# 5.1 Fejer Monotone Sequences

9:29 PM

Definition 5.1. (Fejer monotonicity w.r.t a set)

[ C: nonempty subset of $\mathcal{H}$

$(x_n)_{n\in\mathbb{N}}$ : sequence in $\mathcal{H}$ ]

$(x_n)_{n\in\mathbb{N}}$ : Fejer monotone w.r.t $C \overset{\text{def}}{\longleftrightarrow} \forall_{x\in C} \forall_{n\in\mathbb{N}} \|x_{n+1}-x\| \leq \|x_n - x\|$ (5.1)

Proposition 5.4.

[ C: nonempty subset of $\mathcal{H}$

$(x_n)_{n\in\mathbb{N}}$ : sequence in $\mathcal{H}$, Fejer monotone w.r.t C ] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$ : bounded

(ii) $\forall_{x\in C} (\|x_n - x\|)_{n\in\mathbb{N}}$ : converges

(iii) $(d_C(x_n))_{n\in\mathbb{N}}$ : decreasing, converges.

Proof: From defn (5.1) $\forall_{x\in C} \forall_{n\in\mathbb{N}} \|x_{n+1}-x\| \leq \|x_n - x\|$

consider any $x\in C$

$\|x_1 - x\| \leq \|x_0 - x\|$

$\|x_2 - x\| \leq \|x_1 - x\| \leq \|x_0 - x\|$

$\vdots$

$\|x_n - x\| \leq \|x_0 - x\|$

So, $\forall_{x\in C} \forall_{n\in\mathbb{N}} \|x_n - x\| \leq \|x_0 - x\|$ // recall that $B(c,r) = \{x \mid \|x-c\| \leq r\}$

$\leftrightarrow$ $(x_n)_{n\in\mathbb{N}}$ lies in $B(x; \|x_0 - x\|) \leftrightarrow (x_n)_{n\in\mathbb{N}}$ : bounded ■

(ii)

$\forall_{x\in C} \forall_{n\in\mathbb{N}} \underbrace{\|x_{n+1}-x\|}_{y_{n+1}} \leq \underbrace{\|x_n - x\|}_{y_n}$ : monotone

From (i) $(x_n)_{n\in\mathbb{N}}$ bounded, so $(\|x_n - x\|)_{n\in\mathbb{N}}$ : bounded

} converges /* a bounded monotone sequence always converges */

(iii) given

$\forall_{n\in\mathbb{N}} \forall_{x\in C} \|x_{n+1}-x\| \leq \|x_n - x\|$

$\leftrightarrow \forall_{n\in\mathbb{N}} \underbrace{\inf_{x\in C} \|x_{n+1}-x\|}_{d_C(x_{n+1})} \leq \underbrace{\inf_{x\in C} \|x_n - x\|}_{d_C(x_n)}$ /* order limit theorem */

$\leftrightarrow \forall_{n\in\mathbb{N}} d_C(x_{n+1}) \leq d_C(x_n)$

From (ii) $\forall_{x\in C} \|x_n - x\|_{n\in\mathbb{N}}$: bounded $\Rightarrow d_C(x_n)$: bounded by def. /* a bounded monotone sequence always converges */

$\Rightarrow d_C(x_n)$: converges ■

Theorem 5.5.

[ $(x_n)_{n\in\mathbb{N}}$ : sequence in $\mathcal{H}$, Fejer monotone w.r.t C (Nonempty, $\subseteq \mathcal{H}$).

every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in C ]

$\Rightarrow$

$(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in C.

Proof: $\forall_{x\in C} \|x_n - x\|_{n\in\mathbb{N}}$ : converges /* Proposition 5.4. (ii) */

recall:

/* Lemma 2.39.

C : nonempty, $\subseteq \mathcal{H}$

$(x_n)_{n\in\mathbb{N}}$ :
- sequence in $\mathcal{H}$
- $\forall_{x\in C} \|x_n - x\|$ : converges
- Every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in C.

$\Rightarrow$

Combining these two we see all the givens imply $(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in C.

- Every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in $C$.

$\Rightarrow$

$(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in $C$. */

*Proposition 5.7.

[ $C$: nonempty closed convex subset of $\mathcal{H}$

$(x_n)_{n\in\mathbb{N}}$: sequence in $\mathcal{H}$, Fejer monotone w.r.t. $C$ ] $\Rightarrow$

$\underbrace{(P_C x_n)_{n\in\mathbb{N}}}_{\text{called the shadow sequence}}$ : converges strongly to a point in $C$.

*Corollary 5.8.

[ $C$: nonempty closed convex subset of $\mathcal{H}$

$(x_n)_{n\in\mathbb{N}}$: sequence in $\mathcal{H}$, Fejer monotone w.r.t. $C$, $x_n \rightharpoonup \overbrace{x}^{\in C}$ ]

$\Rightarrow$

$P_C x_n \to \overbrace{x}^{\in C}$.

Proof:

/* Proposition 5.7.

$(x_n)_{n\in\mathbb{N}}$: Fejer monotone w.r.t. $\overbrace{C}^{\text{nonempty, closed, convex}}$, sequence in $\mathcal{H}$

$(P_C x_n)_{n\in\mathbb{N}}$ : $P_C x_n \to y \in C$ */

$\exists_{y\in C}$ $P_C x_n \to y$

$\Rightarrow$ $x - P_C x_n \to x - y$

given

$x_n \rightharpoonup x$

$\Rightarrow$ $x_n - P_C x_n \rightharpoonup x - y$

One of the defining property of projection is $\langle \overbrace{\square}^{\in C} - P_C \square \mid \overbrace{\square}^{\in\mathcal{H}} - P_C \overbrace{\square}^{\in C} \rangle \leq 0$
closed, convex, nonempty

so in our case, $x\in C$, $x_n \in \mathcal{H}$, $P_C x_n \in C$

$\therefore$ $\langle x - P_C x_n \mid x_n - P_C x_n \rangle \leq 0$ $\forall_{n\in\mathbb{N}}$

/* Lemma 2.41. (iii) says:

$(u_n)_{n\in\mathbb{N}} :\subseteq \mathcal{H}$, $u_n \to \overbrace{u}^{\in\mathcal{H}}$

$\langle x_n \mid u_n \rangle \to \langle x \mid u \rangle$

$(x_n)_{n\in\mathbb{N}} :\subseteq \mathcal{H}$, $x_n \rightharpoonup \underbrace{x}_{\in\mathcal{H}}$ */

now,

$$\left.\begin{aligned} (x_n - P_C x_n) &\rightharpoonup (x-y) \\ (x - P_C x_n) &\to \underbrace{(x-y)}_{\in\mathcal{H}} \end{aligned}\right\} \langle x - P_C x_n \mid x_n - P_C x_n \rangle \to \langle x-y \mid x-y \rangle = \|x-y\|^2 \Rightarrow \|x-y\|^2 \leq 0 \Leftrightarrow \|x-y\|^2 = 0$$

$\in\mathcal{H}$ [$\because$ $C$ convex, $x, y\in C$
$x-y$ is not necessarily in $C$ ]

$\therefore$ $x = y$

so, $P_C x_n \to x$ ■

Theorem 5.11.

[ $C$: nonempty closed convex subset of $\mathcal{H}$

$(x_n)_{n\in\mathbb{N}}$: Fejer monotone w.r.t. $C$, $\subseteq\mathcal{H}$ ]

(i) $(x_n)_{n\in\mathbb{N}}$ : converges strongly to a point in $C$ $\Leftrightarrow$

(ii) $(x_n)_{n\in\mathbb{N}}$ : posseses a strong sequential cluster point in $C$ $\Leftrightarrow$

(iii) $\underline{\lim}\ d_C(x_n) = 0$

Proof: /* Proof strategy: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv) */

(i) $\to$ (ii): $(x_n)_{n\in\mathbb{N}}$ : converges strongly to a point in $C$ (say $x$)

Proof: /* Proof strategy: (i) ⇒ (ii), (ii) ⇒ (iii), (iii) ⇒ (i) */

(i) → (ii): $(x_n)_{n\in\mathbb{N}}$ : converges strongly to a point in $C$ (say $x$)

$\Rightarrow$ all subsequences of $(x_n)_{n\in\mathbb{N}}$ converges strongly to that point in $C$ (say $x$) /* Fact 1.9.
[ $(x_a)_{a\in A}$ : net in Hausdorff space $X$, $x_a \to \underbrace{x}_{\in X}$
$(x_{k(b)})_{b\in B}$ : subnet of $(x_a)_{a\in A}$ ] $\Rightarrow x_{k(b)} \to x$ /* if a net in a Hausdorff space converges, then so does its any subset to the same point */ */

$\Rightarrow \exists$ subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges strongly to $x$ in $C$ (say $x$)

$\overset{\text{def}}{\Leftrightarrow}$ $\underbrace{x}_{\in C}$ : strong sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ ☺

