Chapter 4: Part 1

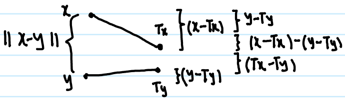
9:13 AM

4.1 Nonexpansive operators.

Definition 4.1. (Different types of nonexpansiveness)

[D: nonempty subset of $\mathcal{H}$

$T: D \to \mathcal{H}$]



- T: firmly nonexpansive $\overset{\text{def}}{\Leftrightarrow} \forall_{x\in D} \forall_{y\in D} \quad \|Tx-Ty\|^2+\|(I-T)x-(I-T)y\|^2 \leq \|x-y\|^2$
- T: nonexpansive $\overset{\text{def}}{\Leftrightarrow} \forall_{x\in D} \forall_{y\in D} \quad \|Tx-Ty\| \leq \|x-y\|$
- T: quasinonexpansive $\overset{\text{def}}{\Leftrightarrow} \forall_{x\in D} \forall_{y\in \operatorname{Fix} T} \quad \|Tx-Ty\| \leq \|x-y\|$
- T: strictly quasinonexpansive $\overset{\text{def}}{\Leftrightarrow} \forall_{x\in D\setminus \operatorname{Fix} T} \forall_{y\in \operatorname{Fix} T} \quad \|Tx-y\| < \|x-y\|$

$$
\begin{array}{ccccc}
\text{firmly nonexpansive} & \Rightarrow & \text{nonexpansive} & \Rightarrow & \text{quasiexpansive} \\
\Downarrow & & & \nearrow & \\
\text{strictly quasinonexpansive} & & & &
\end{array}
$$

- **Proposition 4.2.**

[D: nonempty subset of $\mathcal{H}$

$T: D \to \mathcal{H}$]

(i) T: firmly nonexpansive

$\Updownarrow$

(ii) $(I-T)$: firmly nonexpansive

$\Updownarrow$

(iii) $(2T-I)$: nonexpansive

$\Updownarrow$

(iv) $\forall_{x\in D} \forall_{y\in D} \quad \|Tx-Ty\|^2 \leq \langle x-y \mid Tx-Ty\rangle$

$\Updownarrow$

(v) $\forall_{x\in D} \forall_{y\in D} \quad 0 \leq \langle Tx-Ty \mid (I-T)x-(I-T)y\rangle$

$\Updownarrow$

(vi) $\forall_{x\in D} \forall_{y\in D} \forall_{\alpha\in[0,1]} \quad \|Tx-Ty\| \leq \|\alpha(x-y)+(1-\alpha)(Tx-Ty)\|$

Proof: ((i) $\Leftrightarrow$ (ii)):

T: firmly nonexpansive

$\Leftrightarrow \forall_{x\in D} \forall_{y\in D} \quad \|\underbrace{Tx-Ty}_{(I-(I-T))x-(I-(I-T))y}\|^2+\|(I-T)x-(I-T)y\|^2 \leq \|x-y\|^2$

$\Leftrightarrow \forall_{x\in D} \forall_{y\in D} \quad \|(I-T)x-(I-T)y\|^2+\|(I-(I-T))x-(I-(I-T))y\|^2 \leq \|x-y\|^2$

$\Leftrightarrow (I-T)$: firmly nonexpansive. $\therefore$ (i) $\Leftrightarrow$ (ii)

(i) $\Leftrightarrow$ (iii)

$\forall x,y \in D$

$R = 2T-I$

$\|Rx-Ry\|^2 = \|(2T-I)x-(2T-I)y\|^2 = \|2Tx-x-2Ty+y\|^2$

$R = 2T - I$

$\|Rx - Ry\|^2 = \|(2T-I)x - (2T-I)y\|^2 = \|2Tx - x - 2Ty + y\|^2$

$= \|2(Tx - Ty) - (x-y)\|^2 = \|\underbrace{2}_{\alpha}(Tx-Ty) + \underbrace{(1-2)}_{\alpha}(x-y)\|^2$

/* Corollary 2.14.

