C-x)^{\ominus} = N_C x$ $\qquad T_C^{\ominus} x \subseteq N_C x$ (eq 1)

now let's show: $N_C x \subseteq T_C^{\ominus} x \Leftrightarrow \forall_{u \in N_C x}\ u \in T_C^{\ominus} x = \{\tilde{u} \in \mathcal{H} \mid \sup \langle T_C x \mid \tilde{u}\rangle \leq 0\} = \{\tilde{u} \in \mathcal{H} \mid \langle \overline{\text{cone}}(C-x) \mid \tilde{u}\rangle \leq 0\}$

take $u \in N_C x$ // $x \in C$

$= (C-x)^{\ominus} = \{\tilde{u} \in \mathcal{H} \mid \sup \langle C - x \mid \tilde{u}\rangle \leq 0\}$

$\Leftrightarrow \sup \langle C - x \mid u\rangle \leq 0$

now $\overline{\text{cone}}\,(C-x) = \overline{\text{cone}(C-x)}$ /* Proposition 6.2 */

(iii) $\tilde{C}^{\ominus} = (\text{cone}\, \tilde{C})^{\ominus} = (\text{conv}\, \tilde{C})^{\ominus} = \overline{\tilde{C}}^{\ominus}$ /* to the polar cone operator, a set, its cone, convex hull and closure all are same */

$\tilde{C} := \text{cone}(C-x) \Rightarrow (\text{cone}(C-x))^{\ominus} = (\overline{\text{cone}(C-x)})^{\ominus} = (C-x)^{\ominus}$

now $x \in C$, $T_C x = \overline{\text{cone}}\,(C-x)$

$\Rightarrow T_C x^{\ominus} = (\text{cone}(C-x))^{\ominus} = (C-x)^{\ominus} = \{\tilde{u} \in \tilde{\mathcal{H}} \mid \sup \langle C - x \mid \tilde{u}\rangle \leq 0\}$

$u \in T_C x^{\ominus}$

$\therefore N_C x \subseteq T_C x^{\ominus}$ (eq2)

Combining both $\boxed{N_C x = T_C x^{\ominus}}$ // From (eq1), (eq2)

now $x \in C \Rightarrow T_C x = \overline{\text{cone}}\,(C-x)$: closed convex cone /* using:

* Corollary 6.33. [K: nonempty closed convex cone in $\mathcal{H}$] $\Rightarrow K^{\ominus\ominus} = K$

Proposition 6.2.
[C: subset of $\mathcal{H}$] $\Rightarrow$
(i) $\text{cone}\, C = \mathbb{R}_{++} C$ // most intuitive way creating cone C out of C
(ii) $\overline{\text{cone}\, C} = \overline{\text{cone}}\, C$
(iii) $\text{cone}(\text{conv}\, C) = \text{conv}(\text{cone}\, C)$: smallest convex cone containing C
as $C - x$: convex $\Rightarrow C - x = \text{conv}(C-x)$
(iv) $\overline{\text{cone}}(\text{conv}\, C) = \overline{\text{conv}}(\text{cone}\, C)$: smallest closed convex cone containing C. */

$T_C x^{\ominus\ominus} = T_C x$ $\qquad \therefore T_C x = N_C x^{\ominus}$

$\underbrace{((T_C x)^{\ominus})}_{N_C x \text{ // just proved}}{}^{\ominus}$

(ii) $x \in \text{core}\, C = \{\tilde{x} \in \mathcal{H} \mid \text{cone}(C - \tilde{x}) = \mathcal{H}\}$

$\Leftrightarrow \text{cone}((C-x)) = \mathcal{H} \Leftrightarrow \overline{\text{cone}((C-x))} = \overline{\mathcal{H}} = \mathcal{H} \Leftrightarrow \overline{\text{cone}}\,((C-x)) = \mathcal{H}$

as $x \in C \Rightarrow T_C x = \overline{\text{cone}}\,((C-x)) = \mathcal{H}$

from (i): $N_C x = (T_C x)^{\ominus} = \mathcal{H}^{\ominus} = \{\tilde{u} \in \mathcal{H} \mid \sup \langle \mathcal{H} \mid \tilde{u}\rangle \leq 0\} = \{0\}$