(ii) ⇒ (iii):

(ii): $\underbrace{x}_{\in C}$ : strong sequential cluster point of $(x_n)_{n\in\mathbb{N}}$, now $\underline{\lim}\, d_C(x_n)$ by definition will exist. Say $\underline{\lim}\, d_C(x_n) = p \in \mathbb{R}\cup\{\pm\infty\}$ (0) $\Rightarrow \underline{\lim}\, d_C(x_{k_n}) = p$ (1) for all $(x_{k_n})_{n\in\mathbb{N}}$ that are subsequences of $(x_n)_{n\in\mathbb{N}}$

$\Leftrightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}}}\ x_{k_n} \to x \in C$

now, $\forall_{n\in\mathbb{N}}\ d_C(x_{k_n}) = \inf d(x_{k_n}, C) = \inf_{y\in C} \|x_{k_n} - y\| \leq \|x_{k_n} - x\|$ [by definition, and $x \in C$]

$\Leftrightarrow \forall_{n\in\mathbb{N}}\ d_C(x_{k_n}) \leq \|x_{k_n} - x\| \Rightarrow 0 \leq \underline{\lim}\, d_C(x_{k_n}) \leq \overline{\lim}\, \|x_{k_n} - x\|$

using

now, $\lim \|x_{k_n} - x\| = \overline{\lim}\, \|x_{k_n} - x\| = 0$ $\therefore \underline{\lim}\, d_C(x_{k_n}) = 0$ ...(2)

*Fact 1.3.1. /* these are extendable to nets as well */
(comes handy in dealing with sequences)
- $\forall_{n\in\mathbb{N}}\ a_n \leq a \Rightarrow \overline{\lim}\, a_n \leq a$
- $\forall_{n\in\mathbb{N}}\ b \leq b_n \Rightarrow b \leq \underline{\lim}\, b_n$
- $\forall_{n\in\mathbb{N}}\ a_1 \leq a_n \leq a_2 \Rightarrow$ $a_1 \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq a_2$; $\underline{\lim}\, a_n$ exists in finite norm; $\overline{\lim}\, a_n$ exists in finite norm
- $\forall_{n\in\mathbb{N}}\ a_n \leq b_n \Rightarrow \underline{\lim}\, a_n \leq \overline{\lim}\, b_n$

**
Result 1.2.1. (A very important result) /* verified, uses result 1.2.2, and definition of convergence of a sequence */
[ $(b_n)_{n\in\mathbb{N}} : \subseteq \mathbb{R}$ ] $(0 \leq a \leq b_n,\ b_n \to 0) \Rightarrow a = 0$

Now,

from (0), (1) and (2) we have: $\underline{\lim}\, d_C(x_n) = 0$

$\therefore \underline{\lim}\, d_C(x_n) = 0$ ☺

(iii) ⇒ (i):

/* Proposition 5.4. (iii):

$(x_n)_{n\in\mathbb{N}}$ : Fejer monotone w.r.t $C$; $\subseteq \mathcal{H}$

$(d_{\underset{\substack{\uparrow \\ \text{: nonempty; } \subseteq \mathcal{H}}}{C}}(x_n))_{n\in\mathbb{N}}$ : decreasing and converges */

now (iii) says: $\underline{\lim}\, d_C(x_n) = 0$

as $d_C(x_n)$ : converges $\Rightarrow \lim d_C(x_n) = \underbrace{\underline{\lim}\, d_C(x_n)}_{=0}$

/* recall Fact: 1.15. (iii):

$\underbrace{(\xi_a)_{a\in A}}_{\text{net in } [-\infty,\infty]}$ : converges $\Leftrightarrow \left(\underline{\lim}\, \xi_a = \lim \xi_a = \overline{\lim}\, \xi_a\right)$

as $d_C(x_n) \subseteq [-\infty,\infty]$, we can apply it */

$\Leftrightarrow d_C(x_n) \to 0$

now $d_C(x_n) = \inf_{y\in C} \|x_n - y\|$
$= \|x_n - P_C x_n\|$ /* By definition: $d_C(x) = \|x - P_C x\|$ */
$\to 0$

$\Leftrightarrow x_n - P_C x_n \to 0$

/* now, the shadow sequence of a Fejer monotone sequence w.r.t to a nonempty closed convex set $C$, always converges strongly to a point in $C$ (Proposition 5.7)

$\Leftrightarrow \exists_{x\in C}\ P_C x_n \to x$ */

/* By adding */

$\therefore x_n \to x \in C$ : (i) ☺

■

*A linear convergence result for Fejer monotone sequence.

Theorem 5.12.

[ $C$ : nonempty closed convex subset of $\mathcal{H}$

$(x_n)_{n\in\mathbb{N}} : \subseteq \mathcal{H}$, Fejer monotone w.r.t $C$

$\exists_{K\in[0,1[}\ \forall_{n\in\mathbb{N}}\ d_C(x_{n+1})\leqslant K\,d_C(x_n)\ ]\Rightarrow$

$(x_n)_{n\in\mathbb{N}}$ : converges linearly to a point $x\in C$

More precisely:

$\forall_{n\in\mathbb{N}}\ \|x_n-x\|\leqslant 2K^n d_C(x_0)$

**Proof:** As, $\forall_{n\in\mathbb{N}}\ d_C(x_{n+1})\leqslant \underset{\in[0,1[}{K}\, d_C(x_n) \Leftrightarrow \|x_{n+1}-P_C x_{n+1}\|\leqslant K\|x_n-P_C x_n\|$ /* ( By defn $\|x-P_C x\|=d_C(x)$ )

Now. $0\leqslant d_C(x_{n+1})\leqslant K d_C(x_n)\leqslant K^2 d_C(x_{n-1})\leqslant K^3 d_C(x_{n-2})\leqslant\cdots\leqslant K^{n+1} d_C(x_{n-n})$ (Eq:A)

$\therefore\ \forall_{n\in\mathbb{N}}\ 0\leqslant d_C(x_{n+1}),\quad d_C(x_{n+1})\leqslant K^{n+1} d_C(x_0)$ /* now :

*Fact 1.3.1. /* these are extendable to nets as well */
(comes handy in dealing with sequences)

- $\forall_{n\in\mathbb{N}}\ a_n\leqslant a \Rightarrow \overline{\lim}\, a_n\leqslant a$
- $\forall_{n\in\mathbb{N}}\ b\leqslant b_n \Rightarrow b\leqslant \underline{\lim}\, b_n$
- $\forall_{n\in\mathbb{N}}\ a_1\leqslant a_n\leqslant a_2 \Rightarrow \left|\begin{array}{l} a_1\leqslant \underline{\lim}\, a_n\leqslant \overline{\lim}\, a_n\leqslant a_2 \\ \underline{\lim}\, a_n \text{ exists in finite norm} \\ \overline{\lim}\, a_n \text{ exists in finite norm}\end{array}\right.$
- $\forall_{n\in\mathbb{N}}\ a_n\leqslant b_n \Rightarrow \underline{\lim}\, a_n\leqslant \overline{\lim}\, b_n$

*/

$\Rightarrow\quad 0\leqslant \underline{\lim}\, d_C(x_{n+1}).\quad \underline{\lim}\, d_C(x_{n+1})\leqslant \overline{\lim}\, K^{n+1} d_C(x_0)=\lim K^{n+1} d_C(x_0)=0$

/* now :

*Theorem 5.11.
[ C: nonempty closed convex subset of $\mathcal{H}$
$(x_n)_{n\in\mathbb{N}}$: $\subseteq\mathcal{H}$, Fejer monotone w.r.t. C ]
(i) $(x_n)_{n\in\mathbb{N}}$ : converges strongly to a point in C $\Leftrightarrow$
(ii) $(x_n)_{n\in\mathbb{N}}$ : possesses a strong sequential cluster point in C /* there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that converges strongly to a point in C */
(iii) $\underline{\lim}\, d_C(x_n)=0$ /* recall: limit inferior of net $(\xi_a)_{a\in A}$ : $\underline{\lim}\,\xi_a=\sup_{c\in A}\left[\begin{array}{l}\inf \xi_b\\ b\in A\\ \text{s.t. } b\geqslant c\end{array}\right]$ */

$\therefore\ \underline{\lim}\, d_C(x_{n+1})=0$

*/

$\exists_{x\in C}\ x_n\to x$

Recall definition of Fejer monotone sequence :