$\forall_{x,y \in \mathcal{H}} \ \forall_{\alpha \in \mathbb{R}} \ \|\alpha x + (1-\alpha) y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ */

$= 2\|Tx - Ty\|^2 + (1-2)\|x-y\|^2 - 2(1-2)\underbrace{\|Tx - Ty - x + y\|^2}$

$= \|(I-T)y - (I-T)x\|^2 = \|(I-T)x - (I-T)y\|^2$

$= 2\|Tx - Ty\|^2 - \|x-y\|^2 + 2\|(I-T)x - (I-T)y\|^2$

$\rightarrow \|Rx - Ry\|^2 - \|x-y\|^2 = 2\left(\|Tx-Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x-y\|^2\right)$

// note that this is an identity

T: firmly nonexpansive

$\Leftrightarrow \quad \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2$

$\Leftrightarrow \quad \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x-y\|^2 \leq 0$

$\Leftrightarrow$

$2\left(\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x-y\|^2\right) \leq 0$

$\Leftrightarrow \quad \|\underbrace{R}_{2T-I}x - Ry\|^2 - \|x-y\|^2 \leq 0$

$\Leftrightarrow \quad \|(2T-I)x - (2T-I)y\|^2 \leq \|x-y\|^2$

$\Leftrightarrow \quad \|(2T-I)x - (2T-I)y\| \leq \|x-y\|$

$\Leftrightarrow$ $2T-I$: nonexpansive $\quad \therefore$ (i) $\Leftrightarrow$ (iii)

(i) $\Leftrightarrow$ (iv):

T: firmly nonexpansive

$\Leftrightarrow$

$\forall_{x \in D} \ \forall_{y \in D} \quad \|Tx-Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2$

/* $\|(I-T)x - (I-T)y\|^2 = \|x - Tx - y + Ty\|^2 = \|(x-y) - (Tx-Ty)\|^2$

$= \|x-y\|^2 + \|Tx-Ty\|^2 - 2\langle x-y \,|\, Tx-Ty\rangle$ */

$\Leftrightarrow \forall_{x \in D} \ \forall_{y \in D} \quad \|Tx-Ty\|^2 + \cancel{\|x-y\|^2} + \|Tx-Ty\|^2 - 2\langle x-y \,|\, Tx-Ty\rangle \leq \cancel{\|x-y\|^2}$

$\Leftrightarrow \ \forall_{x \in D} \ \forall_{y \in D} \quad 2\|Tx-Ty\|^2 \leq 2\langle x-y \,|\, Tx-Ty\rangle$

$\Leftrightarrow \ \forall_{x \in D} \ \forall_{y \in D} \quad \|Tx-Ty\|^2 \leq \langle x-y \,|\, Tx-Ty\rangle$

$\therefore$ (i) $\Leftrightarrow$ (iv)

• (iv) $\Leftrightarrow$ (v)

(iv):

$\forall_{x \in D} \ \forall_{y \in D} \quad \|Tx-Ty\|^2 = \langle Tx-Ty \,|\, Tx-Ty\rangle \leq \overbrace{\langle x-y \,|\, Tx-Ty\rangle}^{\langle Tx-Ty \,|\, x-y\rangle}$ /* $\because \langle a|b\rangle = \langle b|a\rangle$ */

$\Leftrightarrow \forall_{x\in D} \forall_{y\in D} \quad \langle Tx-Ty \mid x-y\rangle - \langle Tx-Ty \mid Tx-Ty\rangle = \langle Tx-Ty \mid \underbrace{x-y-Tx-Ty}_{(I-T)x-(I-T)y} \rangle \geq 0$ /* $\langle \alpha a+b\beta \mid c\rangle = \alpha\langle a\mid c\rangle + \beta\langle b\mid c\rangle$ */

$\Leftrightarrow \forall_{x\in D} \forall_{y\in D} \quad \langle Tx-Ty \mid (I-T)x-(I-T)y\rangle \geq 0$

$\therefore$ (iv) $\Leftrightarrow$ (v)

(v) $\Leftrightarrow$ (vi) :

/* Lemma 2.12.