// this is the only value

$\therefore T_C x = \mathcal{H} \Rightarrow N_C(x) = \{0\}$

as $x \in C \Rightarrow T_C x = \operatorname{cone}(C-x) = \mathcal{H}$

from (i): $N_C x = (T_C x)^\ominus = \mathcal{H}^\ominus = \{\tilde{u} \in \mathcal{H} \mid \sup \langle \mathcal{H} \mid \tilde{u} \rangle \leq 0\} = \underbrace{\{0\}}$

// this is the only value

$\therefore T_C x = \mathcal{H} \Rightarrow N_C(x) = \{0\}$

from (i): $N_C^\ominus x = T_C x$

now take $N_C x = \{0\} \Rightarrow$

$T_C x = N_C^\ominus x = \{0\}^\ominus = \{\tilde{u} \in \mathcal{H} \mid \sup \langle 0 \mid \tilde{u} \rangle = 0 \leq 0\} = \mathcal{H}$

$\therefore N_C x = \{0\} \Rightarrow T_C x = \{0\}^\ominus = \mathcal{H}$

$\therefore x \in \operatorname{core} C \Rightarrow N_C x = \{0\} \Leftrightarrow T_C x = \mathcal{H}$.

■

*Proposition 6.46.

[C: nonempty closed convex subset of $\mathcal{H}$

$x, p \in \mathcal{H}$]

$p = P_C x \Leftrightarrow x - p \in N_C p$

Proof: From projection theorem: $(\underbrace{P_C}_{\text{closed convex set}} x \in C,\ \forall_{y \in C} \langle y - P_C x \mid x - P_C x \rangle \leq 0)$ ... (eq:1)

$(C-x)^\ominus$

$\|$

from definition of normal cone to C at x: $N_C x = \begin{cases} \{u \in \mathcal{H} \mid \forall_{y \in C} \langle y - x \mid u \rangle \leq 0\}, & x \in C \\ \emptyset, & \text{otherwise} \end{cases}$ $P_C x := p$

say $p = P_C x$. then

$p \in C$, so $N_C p = \{u \in \mathcal{H} \mid \forall_{y \in C} \langle y - p \mid u \rangle \leq 0\}$

so, $\boxed{\cdot} \in N_C p \Leftrightarrow \forall_{y \in C} \langle y - p \mid \boxed{\cdot} \rangle \leq 0$

set, $\boxed{\cdot} := x - p \hookrightarrow \therefore x - p \in N_C p \Leftrightarrow \forall_{y \in C} \langle y - p \mid x - p \rangle \leq 0$, $\boxed{p \in C}$ now comparing with (eq:1)

$\forall_{y \in C} \langle y - p \mid x - p \rangle \leq 0 \Leftrightarrow x - p \in N_C p$ ■

Corollary 6.44. /* checking membership to interior in terms of tangent cone and normal cone */

[$\mathcal{H}$: finite dimensional,

C: nonempty, convex, $\subseteq \mathcal{H}$.

$x \in C$]

$\underbrace{x \in \operatorname{int} C}_{(i)} \Leftrightarrow \underbrace{T_C x = \mathcal{H}}_{(ii)} \Leftrightarrow \underbrace{N_C x = \{0\}}_{(iii)}$

Proof:

(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii):

recall that $\forall_{C: \text{convex}, \subseteq \mathcal{H}}$ $\operatorname{int} C \subseteq \operatorname{core} C \subseteq \operatorname{sri} C \subseteq \operatorname{ri} C \subseteq \operatorname{qri} C \subseteq C$

as $x \in \operatorname{int} C \subseteq \operatorname{core} C \Rightarrow x \in \operatorname{core} C$

/* recall

Proposition 6.43. /* Relationship between tangent and normal cone */

[C: nonempty convex, $\subseteq \mathcal{H}$; $x \in C$]

(i) $T_C^\ominus x = N_C x$, $N_C^\ominus x = T_C x$

(ii) $x \in \operatorname{core} C \Rightarrow T_C x = \mathcal{H} \Leftrightarrow N_C x = \{0\}$ */

thus (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) is proved.