$\forall_{x\in C}\ \forall_{n\in\mathbb{N}}\ \|x_{n+1}-x\|\leqslant\|x_n-x\|$ now $P_C(x_n)\in C$ by definition

$$x:=P_C(x_n)\Rightarrow \forall_{n\in\mathbb{N}}\ \left.\begin{array}{l}\|x_{n+1}-P_C x_n\|\leqslant\|x_n-P_C x_n\|\\ \|x_{n+2}-P_C x_n\|\leqslant\|x_{n+1}-P_C x_n\|\\ \vdots\\ \|x_{n+m}-P_C x_n\|\leqslant\|x_{n+m-1}-P_C x_n\|\end{array}\right\}\Rightarrow \|x_{n+m}-P_C x_n\|\leqslant\|x_n-P_C x_n\|=d_C(x_n)$$

now $\forall_{n,m\in\mathbb{N}}$

$$\begin{aligned}\|x_n-x_{n+m}\| &= \|x_n-P_C x_n-x_{n+m}+P_C x_n\|\\ &=\|(x_n-P_C x_n)+(-(x_{n+m}-P_C x_n))\|\\ &\leqslant\|(x_n-P_C x_n)\|+\|-(x_{n+m}-P_C x_n)\|\\ &=\underbrace{\|x_n-P_C x_n\|}_{=d_C(x_n)}+\underbrace{\|x_{n+m}-P_C x_n\|}_{\leqslant d_C(x_n)}\leqslant 2d_C(x_n)\end{aligned}$$

let $m\to\infty$ then $x_{n+m}\to x$

$\therefore\ \forall_{n\in\mathbb{N}}\ \|x_n-x\|\leqslant 2d_C(x_n)$

But given: $d_C(x_n)\leqslant K\, \underset{\leqslant K d_C(x_{n-2})}{d_C(x_{n-1})}\leqslant K^2 d_C(x_{n-2})\cdots\leqslant K^n d_C(\underset{n-n}{x_0})$

From (Eq:A)

$\therefore\ \forall_{n\in\mathbb{N}}\ \|x_n-x\|\leqslant 2K^n d_C(x_0)$

■

# 5.2 KM iteration

9:39 AM

$x_{n+1} = \underset{\substack{\downarrow \\ \text{nonexpansive}}}{T} x_n$ may fail to produce a fixed point

e.g., $T = -Id$ want to $(-Id)x = x$ whic is 0

now $x_0 \neq 0$, then, $x_1 = -Id\, x_0 = -x_0$

$x_2 = -Id\, x_1 = -Id(-x_0) = x_0$

$x_3 = -Id\, x_2 = -x_0$

$\vdots$

so $x_n = \begin{cases} x_0, & \text{if } n\text{: even} \\ -x_0, & \text{if } n\text{: odd} \end{cases}$

so it is not going to 0, rather fluctuating between $\pm x_0$

* The reason behind not converging: as $x_n - Tx_n \not\to 0$

$\underbrace{x_n - Tx_n \to 0}$ /* in this case, $x_n - Tx_n = x_n - x_{n+1} = \pm 2x_0 \neq 0$

is called asympotic regularity property, this needs to hold for convergence.

if it holds then $x_{n+1} = Tx_n$ converges to a fixed point.

* Theorem 5.13. (Necessity of asympotic regularity for classic iteration scheme to work)

[ D: nonempty closed convex subset of $\mathcal{H}$,

$T: D \to D$, nonexpansive, Fix $T \neq \emptyset$, $x_n - Tx_n \to 0$, $\forall_{n \in \mathbb{N}}\ x_{n+1} = Tx_n$

$x_0 \in D$ ] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in Fix T.

(ii) $\left.\begin{array}{l} D = -D, \\ T\text{: odd} \overset{\text{def}}{\Leftrightarrow} \forall_{x\in D}\ T(-x) = -Tx \end{array}\right\} \Rightarrow (x_n)_{n\in\mathbb{N}}$: converges strongly to a point in Fix T.

* Theorem 5.14. (KM algorithm)

[ D: nonempty closed convex subset of $\mathcal{H}$

$T: D \to D$, nonexpansive, Fix $T \neq \emptyset$,

$(\lambda_n)_{n\in\mathbb{N}}$: sequence in $[0,1]$, $\sum_{n\in\mathbb{N}} \lambda_n(1-\lambda_n) = +\infty$

$x_0 \in D$

$\forall_{n\in\mathbb{N}}\ x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n$ /* KM iteration */

] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$: Fejer monotone w.r.t Fix T

(ii) $(Tx_n - x_n)_{n\in\mathbb{N}}$: converges strongly to 0 /* so, Asympotic regularity property holds */

(iii) $(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in Fix T.

Proof: Before starting the proof we show that $(x_n)_{n\in\mathbb{N}}$: well defined sequence in D

$x_0 \in D$, $Tx_0 \in D$ ($\because T: D \to D$, so. dom$T = D$, ran$T \subseteq D$)

$x_1 = (1-\lambda_0)\underbrace{x_0}_{\in D} + \lambda_0 \underbrace{Tx_0}_{\in D} \in D$ [$\because$ D: convex]

$\vdots$

$x_{n+1} = (1-\lambda_n)x_n + Tx_n \in D$

$\therefore (x_n)_{n\in\mathbb{N}}$ is a sequence in D.

(i)

$\forall_{y\in \text{Fix}T}\ \forall_{n\in\mathbb{N}}$

$\|x_{n+1} - y\|^2$

$= \| (1-\lambda_n) x_n + \lambda_n T x_n \underbrace{- y}_{-(1-\lambda_n) y - \lambda_n y} \|^2$

$= \| (1-\lambda_n) x_n - (1-\lambda_n) y + \lambda_n T x_n - \lambda_n y \|^2$

$= \| (1-\lambda_n)(x_n - y) + \lambda_n (T x_n - y) \|^2$

$= (1-\lambda_n) \|x_n - y\|^2 + \lambda_n \|T x_n - y\|^2 - \lambda_n (1-\lambda_n) \| x_n \cancel{-y} \cancel{+y} - T x_n \|^2$ /* Corollary 2.14. $\forall_{x,y \in \mathcal{H}} \ \forall_{\alpha \in \mathbb{R}} \ \|\alpha x + (1-\alpha) y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ */

$= (1-\lambda_n) \|x_n - y\|^2 + \lambda_n \underbrace{\|T x_n - \underset{\underset{Ty}{\parallel}}{y}\|^2}_{} - \lambda_n (1-\lambda_n) \|x_n - T x_n\|^2$

$Ty$ [$\because y \in \operatorname{Fix} T$]

\# $\|T x_n - Ty\|^2 \leq \|x_n - y\|^2$ [$\because T$: nonexpansive]

$\leq (1-\lambda_n)\|x_n - y\|^2 + \lambda_n \|x_n - y\|^2 - \lambda_n (1-\lambda_n) \|T x_n - x_n\|^2 \quad \ldots (5.13)$

$= \|x_n - y\|^2 \underbrace{- \lambda_n (1-\lambda_n) \|T x_n - x_n\|^2}$

: negative so removing it will increase the value

$\leq \|x_n - y\|^2$

$\therefore \forall_{y \in \operatorname{Fix} T} \ \forall_{n \in \mathbb{N}} \ \|x_{n+1} - y\|^2 \leq \|x_n - y\|^2 \Leftrightarrow (x_n)_{n \in \mathbb{N}}$ : Fejer monotone w.r.t. Fix T. ☺

(ii) From 5.13 we have

$\forall_{y \in \operatorname{Fix} T} \ \forall_{n \in \mathbb{N}} \ \|x_{n+1} - y\|^2 \leq \|x_n - y\|^2 - \lambda_n (1-\lambda_n) \|T x_n - x_n\|^2$

$\Leftrightarrow \lambda_n (1-\lambda_n) \|T x_n - x_n\|^2 \leq \|x_n - y\|^2 - \|x_{n+1} - y\|^2$ : telescopic term

$\Rightarrow \sum_{n=0}^{m} \lambda_n (1-\lambda_n) \|T x_n - x_n\|^2 \leq \sum_{n=0}^{m} \left( \|x_n - y\|^2 - \|x_{n+1} - y\|^2 \right) = \|x_0 - y\|^2 - \cancel{\|x_1 - y\|^2} +$

$\cancel{\|x_1 - y\|^2} - \|x_2 - y\|^2 +$

$\vdots$

$\|x_m - y\|^2 - \|x_{m+1} - y\|^2$

$= \|x_0 - y\|^2 - \|x_{m+1} - y\|^2$

$\Leftrightarrow \sum_{n=0}^{m} \lambda_n (1-\lambda_n) \|T x_n - x_n\|^2 \leq \|x_0 - y\|^2 \underbrace{- \|x_{m+1} - y\|^2}$

negative number so removing it only increase the rest

$\Rightarrow \sum_{n=0}^{m} \lambda_n (1-\lambda_n) \|T x_n - x_n\|^2 \leq \|x_0 - y\|^2$

$\Rightarrow \quad 0 \leq \sum_{\substack{n \in \mathbb{N} \\ n \in \{0,1,\ldots,\infty\}}} \underbrace{\lambda_n (1-\lambda_n)} \|T x_n - x_n\|^2 \leq \|x_0 - y\|^2$

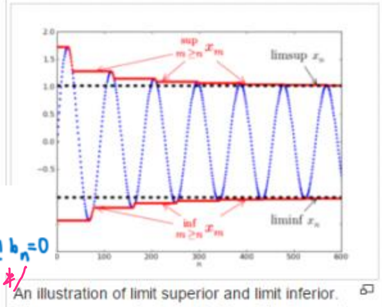
\# $\sum_{n \in \mathbb{N}} \lambda_n (1-\lambda_n) = +\infty$ : given

($\underline{\lim}$, $\overline{\lim}$ always exists for real number sequences)

now we donot know if $\lim \|T x_n - x_n\|^2$ exists but $\underline{\lim} \|T x_n - x_n\|^2$ must exist, and must be 0, otherwise the sequence wouldnot stay bounded.