(i) $\forall_{x,y\in\mathcal{H}} \quad \langle x\mid y\rangle \leq 0 \Leftrightarrow \forall_{\alpha\in\mathbb{R}_+} \ \|x\| \leq \|x-\alpha y\| \Leftrightarrow \forall_{\alpha\in[0,1]} \ \|x\| \leq \|x-\alpha y\|$

(ii) $\forall_{x,y\in\mathcal{H}} \quad x \perp y \Leftrightarrow \forall_{\alpha\in\mathbb{R}} \ \|x\| \leq \|x-\alpha y\| \Leftrightarrow \forall_{\alpha\in[-1,1]} \ \|x\| \leq \|x-\alpha y\|$

*/

(v):

$\forall_{x\in D} \forall_{y\in D} \quad \langle Tx-Ty \mid (I-T)x-(I-T)y\rangle \geq 0$

$\Leftrightarrow - \langle Tx-Ty \mid (I-T)x-(I-T)y\rangle \leq 0$

$\Leftrightarrow \langle -(Tx-Ty) \mid (I-T)x-(I-T)y\rangle \leq 0$

$\Leftrightarrow \forall_{\alpha\in[0,1]} \quad \|-(Tx-Ty)\| = \|Tx-Ty\| \leq \|(-Tx+Ty) - \alpha((I-T)x-(I-T)y)\|$

$= \|-Tx+Ty-\alpha(x-Tx-y+Ty)\|$

$= \|-Tx+Ty-\alpha x+\alpha Tx+\alpha y-\alpha Ty\|$

$= \|-\alpha(x-y)-(1-\alpha)Tx+(1-\alpha)Ty\|$

$= \|-(\alpha(x-y)+\underbrace{(1-\alpha)Tx-(1-\alpha)Ty}_{(1-\alpha)(Tx-Ty)})\|$

$= \|\alpha(x-y)+(1-\alpha)(Tx-Ty)\|$

$\Leftrightarrow \forall_{x\in D} \forall_{y\in D} \forall_{\alpha\in[0,1]} \quad \|Tx-Ty\| \leq \|\alpha(x-y)+(1-\alpha)(Tx-Ty)\|$: (vi)

$\therefore$ (v) $\Leftrightarrow$ (vi)

* Corollary 4.3.

[$T\in\mathcal{B}(\mathcal{H})$]

(i) T: firmly nonexpansive $\Leftrightarrow$

(ii) $\|2T-I\|\leq 1 \Leftrightarrow$

(iii) $\forall_{x\in\mathcal{H}} \quad \|Tx\|^2 \leq \langle x\mid Tx\rangle \Leftrightarrow$

(iv) T^*: firmly nonexpansive $\Leftrightarrow$

(v) $T+T^*-2T^*T$: positive

/* $\forall_{x\in\mathcal{H}} \quad \langle x \mid (T+T^*-2T^*T)x\rangle \geq 0$ */

* Definition: 4.4. (β-cocoercive / β-inverse strongly monotone)

⟦ D : nonempty subset of $\mathcal{H}$
$T : D \to \mathcal{H}$, $\beta \in \mathbb{R}_{++}$ ⟧

$T : \beta$-cocoercive $\overset{\text{def}}{\leftrightarrow}$ βT : firmly nonexpansive
(β-inverse strongly monotone)

$$\overset{\text{def}}{\leftrightarrow} \forall_{x \in D} \forall_{y \in D} \ \langle x-y \mid Tx-Ty \rangle \geqslant \beta \|Tx-Ty\|^2$$