All we need to prove (iii) $\Rightarrow$ (i)

(iii) $\Rightarrow$ (i)

$N_C(x) = \{0\}$

$U := \operatorname{aff}(C)$ /* smallest affine subspace containing C, $U = \lambda U + (1-\lambda) U$ */ $\lambda \in \mathbb{R}$

C: affine subspace $\Leftrightarrow \begin{cases} V = C - C: \text{linear subspace parallel to } C \\ \forall_{x \in C}\ C = x + V \end{cases}$

$V = U - U$: linear subspace parallel to U,

$\therefore \forall_{x \in U}\ x + V = U$

$C \subseteq U$

$\Rightarrow \underbrace{C - x}_{\in C \text{ by given}} \subseteq U - x$ /* both sets are shifted by x */

$= V$

$C - x \subseteq V$ (1)

recall proposition 6.23.

* Proposition 6.23. [C: subset of $\mathcal{H}$] $\Rightarrow$ (i) $D \subseteq C \Rightarrow$ $\cdot\ C^\ominus \subseteq D^\ominus$ /* inclusion flips */ $\cdot\ C^\oplus \subseteq D^\oplus$

$\underbrace{V^\perp = V^\ominus}$ $\subseteq (C-x)^\ominus = N_C x$ /* recall, $N_C x = \begin{cases} (C-x)^\ominus, & \text{if } x \in C \\ \emptyset, & \text{else} \end{cases}$ */

/* for linear subspac

$V^\perp = V^\ominus$

now given, $N_C x = \{0\} \Rightarrow V^\perp = \{0\} \Rightarrow \underbrace{V}_{U - x} = \mathcal{H} \Rightarrow U - x = \mathcal{H}$

$\Leftrightarrow U = x + \mathcal{H} = \mathcal{H}$

$\therefore \operatorname{aff}(C) = \mathcal{H}$ (2)

/* Recall:

/* Required info: • seperable space: A space is seperable if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence */

* Fact 6.14. [C: nonempty convex subset of $\mathcal{H}$] * * *

$\lceil$ $\operatorname{ri} C$: interior of C relative to $\operatorname{aff} C$, $\operatorname{ri} C \neq \emptyset$

(infinite) of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence */

✱ Fact 6.14. [C: nonempty convex subset of $\mathcal{H}$] ✱ ✱ ✱

(i) $\mathcal{H}$: finite dimensional $\Rightarrow$ $\begin{cases} \text{ri}\, C : \text{interior of } C \text{ relative to aff}\, C,\ \text{ri}\, C \neq \emptyset \\ \overline{\text{ri}\, C} = \overline{C},\ \text{ri}\, \overline{C} = \text{ri}\, C,\ \text{sri}\, C = \text{ri}\, C = \text{qri}\, C \end{cases}$

*/

and

$\text{ri}\, C = \{x \in C \mid \text{cone}(C - x) = \text{span}(C - x)\}$ */

$\underset{\neq \emptyset}{\text{ri}\, C} : \underbrace{\text{interior of } C \text{ relative to aff}\, C = \mathcal{H}}_{= \text{int}\, C}$

$\Rightarrow \text{ri}\, C = \text{int}\, C \neq \emptyset$

So we have shown that

$N_C x = \{0\} \Rightarrow \text{int}\, C \neq \emptyset$

/* Fact 6.14. [C: $\neq \emptyset$, convex, $\subseteq \mathcal{H}$]

(iii) $\text{int}\, C \neq \emptyset \Rightarrow \text{int}\, C = \text{core}\, C = \text{sri}\, C = \text{ri}\, C = \text{qri}\, C$ */

✱ Proposition 6.43. /* Relationship between tangent and normal cone */

[C: nonempty convex, $\subseteq \mathcal{H}$; $x \in C$]

(i) $T_C^{\ominus} x = N_C x$, $N_C^{\ominus} x = T_C x$

(ii) $x \in \text{core}\, C \Rightarrow T_C x = \mathcal{H} \Leftrightarrow N_C x = \{0\}$

$\text{int}\, C = \text{core}\, C \neq \emptyset$

/*

recall that:

(2nd edition)

Proposition 6.45 *Let C be a convex subset of $\mathcal{H}$ such that* $\text{int}\, C \neq \varnothing$ *and let $x \in C$. Then* $x \in \text{int}\, C \Leftrightarrow T_C x = \mathcal{H} \Leftrightarrow N_C x = \{0\}$.