/* Fact 1.3.2. (Establishing $\underline{\lim}(\cdot) = 0$)
$[a_n, b_n \geq 0]$ $(0 \leq \sum_{n=0}^{\infty} a_n b_n \leq \mu, \ \sum_{n=0}^{\infty} a_n = +\infty) \Rightarrow \underline{\lim}\, b_n = 0$ */

$\therefore \underline{\lim} \|T x_n - x_n\|^2 = 0 \Leftrightarrow \underline{\lim} \|T x_n - x_n\| = 0$




An illustration of limit superior and limit inferior. The sequence $x_n$ is shown in blue. The two red curves approach the limit superior and limit inferior of $x_n$, shown as dashed black lines. In this case, the sequence *accumulates* around the two limits. The superior limit is the larger of the two, and the inferior limit is the smaller of the two. The inferior and superior limits agree if and only if the sequence is convergent (i.e., when there is a single limit).

now $\forall_{n \in \mathbb{N}}$

$\|T x_{n+1} - \underset{\underset{(1-\lambda_n) x_n + \lambda_n T x_n}{\parallel}}{x_{n+1}}\| = \|T x_{n+1} - (1-\lambda_n) x_n - \lambda_n T x_n\| = \|T x_{n+1} - T x_n + \underbrace{T x_n - (1-\lambda_n) x_n - \lambda_n T x_n}_{(1-\lambda_n) T x_n - (1-\lambda_n) x_n = (1-\lambda_n)(T x_n - x_n)}\|$

$= \|T x_{n+1} - T x_n + (1-\lambda_n)(T x_n - x_n)\|$

$\leq \underbrace{\|T x_{n+1} - T x_n\|} + (1-\lambda_n) \|T x_n - x_n\|$ [triangle inequality]

$\leq \underbrace{\|Tx_{n+1} - Tx_n\|}_{\leq \|x_{n+1}-x_n\| \ [T:\text{ nonexpansive}]} + (1-\lambda_n)\|Tx_n - x_n\|$ [triangle inequality]

$\leq \underbrace{\|x_{n+1} - x_n\|}_{\lambda_n(Tx_n - x_n)} + (1-\lambda_n)\|Tx_n - x_n\|$ /* $x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n = x_n + \lambda_n(Tx_n - x_n)$

$\Leftrightarrow x_{n+1} - x_n = \lambda_n(Tx_n - x_n)$ */

$= \lambda_n\|Tx_n - x_n\| + (1-\lambda_n)\|Tx_n - x_n\| = \|Tx_n - x_n\|$

$\therefore \forall_{n\in\mathbb{N}} \ \|Tx_{n+1} - x_{n+1}\| \leq \|Tx_n - x_n\|$ So, $(\|Tx_n - x_n\|)_{n\in\mathbb{N}}$ : monotonically decreasing sequence

$(\|Tx_n - x_n\|)_{n\in\mathbb{N}}$ : bounded too, as $\|Tx_0 - x_0\| \geq \|Tx_1 - x_1\| \geq \cdots \geq \|Tx_n - x_n\| \geq \cdots \geq 0$

/* now

Result 1.2.4 (Convergence of monotonic sequence)
[ $(x_n)_{n\in\mathbb{N}}$ : monotonic ]
$(x_n)_{n\in\mathbb{N}}$ : converges $\leftrightarrow$ $(x_n)_{n\in\mathbb{N}}$ : bounded

/* $(x_n)_{n\in\mathbb{N}}$ : monotonically increasing $\overset{\text{def}}{\Leftrightarrow} \forall_{n\in\mathbb{N}} \ x_{n+1} \geq x_n$
$(x_n)_{n\in\mathbb{N}}$ : monotonically decreasing $\overset{\text{def}}{\Leftrightarrow} \forall_{n\in\mathbb{N}} \ x_{n+1} \leq x_n$ */ */

$\therefore$ So, $(\|Tx_n - x_n\|)_{n\in\mathbb{N}}$ : converges

$\therefore \lim \|Tx_n - x_n\| = \underbrace{\underline{\lim} \|Tx_n - x_n\|}_{= 0} = \overline{\lim}\|Tx_n - x_n\| \Leftrightarrow Tx_n - x_n \to 0$

:)

(iii)

in (i) we have shown that : $(x_n)_{n\in\mathbb{N}}$ : Fejer monotone w.r.t Fix T

$\Rightarrow (x_n)_{n\in\mathbb{N}}$ : bounded /* Proposition 5.4.

• Proposition 5.4.
[ C : nonempty subset of $\mathcal{H}$
$(x_n)_{n\in\mathbb{N}}$ : Fejer monotone sequence w.r.t $C$, $\subseteq \mathcal{H}$ ] $\Rightarrow$
(i) $(x_n)_{n\in\mathbb{N}}$ : bounded
(ii) $\forall_{z\in C} (\|x_n - z\|)_{n\in\mathbb{N}}$ : converges
(iii) $(d_C(x_n))_{n\in\mathbb{N}}$ : decreasing and converges

$\Rightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}} \ (x_{k_n})_{n\in\mathbb{N}}$ : weakly convergent /*

*Lemma 2.37.
[ $(x_n)_{n\in\mathbb{N}}$ : bounded sequence in $\mathcal{H}$ ] $\Rightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}} : \text{ subsequence of } (x_n)_{n\in\mathbb{N}}} \ (x_{k_n})_{n\in\mathbb{N}}$ : weakly convergent */

now as $(x_n)_{n\in\mathbb{N}}$ stays in D, $(x_{k_n})_{n\in\mathbb{N}}$ will also stay in D

$\therefore x_{k_n}$ weakly converges to some point (say x) in D $\therefore x_{k_n} \rightharpoonup x \in D$

and by definition x : weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$

from (ii) we have $Tx_n - x_n \to 0$

$\Leftrightarrow x_n - Tx_n \to 0$

$\Rightarrow x_{k_n} - Tx_{k_n} \to 0$ /* If a net in a Hausdorff space converges, then so does any of its subnet and to the same point (Fact 1.9) */

Now we use Corollary 4.18.

/* Corollary 4.18. ☆
[ D : nonempty closed convex subset of $\mathcal{H}$
$T : (D \to \mathcal{H}$, nonexpansive$)$, $x$ : some point in D
$(x_n)_{n\in\mathbb{N}}$ : sequence in D, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ ] $\Rightarrow x \in \text{Fix}\,T$ */

$\therefore x$ : Fix T

so we have shown that any weakly convergent subsequence of $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in Fix T

$\Updownarrow$

any weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in Fix T

Now we use Theorem 5.5

any weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in Fix T

Now we use Theorem 5.5.

Theorem 5.5.

[ $(x_n)_{n\in\mathbb{N}}$ : sequence in $\mathcal{H}$, Fejer monotone w.r.t. ~~C~~ $\neq\emptyset, \subseteq\mathcal{H}$ Fix T [showed in (i)], every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in ~~C~~ Fix T ] $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in ~~C~~ Fix T

$$x_n=(1-\lambda_{n-1})x_{n-1}+\lambda_{n-1}Tx_{n-1}$$

$\therefore$ $(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in Fix T. ☺

■

*** Proposition 5.15.** (Variant of KM algorithm for $\alpha$-averaged operator)

[ $\alpha\in]0,1[$

$T:\mathcal{H}\to\mathcal{H}$, $\alpha$-averaged operator, Fix $T\neq\emptyset$

$(\lambda_n)_{n\in\mathbb{N}}:\subseteq[0,1/\alpha]$, $\sum_{n\in\mathbb{N}}\lambda_n(1-\alpha\lambda_n)=+\infty$

$x_0\in\mathcal{H}$

$\forall_{n\in\mathbb{N}}$ $x_{n+1}=x_n+\lambda_n(Tx_n-x_n)$ ] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$ : Fejer monotone w.r.t. Fix T

(ii) $(Tx_n-x_n)_{n\in\mathbb{N}}$ : converges strongly to 0

(iii) $(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in Fix T.