/* Recall from Proposition 4.2 (i), (iv): T: firmly nonexpansive $\Leftrightarrow \forall_{x \in D} \forall_{y \in D} \ \langle x-y \mid Tx-Ty \rangle \geqslant \|Tx-Ty\|^2$

$\therefore$ βT: firmly nonexpansive $\Leftrightarrow \forall_{x \in D} \forall_{y \in D} \ \underbrace{\langle x-y \mid \beta Tx-\beta Ty \rangle}_{\langle x-y \mid \beta(Tx-Ty) \rangle} \geqslant \underbrace{\| \beta Tx-\beta Ty \|^2}_{\| \beta(Tx-Ty)\|^2 = \beta^2 \|Tx-Ty\|^2}$

$= \underbrace{\beta}_{\in \mathbb{R}_{++}} \langle x-y \mid Tx-Ty \rangle$ [$\because \langle a \mid \alpha b \rangle = \alpha \langle a \mid b \rangle$]

$\Leftrightarrow \beta \langle x-y \mid Tx-Ty \rangle \geqslant \beta^2 \|Tx-Ty\|^2$

$\Leftrightarrow \langle x-y \mid Tx-Ty \rangle \geqslant \beta \|Tx-Ty\|^2$: this is what is given in the definition. */

* Proposition 4.5.

⟦ $\mathcal{K}$: real Hilbert space
$\beta \in \mathbb{R}_{++}$
$T : \mathcal{K} \to \mathcal{K}$, β-cocoercive
$L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$: $L \neq 0$, $\|L\|^2 = \frac{\beta}{\gamma}$ ⟧

$L^* T L$: γ-cocercive

[Proposition 4.5 Proof] Proof: $\forall_{x,y \in \mathcal{H}}$

We want to prove:

$L^* T L$: γ-cocercive

$$\Leftrightarrow \forall_{x,y \in \mathcal{H}} \ \langle x-y \mid \underbrace{L^*TLx - L^*TLy}\rangle \geqslant \gamma \| L^*TLx - L^*TLy \|$$

now:

$$\forall_{x,y \in \mathcal{H}} \ \underbrace{\langle x-y \mid \underbrace{L^*TLx - L^*TLy}_{L^*(TLx-TLy)} \rangle}_{\langle x-y \mid L^*(TLx-TLy) \rangle}$$

[$\because L^*$: linear, continuous]

$= \langle L(x-y) \mid TLx - TLy \rangle$ [By definition of adjoint operator: $\forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} \ \langle Tx \mid y \rangle = \langle x \mid T^* y \rangle$]

$= \langle Lx - Ly \mid TLx - TLy \rangle$ [$\because$ L : linear, continuous]

$= \langle (Lx)-(Ly) \mid T(Lx)-T(Ly) \rangle$ [$\because$ T: linear, continuous]

$\because L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ $\therefore Lx, Ly \in \mathcal{K}$

$T : \mathcal{K} \to \mathcal{K}$, β-cocoercive $\overset{\text{def}}{\Leftrightarrow} \forall_{x \in \mathcal{K}} \forall_{y \in \mathcal{K}} \ \langle x-y \mid Tx-Ty \rangle \geqslant \beta \|Tx-Ty\|^2$
set: $x := Lx$, $y := Ly$

$\langle Lx-Ly \mid T(Lx)-T(Ly) \rangle \geqslant \beta \| T(Lx) - T(Ly) \|^2$

$= \gamma \|L\|^2 \|TLx - TLy\|^2$ /* $\because \beta = \gamma \|L\|^2$ */

$= \gamma (\|L\| \, \|TLx - TLy\|)^2$

$$= \gamma \|L\|^2 \|TLx - TLy\|^2 \quad /* \because \beta = \gamma\|L\|^2 */$$

$$= \gamma \big(\underbrace{\|L\|}_{\substack{\text{operator,} \\ \in \mathcal{B}(\mathcal{H},\mathcal{K})}} \underbrace{\|TLx - TLy\|}_{\in \mathcal{K}} \big)^2$$