*/

thus $N_C(x) = \{0\} \Rightarrow x \in \text{int}\, C$

$\therefore$ (iii) $\Rightarrow$ (i)

Thus: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

■

6.5 Recession and Barrier Cones

11:15 AM

$\ast$ Proposition 6.48.

[C: nonempty convex subset of $\mathcal{H}$]

(i) $\operatorname{rec} C$: convex cone, $0 \in \operatorname{rec} C$

(ii) $\operatorname{bar} C$: convex cone, $C^{\ominus} \subseteq \operatorname{bar} C$

(iii) C: bounded $\Rightarrow$ $\operatorname{bar} C = \mathcal{H}$

(iv) C: cone $\Rightarrow$ $\operatorname{bar} C = C^{\ominus}$

(v) C: closed $\Rightarrow (\operatorname{bar} C)^{\ominus} = \operatorname{rec} C$

Proof: (not complete).

(i) $\operatorname{rec} C = \{ x \in \mathcal{H} \mid x + C \subseteq C \}$

so, $\boxed{\cdot} \in \operatorname{rec} C \Leftrightarrow \boxed{\cdot} \in \mathcal{H},\ \boxed{\cdot} + C \subseteq C$

as. $0 \in \mathcal{H},\ 0 + C = C \subseteq C \Leftrightarrow 0 \in \operatorname{rec} C$.

first, $\forall_{x_1 \in \operatorname{rec} C}\ \forall_{x_2 \in \operatorname{rec} C}$ $\qquad x_1 + x_2 = z_1 + z_2 - y_1 - y_2$

$\updownarrow \qquad \updownarrow$

$\forall_{y_1 \in C}\ \exists_{z_1 \in C}\ x_1 + y_1 = z_1 \qquad \forall_{y_2 \in C}\ \forall_{z_2 \in C}\ x + y_2 = z_2$

By definition of a convex set, $\forall_{\alpha \in]0,1[}\ \alpha C + (1-\alpha) C = C$

$\forall_{x,y \in C}\ \forall_{\alpha \in]0,1[}\ \alpha x + (1-\alpha) y \in C$

suppose $x \in \operatorname{rec} C \Leftrightarrow \forall_{z \in C}\ x + z \in C \leftrightarrow \forall_{z \in C}\ \exists_{\eta_1 \in C}\ x + z = \eta_1 \rightarrow \overset{\in]0,1[}{\alpha} x + \alpha z = \alpha \eta_1$

$y \neq x$, $y \in \operatorname{rec} C \Leftrightarrow \forall_{w \in C}\ y + w \in C \leftrightarrow \forall_{w \in C}\ \exists_{\eta_2 \in C}\ y + w = \eta_2 \rightarrow (1-\alpha) y + (1-\alpha) w = (1-\alpha) \eta_2$

$$\alpha x + (1-\alpha) y + \underbrace{\alpha z + (1-\alpha) w}_{\bar{z} \in C} = \underbrace{\alpha \eta_1 + (1-\alpha) \eta_2}_{\bar{\eta} \in C}$$

[as. $z, w \in \underbrace{C}_{\text{convex}}$] $\qquad$ [as $\eta_1, \eta_2 \in C$]

$\leftrightarrow \alpha x + (1-\alpha) y + \bar{z} = \hat{\eta} \in C$

now $\bar{z}$: arbitrary as this is a convex combination of arbitrary points $z, w \in C$

So we have: $\forall_{x \in \operatorname{rec} C}\ \forall_{y \in \operatorname{rec} C}\ \alpha x + (1-\alpha) y + C \subseteq C \leftrightarrow \alpha x + (1-\alpha) y \in \operatorname{rec} C$

$\therefore \operatorname{rec} C$: convex

/$\ast$ Proposition 6.3. (ii)

$[\![C: \text{convex}, 0 \in C]\!]$ $C + C \subseteq C \Leftrightarrow C: \text{cone}$ */

We have shown, $\text{rec}\,C: \text{convex}, \ni 0$, $\text{rec}\,C + \text{rec}\,C \subseteq \text{rec}\,C$