Proof:

Set, $R:=(1-\frac{1}{\alpha})Id+\frac{1}{\alpha}T$ : nonexpansive /* T- $\alpha$ averaged $\Leftrightarrow$ $(1-\frac{1}{\alpha})Id+\frac{1}{\alpha}T$ : nonexpansive (Proposition 4.25) */

$\mu_n:=\alpha\lambda_n$

now Fix T = Fix R /* $\forall x$ $x\in$ Fix R $\Leftrightarrow Rx=((1-\frac{1}{\alpha})Id+\frac{1}{\alpha}T)x=x$

$\Leftrightarrow (1-\frac{1}{\alpha})x+\frac{1}{\alpha}Tx=x-\frac{1}{\alpha}x+\frac{1}{\alpha}Tx=x$

$\Leftrightarrow Tx=x \Leftrightarrow x\in$ Fix T $\therefore$ Fix R = Fix T */

now: the given iteration is:

$$x_{n+1}=x_n+\lambda_n(Tx_n-x_n)$$

/* now, $R=(1-\frac{1}{\alpha})Id+\frac{1}{\alpha}T$

$\Leftrightarrow T=\alpha\left(R-(1-\frac{1}{\alpha})Id\right)$ */

$$=x_n+\lambda_n\left(\underbrace{\alpha(R-(1-\tfrac{1}{\alpha})Id)x_n}-x_n\right)$$

$$\underbrace{\alpha(Rx_n-x_n+\tfrac{1}{\alpha}x_n)}$$

$$(\alpha Rx_n-\alpha x_n+x_n-x_n)=\alpha(Rx_n-x_n)$$

$$=x_n+\underbrace{\alpha\lambda_n}_{\mu_n\in[0,1]}(Rx_n-x_n)=x_n+\mu_n(\underbrace{R}_{\text{nonexpansive}}x_n-x_n)$$

/* $\because$ $\alpha\in[0,1]$, $\lambda_n\in[0,\frac{1}{\alpha}]\Rightarrow\underbrace{\alpha\lambda_n}_{\mu_n}\in\alpha[0,\frac{1}{\alpha}]=[0,1]$ */

/* Set: $\tilde{D}:=\mathcal{H}$, $\tilde{T}:=R$, $\tilde{\lambda}_n=\mu_n$

*Theorem 5.14. (KM algorithm): ★★

[ $\tilde{D}$: nonempty closed convex subset of $\mathcal{H}$,

$\tilde{T}:\tilde{D}\to\tilde{D}$, nonexpansive, Fix $\tilde{T}\neq\emptyset$

$(\tilde{\lambda}_n)_{n\in\mathbb{N}}$ : sequence in $[0,1]$, $\sum_{n\in\mathbb{N}}\tilde{\lambda}_n(1-\tilde{\lambda}_n)=+\infty$

$x_0\in\tilde{D}$

$\forall_{n\in\mathbb{N}}$ $x_{n+1}=(1-\tilde{\lambda}_n)x_n+\lambda_n\tilde{T}x_n$ /* KM iteration */ ] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$ : Fejer monotone w.r.t. Fix $\tilde{T}$

(ii) $(\tilde{T}x_n-x_n)_{n\in\mathbb{N}}$ : converges strongly to 0

(iii) $(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in Fix $\tilde{T}$ */

the goals follow immediately. ■

*** Corollary 5.18.** (KM iteration for composition and convex combinations of nonexpansive operators)

[ $(T_i)_{i\in I}$ : finite family of $\underbrace{\alpha_i}_{\in ]0,1[}$-averaged operators, $\forall_{i\in I}\ T_i : \mathcal{H}\to\mathcal{H}$, $\bigcap_{i\in I} \operatorname{Fix} T_i \neq \emptyset$,

$\tilde{i} : \begin{cases} \{(k,l) \mid k\in\{1,\dots,\overset{\in \mathbb{Z}_{++}}{p}\},\ l\in\{1,\dots,\overset{\in\mathbb{Z}_{++}}{m_k}\}\} \to I \\ \text{onto} \quad /\ast \ \operatorname{range}(\tilde{i}) = I \ \ast/ \end{cases}$

$\underbrace{(\omega_k)_{k=1}^{p}}_{>0} : \sum_{k=1}^{p} \omega_k = 1$

$\forall_{k\in\{1,\dots,p\}} \begin{cases} I_k = \{\tilde{i}(k,1), \tilde{i}(k,2), \dots, \tilde{i}(k,m_k)\} \\ \rho_k = \dfrac{m_k}{(m_k - 1) + \dfrac{1}{\max_{i\in I_k} \alpha_i}} \end{cases}$, $\alpha = \max_{1\le k\le p} \rho_k$

$(\lambda_n)_{n\in\mathbb{N}} : \subseteq [0, \frac{1}{\alpha}]$, $\sum_{n\in\mathbb{N}} \lambda_n (1-\alpha\lambda_n) = +\infty$

$\begin{cases} x_0 \in \mathcal{H} \\ \forall_{n\in\mathbb{N}}\ x_{n+1} = (1-\lambda_n)x_n + \lambda_n \sum_{k=1}^{p} \omega_k \Big( \underbrace{\prod_{j\in I_k} T_j}_{T_{\tilde{i}(k,1)}\cdots T_{\tilde{i}(k,m_k)}} \Big) x_n \end{cases}$ /* KM iteration for composition of nonexpansive operators */

] $\Rightarrow$

$(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in $\bigcap_{i\in I} \operatorname{Fix} T_i$

Proof:

define $T = \sum_{k=1}^{p} \omega_k \Big( \prod_{j\in I_k} T_j \Big)$

then we can write the modified KM iteration as:

$\forall_{n\in\mathbb{N}}\ x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n$

• Proof-strategy: we will show that, (i) $T$ : $\alpha$-averaged

$\in ]0,1[$, /* $\alpha \in ]0,1[$

Proof: $\forall_k\ \rho_k = \dfrac{m_k}{m_k - 1 + \dfrac{1}{\max_{i\in I_k} \underbrace{\alpha_i}_{\in ]0,1[}}}$

so, $\Big(\max_{i\in I_k} \alpha_i\Big) \in ]0,1[ \Rightarrow \dfrac{1}{\max_{i\in I_k}\alpha_i} > 1$

$\Rightarrow \dfrac{1}{\max_{i\in I_k}\alpha_i} - 1 > 0 \Rightarrow \Big( \underbrace{m_k - 1}_{\in\mathbb{Z}_{++}} + \dfrac{1}{\max_{i\in I_k}\alpha_i} \Big) > \underbrace{m_k}_{\in\mathbb{Z}_{++}}$

$\Rightarrow \rho_k = \dfrac{m_k}{m_k - 1 + \dfrac{1}{\max_{i\in I_k}\alpha_i}} \in ]0,1[ \Rightarrow \alpha = \max_{1\le k\le p} \rho_k \in ]0,1[$ ■ */

once we have done that we can just plug in KM iteration convergence theorem for averaged operator:

*** Proposition 5.15.** (Variant of KM algorithm for $\alpha$-averaged operator)

[ $\alpha \in ]0,1[$

$T : \mathcal{H}\to\mathcal{H}$, $\alpha$-averaged operator, $\operatorname{Fix} T \neq \emptyset$

$(\lambda_n)_{n\in\mathbb{N}} : \subseteq [0, \frac{1}{\alpha}]$, $\sum_{n\in\mathbb{N}} \lambda_n(1-\alpha\lambda_n) = +\infty$

$x_0 \in \mathcal{H}$

$\forall_{n\in\mathbb{N}}\ x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n$ ] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$ : Fejer monotone w.r.t $\operatorname{Fix} T$

(ii) $(Tx_n - x_n)_n$ : converges strongly to 0

(iii) $(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in $\operatorname{Fix} T$.

and then we show that (ii) $\operatorname{Fix} T = \bigcap_{i\in I} \operatorname{Fix} T_i$

Now let's start the proof: $\forall_{k\in\{1,\dots,p\}}\ R_k = \prod_{j\in I_k} \overbrace{T_j}^{\alpha_j\text{-averaged}}$

Recall:

*** Proposition 4.32.** (composition of averaged operators) ★★★

[ $D$ : nonempty subset of $\mathcal{H}$

$m$ : strictly positive integers

$I = \{1,\dots,m\}$

$(\alpha_i)_{i\in I}$ : real numbers, $\alpha_i \in ]0,1[$

$(T_i)_{i\in I}$ : (family of operators, : $D\to D$, $\forall_{i\in I}\ T_i$ : $\alpha_i$-averaged) ]

$\Big( T = T_1 \cdots T_m,\ \alpha = \dfrac{m}{m - 1 + \dfrac{1}{\max_{i\in I}\alpha_i}} \Big) \Rightarrow T$ : $\alpha$-averaged

So: imposing our condition for this case we have:

*** Proposition 4.32.** (composition of averaged operators) ★★★

[ $D$ : nonempty subset of $\mathcal{H}$

$\mathcal{H}$

$m$ : strictly positive integers

* Proposition 4.32. (Composition of averaged operators) ★★★

[ $\cancel{D}\ \mathcal{H}$: nonempty subset of $\mathcal{H}$

$\cancel{m}\ m_k$: strictly positive integers

$\cancel{I}\ I_k = \{1,\dots,\cancel{m}\ m_k\}$

$(\alpha_i)_{i\in I}$: real numbers, $\alpha_i \in\, ]0,1[$

$(T_i)_{i\in \cancel{I}\, I_k}$: (family of operators, $\cancel{D}\to\cancel{D}$ $(\mathcal{H}\to\mathcal{H})$, $\forall_{\cancel{i\in I}\ i\in I_k}$ $T_i$: $\alpha_i$-averaged) ]

$$\left(\cancel{T}\ R_k = T_1\cdots \cancel{T_m}\ T_{m_k},\quad \cancel{\alpha}\ \rho_k = \underbrace{\frac{\cancel{m}\ m_k}{\cancel{m}\ m_k - 1 + \dfrac{1}{\max_{\cancel{i\in I}\ i\in I_k} \alpha_i}}}_{\text{by given, and also } \rho_k\in]0,1[}\right) \Rightarrow \cancel{T}\ R_k : \cancel{\alpha}\ \rho_k\text{-averaged}$$

∴ $R_k$: $\rho_k$-averaged

Now recall corollary 4.37: Corollary 4.37.

averaged operators, $:D\to D$, $\bigcap_{i\in I}\operatorname{Fix}T_i \neq \emptyset$

$$\operatorname{Fix}\left(\prod_{i=1}^{m} T_i\right) = \bigcap_{i=1}^{m} (\operatorname{Fix} T_i)$$

using this:

$$\operatorname{Fix}(R_k) = \operatorname{Fix}\left(\prod_{j\in I_k} T_j\right) = \bigcap_{j\in I_k} (\operatorname{Fix}T_j)$$

[∵ $\bigcap_{i\in I}\operatorname{Fix}T_i \neq \emptyset$ (given)

∴ $\bigcap_{i\in I_k}\operatorname{Fix}T_i\neq\emptyset$ as $I_k\subseteq I$ ]

So, we have $R_k$: $\rho_k$-averaged, $\operatorname{Fix}(R_k) = \bigcap_{i\in I_k}(\operatorname{Fix}T_i)$

Now, $T = \sum_{k=1}^{p}\omega_k\left(\prod_{j\in I_k}T_j\right) = \sum_{k=1}^{p}\omega_k R_k$

given: $(\underbrace{\omega_k}_{>0})_{k=1}^{p}$ : $\sum_{k=1}^{p}\omega_k = 1$

now we plug in Proposition 4.30 regarding addition of averaged operator

* Proposition 4.30. (Addition of averaged operators) ★

[ D: nonempty subset of $\mathcal{H}$

$(\underbrace{T_i}_{:D\to\mathcal{H}})_{i\in I}$ : ( • finite family of nonexpansive operators

• $\forall_{i\in I}$ $T_i$ : $\underbrace{\alpha_i}_{\in]0,1]}$-averaged )

$(\underbrace{\omega_i}_{\in]0,1]})_{i\in I}$ : $\sum_{i\in I}\omega_i = 1$ ] $\Rightarrow$ $\sum_{i\in I}\omega_i T_i$ : $(\max_{i\in I}\alpha_i)$ :averaged.

* Proposition 4.30. (Addition of averaged operators) ★

[ $\cancel{D}\ \mathcal{H}$: nonempty subset of $\mathcal{H}$

$(\underbrace{\cancel{T_i}\ R_k}_{:\cancel{D\to\mathcal{H}}\ \mathcal{H}\to\mathcal{H}})_{\cancel{i\in I}\ k\in\{1,\dots,p\}}$ : ( • finite family of nonexpansive operators

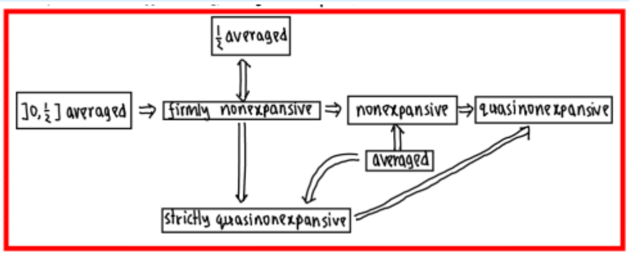
• $\forall_{\cancel{i\in I}\ k\in\{1,\dots,p\}}$ $\cancel{T_i}$ : $\underbrace{\cancel{\alpha_i}}_{\in]0,1]}$-averaged ) ($R_k$ : $\rho_k$-averaged)

$(\underbrace{\omega_i}_{\in]0,1]})_{\cancel{i\in I}\ k\in\{1,\dots,p\}}$ : $\sum_{\cancel{i\in I}\ k=1}^{p}\cancel{\omega_i}\ \omega_k = 1$ ] $\Rightarrow$ $\underbrace{T = \sum_{\cancel{i\in I}\ k=1}^{p}\cancel{\omega_i T_i}\ \omega_k R_k : (\max_{\cancel{i\in I}\ 1\le k\le p}\cancel{\alpha_i}\ \rho_k) \text{ :averaged.}}$

∴ $T$: $(\underbrace{\max_{1\le k\le p}\rho_k}_{\alpha})$ : averaged

$\Leftrightarrow$ $T$: $\alpha$-averaged

Now let us work on goal: (ii): show $\operatorname{Fix}T = \bigcap_{i\in I}\operatorname{Fix}T_i \neq \emptyset$   first look at the diagram:




} $\underbrace{R_k : \rho_k\text{-averaged}}$

⇓

nonexpansive

⇓

quasinonexpansive

as $\tilde{i}$ onto $\bigcap_{k=1}^{p}\bigcap_{i\in I_k} \equiv \bigcap_{i\in I}$

now recall proposition 4.34. and plug it in for our case

$\Downarrow$ $\Downarrow$

$\boxed{\text{strictly quasinonexpansive}}$

now recall proposition 4.34. and plug it in for our case

**Proposition 4.34.**

[ $\cancel{D}$ $\mathcal{H}$: nonempty subset of $\mathcal{H}$

$R_k$: $\rho_k$-averaged

$\Downarrow$

$R_k \cancel{(T_i)_{i\in I}}_{k\in\{1,\dots,p\}}$: finite family of quasinonexpansive operators, : $\cancel{D}\to\mathcal{H}$, $\bigcap_{k\in\{1,\dots,p\}} \operatorname{Fix} \cancel{T_i} R_k \neq \emptyset$

$\cancel{(w_i)_{i\in I}} \, (w_k)_{k\in\{1,\dots,p\}}$: strictly positive real numbers, $\sum_{k\in\{1,\dots,p\}} w_k = 1$ ] : given

$\Rightarrow \operatorname{Fix} \sum_{k\in\{1,\dots,p\}} w_k R_k = \bigcap_{k\in\{1,\dots,p\}} \operatorname{Fix} R_k \neq \emptyset$

proof: $\bigcap_{k=1}^{p} \operatorname{Fix} R_k = \underbrace{\bigcap_{k=1}^{p} \bigcap_{i\in I_k}}_{\bigcap_{i\in I}} \operatorname{Fix} T_i = \bigcap_{i\in I} \operatorname{Fix} T_i \neq \emptyset$ [given]

as $\tilde{i}$ onto $\bigcap_{k=1}^{p} \bigcap_{i\in I_k} \equiv \bigcap_{i\in I}$

and as $\tilde{\imath}$ : onto

$\bigcap_{i\in I_k} \operatorname{Fix} T_i$

$\therefore \operatorname{Fix} T = \operatorname{Fix}\left(\sum_{k=1}^{p} w_k R_k\right) = \bigcap_{k=1}^{p} \operatorname{Fix} R_k = \bigcap_{i\in I} \operatorname{Fix} T_i \neq \emptyset$ reached

Now we apply proposition 5.15:

*** Proposition 5.15. (Variant of KM algorithm for $\alpha$-averaged operator)**

[ $\alpha \in ]0,1[$ ✓

$T: \mathcal{H}\to\mathcal{H}$, $\alpha$-averaged operator, $\operatorname{Fix} T \neq \emptyset$ ✓

$(\lambda_n)_{n\in\mathbb{N}}$ : $\subseteq [0, \frac{1}{\alpha}]$, $\sum_{n\in\mathbb{N}} \lambda_n(1-\alpha\lambda_n) = +\infty$ ✓

$x_0 \in \mathcal{H}$ ✓

$\forall_{n\in\mathbb{N}}$ $x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n$ ✓ ] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$ : Fejer monotone w.r.t $\operatorname{Fix} T$ ✓

(ii) $(Tx_n - x_n)_n$ : converges strongly to 0 ✓

(iii) $(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in $\operatorname{Fix} T$. ✓

$\operatorname{Fix} T = \bigcap_{i\in I} \operatorname{Fix} T_i$

$\therefore (x_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\bigcap_{i\in I} \operatorname{Fix} T_i$ ∎

# 5.3 Iterating Compositions of Averaged Operators

7:22 PM

**Theorem 5.22.**

[$D$: nonempty weakly sequentially closed subset of $\mathcal{H}$ /* weakly sequentially closed set: weak limit of every weakly convergent sequence in the set is also in the set.