$[\because x \in \mathcal{H},\ L \in \mathcal{B}(\mathcal{H},\mathcal{K})$
$\therefore Lx \in \mathcal{K}$
$T \in \mathcal{B}(\mathcal{K},\mathcal{K}) \therefore TLx \in \mathcal{K} \Rightarrow TLx - TLy \in \mathcal{K}]$

/* with $L \in \mathcal{B}(\mathcal{H},\mathcal{K})$ we cannot apply $\|T\|\|x\| \geq \|Tx\|$ as $TLx - TLy \in \mathcal{K}$, but if we take L^* as $\|L^*\| = \|L\|$ for linear continuous operator then it would work, as $L^* \in \mathcal{B}(\mathcal{K},\mathcal{H})$ */

$$= \gamma \big(\underbrace{\|L^*\|}_{\in \mathcal{B}(\mathcal{K},\mathcal{H})} \underbrace{\|TLx - TLy\|}_{\in \mathcal{K}} \big)^2$$

$$\geq \|L^*(\underbrace{TLx - TLy}_{L^*TLx - L^*TLx})\|$$

/* $\forall_{T \in \mathcal{B}(\mathcal{H},\mathcal{K})}$ $\|T\| = \sup_{\substack{x \in \mathcal{X} \\ \|x\| \leq 1}} \|Tx\| = \sup_{x \neq 0} \dfrac{\|Tx\|}{\|x\|}$

* An useful inequality: By definition $\forall_{x \neq 0}\ \|T\| \geq \dfrac{\|Tx\|}{\|x\|} \leftrightarrow \forall_{x \neq 0}\ \|T\|\|x\| \geq \|Tx\|$

$\leftrightarrow \boxed{\forall_x\ \|T\|\|x\| \geq \|Tx\|}$ /* as for $x=0$, equality will hold */ */

$$\geq \gamma \|L^*TLx - L^*TLy\|^2$$

$$\Leftrightarrow \forall_{x,y \in \mathcal{H}} \langle x - y \mid \underbrace{L^*TLx - L^*TLy}\rangle \geq \gamma \|L^*TLx - L^*TLy\|$$

$\therefore L^*TL : \gamma$-cocoercive. ∎

Corollary 4.6.

[[$\mathcal{K}$: real Hilbert space
$T: \mathcal{K} \to \mathcal{K}$, firmly nonexpansive
$L \in \mathcal{B}(\mathcal{H},\mathcal{K}) : \|L\| \leq 1$]] $\Rightarrow L^*TL$: firmly nonexpansive

4.2. Projectors and Convex Sets.

Proposition 4.8.

[[C: nonempty closed convex set of $\mathcal{H}$]] $\Rightarrow P_C$: firmly nonexpansive

Corollary 4.10.

[[C: nonempty closed convex set of $\mathcal{H}$]] $\Rightarrow$ • $\mathrm{I} - P_C$: firmly nonexpansive

• $2P_C - \mathrm{I}$: nonexpansive

Proof: Comes from Proposition 4.8, Proposition 4.2.

Proposition 4.11.

[[C: closed affine subspace of $\mathcal{H}$]]

$\Rightarrow$

(i) P_C : weakly continuous

(ii) $\forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}}\ \|P_C x - P_C y\|^2 = \langle x - y \mid P_C x - P_C y\rangle$

4.3. Fixed Points of Nonexpansive Operators:

Proposition 4.13.

4.3. Fixed Points of Nonexpansive Operators:

Proposition 4.13.