$(T_i)_{\underbrace{i\in I}_{=\{1,2,\dots,m\}}}$: family of $\underbrace{\alpha_i}_{\in ]0,1[}$-averaged operators, $: D\to D$, $\mathrm{Fix}(T_1\cdots T_m)\neq\emptyset$

$x_0\in D$

/* note that this is classic iteration, not the KM iteration */

$\forall_{n\in\mathbb{N}}\ x_{n+1}=(T_1\cdots T_m)x_n$ ] $\Rightarrow$

- $x_n-(T_1\cdots T_m)x_n\to 0$ (5.20)

- $\begin{pmatrix} \exists\, y_1\in \mathrm{Fix}\, T_1T_2\cdots T_{m-1}T_m \\ \exists\, y_2\in \mathrm{Fix}\, T_2T_3\cdots T_mT_1 \\ \vdots \\ \exists\, y_m\in\mathrm{Fix}\, T_mT_1\cdots T_{m-2}T_{m-1}\end{pmatrix}$:

$x_n\rightharpoonup y_1=T_1y_2$ (5.21)

$T_m x_n\rightharpoonup y_m=T_my_1$ (5.22)

$T_{m-1}T_m x_n\rightharpoonup y_{m-1}=T_{m-1}y_m$ (5.23)

$\vdots$

$T_3\cdots T_m x_n\rightharpoonup y_3=T_3y_4$ (5.24)

$T_2T_3\cdots T_m x_n\rightharpoonup y_2=T_2y_3$ (5.25)

**Proof:**

Let $T=T_1\cdots T_m$, then the iteration becomes $x_{n+1}=Tx_n$

$\beta_i=(1-\alpha_i)/\alpha_i$

$y\in\mathrm{Fix}\,T$

then $\|x_{n+1}-y\|^2=\|Tx_n-Ty\|^2=\|T_1\overbrace{(T_2\cdots T_m)x_n}^{\alpha}-T_1\overbrace{(T_2\cdots T_m)y}^{\beta}\|^2$

$=\|T_1\alpha-T_1\beta\|^2\leq\|\alpha-\beta\|^2-\underbrace{\frac{1-\alpha_1}{\alpha_1}}_{\beta_1}\|(\mathrm{Id}-T_1)\alpha-(\mathrm{Id}-T)\beta\|^2$

$=\|(T_2T_3\cdots T_m)x_n-(T_2T_3\cdots T_m)y\|^2-$

$\beta_1\|(\mathrm{Id}-T_1)(T_2\cdots T_m)x_n-(\mathrm{Id}-T_1)(T_2\cdots T_m)y\|^2$

now $\|(T_2T_3\cdots T_m)x_n-(T_2T_3\cdots T_m)y\|^2$

$=\|T_2(T_3\cdots T_m)x_n-T_2(T_3\cdots T_m)y\|^2$

$\leq\|(T_3\cdots T_m)x_n-(T_3\cdots T_m)y\|^2-$

$\beta_2\|(\mathrm{Id}-T_2)(T_3\cdots T_m)x_n-(\mathrm{Id}-T_2)(T_3\cdots T_m)y\|^2$

(using Proposition 4.25 again)

> **Proposition 4.25.** (Different faces of an $\alpha$-averaged operator) ★★★
>
> [ $D$: nonempty subset of $\mathcal{H}$
>
> $T: D\to\mathcal{H}$, nonexpansive
>
> $\alpha\in]0,1[$ ]
>
> (i) $T$: $\alpha$-averaged $\Leftrightarrow$
>
> (ii) $\left(1-\frac{1}{\alpha}\right)\mathrm{Id}+\left(\frac{1}{\alpha}\right)T$: nonexpansive $\Leftrightarrow$
>
> (iii) $\forall_{x\in D}\ \forall_{y\in D}\ \|Tx-Ty\|^2\leq\|x-y\|^2-\frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2\Leftrightarrow$
>
> (iv) $\forall_{x\in D}\ \forall_{y\in D}\ \|Tx-Ty\|^2+(1-2\alpha)\|x-y\|^2\leq 2(1-\alpha)\langle x-y\mid Tx-Ty\rangle$

then,

$\|x_{n+1}-y\|^2\leq\|T_3\cdots T_mx_n-T_3\cdots T_my\|^2$

$-\beta_2\|(\mathrm{Id}-T_2)(T_3\cdots T_m)x_n-(\mathrm{Id}-T_2)(T_3\cdots T_m)y\|^2$

$-\beta_1\|(\mathrm{Id}-T_1)(T_2\cdots T_m)x_n-(\mathrm{Id}-T_1)(T_2\cdots T_m)y\|^2$

$=\|T_3\cdots T_mx_n-T_3\cdots T_my\|^2-\sum_{i=2}^{1}\beta_i\|(\mathrm{Id}-T_i)\left(\prod_{j=i+1}^{m}T_j\right)x_n-(\mathrm{Id}-T_i)\left(\prod_{j=i+1}^{m}T_j\right)y\|^2$

$\vdots$

/* similar logic */

$\leq\|x_n-y\|^2-\beta_m\|(\mathrm{Id}-T_m)x_n-(\mathrm{Id}-T_m)y\|^2$

$-\beta_{m-1}\|(\mathrm{Id}-T_{m-1})T_mx_n-(\mathrm{Id}-T_{m-1})T_my\|^2$

$\vdots$

$-\beta_2\|(\mathrm{Id}-T_2)(T_3\cdots T_m)x_n-(\mathrm{Id}-T_2)(T_3\cdots T_m)y\|^2$

$-\beta_1\|(\mathrm{Id}-T_1)(T_2\cdots T_m)x_n-(\mathrm{Id}-T_1)(T_2\cdots T_m)y\|^2$ ... (eq:0)

$=\|x_n-y\|^2-\underbrace{\sum_{i=m}^{1}\beta_i\|(\mathrm{Id}-T_i)\left(\prod_{j=i+1}^{m}T_j\right)x_n-(\mathrm{Id}-T_i)\left(\prod_{j=i+1}^{m}T_j\right)y\|^2}_{\text{negative number, so removing it will make the rest only larger}}$ (eq:1)

$\leq\|x_n-y\|^2$

$\therefore\ \forall_{y\in\mathrm{Fix}\,T}\ \|x_{n+1}-y\|^2\leq\|x_n-y\|^2\Leftrightarrow(x_n)_{n\in\mathbb{N}}$: Fejer monotone w.r.t $\mathrm{Fix}\,T$

$\Rightarrow\forall_{y\in\mathrm{Fix}\,T}\ (\|x_n-y\|)_{n\in\mathbb{N}}$: converges

> ★ **Proposition 5.4.** (Some key properties of Fejer monotone sequences)
>
> [ $C$: nonempty subset of $\mathcal{H}$
>
> $(x_n)_{n\in\mathbb{N}}$: Fejer monotone sequence w.r.t $C$, $\subseteq\mathcal{H}$ ] $\Rightarrow$
>
> (i) $(x_n)_{n\in\mathbb{N}}$: bounded
>
> (ii) $\forall_{x\in C}\ (\|x_n-x\|)_{n\in\mathbb{N}}$: converges

now $\forall_{y\in\mathrm{Fix}\,T}\ \sum_{i=m}^{1}\beta_i\|(\mathrm{Id}-T_i)\left(\prod_{j=i+1}^{m}T_j\right)x_n-(\mathrm{Id}-T_i)\left(\prod_{j=i+1}^{m}T_j\right)y\|^2\leq$

$(x_n)_{n\in\mathbb{N}}$ : Fejer monotone sequence w.r.t $C$, $\subseteq \mathcal{H}$ ] $\Rightarrow$

(i) $(x_n)_{n\in\mathbb{N}}$ : bounded

(ii) $\forall_{z\in C} \left(\|x_n - z\|\right)_{n\in\mathbb{N}}$ : converges

(iii) $(d_C(x_n))_{n\in\mathbb{N}}$ : decreasing and converges */

now $\forall_{y\in \operatorname{Fix} T}$ $\displaystyle\sum_{i=m}^{1} \beta_i \left\| (Id - T_i)\left(\prod_{j=i+1}^{m} T_j\right) x_n - (Id - T_i)\left(\prod_{j=i+1}^{m} T_j\right) y \right\|^2 \leqslant$

$\|x_n - y\|^2 - \|x_{n+1} - y\|^2$

(eq.:0) $\Rightarrow$ // removing negative terms associated with

// $\beta_{m-1}, \ldots, \beta_1$

$$\|x_{n+1} - y\|^2 \leqslant \|x_n - y\|^2 - \beta_m \|(Id - T_m)x_n - (Id - T_m)y\|^2$$

$$\Leftrightarrow 0 \leqslant \beta_m \|(Id - T_m)x_n - (Id - T_m)y\|^2 \leqslant \|x_n - y\|^2 - \|x_{n+1} - y\|^2$$

/* now Fact 13.1: $a_n \leqslant b_n \Rightarrow \underline{\lim}\, a_n \leqslant \underline{\lim}\, b_n$, $\overline{\lim}\, a_n \leqslant \overline{\lim}\, b_n$ */

$$0 \leqslant \underline{\lim}\, \beta_m \|(Id - T_m)x_n - (Id - T_m)y\|^2 \leqslant \underline{\lim}\left(\|x_n - y\|^2 - \|x_{n+1} - y\|^2\right) = \lim\left(\|x_n - y\|^2 - \|x_{n+1} - y\|^2\right) = 0$$

$$0 \leqslant \overline{\lim}\, \beta_m \|(Id - T_m)x_n - (Id - T_m)y\|^2 \leqslant \overline{\lim}\left(\|x_n - y\|^2 - \|x_{n+1} - y\|^2\right) = \lim\left(\|x_n - y\|^2 - \|x_{n+1} - y\|^2\right) = 0$$

$$\underline{\lim}\, \beta_m \|(Id - T_m)x_n - (Id - T_m)y\|^2 = \overline{\lim}\, \beta_m \|(Id - T_m)x_n - (Id - T_m)y\|^2 = \lim \underbrace{\beta_m}_{>0} \|(Id - T_m)x_n - (Id - T_m)y\|^2 = 0$$

$\therefore \|(Id - T_m)x_n - (Id - T_m)y\|^2 \to 0 \overset{eq}{\Leftrightarrow} (Id - T_m)x_n - (Id - T_m)y = x_n - y - T_m x_n + T_m y \to 0$ (5.27)

similarly, $(Id - T_{m-1})T_m x_n - (Id - T_{m-1})T_m y = T_m x_n - T_m y - T_{m-1}T_m x_n + T_{m-1}T_m y \to 0$ (5.28)

$\vdots$

$(Id - T_2)T_3 \cdots T_m x_n - (Id - T_2)T_3 \cdots T_m y = T_3 \cdots T_m x_n - T_3 \cdots T_m y - T_2 \cdots T_m x_n + T_2 \cdots T_m y \to 0$ (5.29)

$(Id - T_1)T_2 \cdots T_m x_n - (Id - T_1)T_2 \cdots T_m y = T_2 \cdots T_m x_n - T_2 \cdots T_m y - \underbrace{T_1 \cdots T_m}_{T} x_n + \underbrace{T_1 \cdots T_m}_{T} y \to 0$ (5.30)

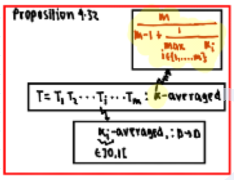
adding

$$x_n - y + Tx_n - \underbrace{Ty}_{y \,//\, y\in \operatorname{Fix} T} = x_n - Tx_n \to 0$$

$\therefore x_n - Tx_n \to 0$ : asympotic regularity property holds.

recall that $T = T_1 \cdots T_m$ : $\dfrac{m}{m-1+\dfrac{1}{\max\limits_{i=1,\ldots,m} \alpha_i}}$ averaged

$\Rightarrow$ T: nonexpansive




Now we apply Theorem 5.13:

/* Theorem 5.13. (necessity of asympotic regularity for classic nonexpansive iteration to work)

[ D: nonempty closed convex subset of $\mathcal{H}$ ✓ /* Showing this property is the key */

$T: D \to D$, nonexpansive operator. Fix $T \neq \emptyset$, $x_n - Tx_n \to 0$ ✓

$x_0 \in D$, $\forall_{n\in\mathbb{N}}$ $x_{n+1} = Tx_n$ ] $\Rightarrow$ (given)

(i) $(x_n)_{n\in\mathbb{N}}$ : converges weakly to a point in Fix $T$ */

so, $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in Fix $T$. Suppose the point is $y_1$.

so, $x_n \rightharpoonup y_1$, $y_1 \in \operatorname{Fix} T = T_1 \cdots T_m$ (goal 5.21 reached)

from 5.27.

$x_n - y + T_m x_n + T_m y \to 0 \Rightarrow x_n - T_m x_n \to y - T_m y \Leftrightarrow T_m x_n - x_n \to T_m y - y$ $\forall y \in \operatorname{Fix} T$

$\xRightarrow{\text{set } y := y_1 \in \operatorname{Fix} T}$ $T_m x_n - x_n \to T_m y_1 - y_1$

$\Rightarrow$ $T_m x_n - x_n \rightharpoonup T_m y_1 - y_1$

$\therefore T_m x_n \rightharpoonup T_m y_1$ /* now $y_1 \in \operatorname{Fix} T \Leftrightarrow Ty_1 = T_1 \cdots T_m y_1 = y_1$ //say

$\Rightarrow T_m (T_1 \cdots T_{m-1}) T_m y_1 = T_m y_1 \Leftrightarrow \underbrace{T_m y_1}_{} = y_m \in \operatorname{Fix} T_m T_1 \cdots T_{m-1}$ */

$\therefore T_m x_n \rightharpoonup T_m y_1 = y_m$, $y_m \in \operatorname{Fix} T_m T_1 \cdots T_{m-1}$ : goal (5.22) proved

from (5.28)

$(Id - T_{m-1})T_m x_n - (Id - T_{m-1})T_m y = T_m x_n - T_m y - T_{m-1}T_m x_n + T_{m-1}T_m y \to 0$

From (5.28)

$$(Id-T_{m-1})T_m x_n - (Id-T_{m-1})T_m y = T_m x_n - T_m y - T_{m-1}T_m x_n + T_{m-1}T_m y \to 0$$

$$\Rightarrow \quad T_m x_n - T_{m-1}T_m x_n \to T_m y - T_{m-1}T_m y \qquad \forall y \in \operatorname{Fix} T$$

set $y := y_1 \in \operatorname{Fix} T$

$$\Rightarrow \quad T_m x_n - T_{m-1}T_m x_n \to T_m y_1 - T_{m-1}T_m y_1$$

$$\Leftrightarrow \quad T_{m-1}T_m x_n - T_m x_n \to T_{m-1}T_m y_1 - T_m y_1$$

$$\Rightarrow \quad T_{m-1}T_m x_n - T_m x_n \to T_{m-1}T_m y_1 - T_m y_1$$

$$T_{m-1}T_m x_n \to T_{m-1}\underbrace{T_m y_1}_{y_m :\ y_m \in \operatorname{Fix} T_m T_1 \cdots T_{m-1}} = T_{m-1} y_m$$

// now $T_m T_1 \cdots \underbrace{T_{m-1} y_m}_{y_{m-1}\ \text{// say}} = y_m \Rightarrow T_m T_1 \cdots T_{m-2}\, y_{m-1} = y_m \Rightarrow (T_{m-1}T_m T_1 \cdots T_{m-2})\, y_{m-1} = T_{m-1} y_m = y_{m-1}$

$\therefore y_{m-1} = T_{m-1} y_m \in \operatorname{Fix} T_{m-1}T_m T_1 \cdots T_{m-2}$ //

$\therefore T_{m-1}T_m x_n \to y_{m-1} = T_{m-1} y_m,\ y_{m-1} \in \operatorname{Fix} T_{m-1}T_m T_1 \cdots T_{m-2}$ (goal (5.23) proved)

Proceeding like this we can construct rest of 5.21-5.25. ∎

* Corollary 5.23 (POCS algorithm)

[ $(C_i)_{i\in I}$ : family of nonempty closed convex subsets of $\mathcal{H}$

$(P_i)_{i\in I}$ : projectors on $(C_i)_{i\in I}$, $\operatorname{Fix}(P_1 P_2 \cdots P_m) \neq \emptyset$

$\quad \hookrightarrow |I| = m$

$x_0 \in \mathcal{H}$

$\forall_{n\in\mathbb{N}}\ x_{n+1} = P_1 P_2 \cdots P_m x_n$ ] $\Rightarrow$

$$\exists_{(y_1,\dots,y_m)\in C_1\times\dots\times C_m} \begin{cases} x_n \rightharpoonup y_1 = P_1 y_2 \\ P_m x_n \rightharpoonup y_m = P_m y_1 \\ P_{m-1}P_m x_n \rightharpoonup y_{m-1} = P_{m-1} y_m \\ \quad \vdots \qquad\qquad \vdots \\ P_3 \cdots P_m x_n \rightharpoonup y_3 = P_3 y_4 \\ P_2 \cdots P_m x_n \rightharpoonup y_2 = P_2 y_3 \end{cases}$$