⟦ D: nonempty convex subset of $\mathcal{H}$

$T: D \to \mathcal{H}$, quasinonexpansive ⟧ $\Rightarrow$

Fix T : convex

Proof:

$x, y \in \text{Fix}\, T$; $\alpha \in]0,1[$, $z = \alpha x + (1-\alpha) y \in D$ // ∵ D: nonconvex

now: $\|Tz - z\|^2$

$= \| \alpha Tz - \alpha Tz - \alpha x + \alpha x + Tz - z \|^2$

$= \| \alpha Tz - \alpha Tz - \alpha x + \cancel{\alpha x} + Tz - \cancel{\alpha x} - (1-\alpha) y \|^2$

$= \| \alpha Tz - \alpha x + (1-\alpha) Tz - (1-\alpha) y \|^2$

$= \| \alpha (Tz - x) + (1-\alpha)(Tz - y) \|^2$ /* Corollary 2.14. $\forall_{\alpha \in \mathbb{R}}$ $\|\alpha x + (1-\alpha) y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ */

$= \alpha \|Tz - \underbrace{x}_{Tx}\|^2 + (1-\alpha)\|Tz - \underbrace{y}_{Ty}\|^2 - \alpha(1-\alpha)\|\cancel{Tz} - x - \cancel{Tz} + y\|^2$ /* ∵ $x, y \in \text{Fix}\, T$ */

$= \alpha \underbrace{\|Tz - Tx\|^2}_{\leq \|z - x\|^2} + (1-\alpha)\underbrace{\|Tz - Ty\|^2}_{\leq \|z - y\|^2} - \alpha(1-\alpha)\|x - y\|^2$ /* T: quasinonexpansive $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{\tilde{x} \in D}$ $\forall_{\tilde{y} \in \text{Fix}\, T}$ $\|T\tilde{x} - T\tilde{y}\| \leq \|\tilde{x} - \tilde{y}\|^2$ */

$\leq \alpha \|z - x\|^2 + (1-\alpha)\|z - y\|^2 - \alpha(1-\alpha)\|x - y\|^2$ /* now Corollary 2.14. in opposite direction */

$= \|\alpha(z - x) + (1-\alpha)(z - y)\|^2 = \|\cancel{\alpha z} - \alpha x + \underbrace{z}_{\alpha x + (1-\alpha) y} - y - \cancel{\alpha z} + \alpha y\|^2 = \|-\alpha x + \alpha x + (1-\alpha) y - (1-\alpha) y\|^2 = 0$

∴ $Tz = z \Leftrightarrow z \in \text{Fix}\, T$

∴ $\left(\forall_{x, y \in \text{Fix}\, T} \ \forall_{\alpha \in]0,1[} \ z \in \text{Fix}\, T\right) \Leftrightarrow$ Fix T : convex

■

Proposition 4.14.

⟦ D: nonempty closed subset of $\mathcal{H}$,

$T: D \to \mathcal{H}$, continuous ⟧ $\Rightarrow$

Fix T : closed.

Proof: /* A: closed $\Rightarrow$ A: sequentially closed $\Leftrightarrow \forall_{(x_n)_{n \in \mathbb{N}} \subseteq A : x_n \to x}$ $x \in A$ */

Suppose, $(x_n)_{n \in \mathbb{N}} : \subseteq \text{Fix}\, T$, $x_n \to x$ // goal: $x \in D$

$\Updownarrow$

$Tx_n = x_n \in D$

So, $(x_n)_{n \in \mathbb{N}} : \subseteq D$, $x_n \to x$ $\overset{D:\text{closed}}{\Rightarrow}$ $x \in D$

As, $T: D \to \mathcal{H}$, continuous $\overset{\text{def}}{\Leftrightarrow}$ T: continuous on every point in D

$\Rightarrow$ T: continuous at $x \in D$ $\Leftrightarrow \forall_{(x_n)_{n \in \mathbb{N}} : x_n \to x}$ $Tx_n \to \underbrace{Tx}_{x} = x \in D$ // as $x \in \text{Fix}\, T$

∴ $x \in D$: goal achieved. ■

Corollary 4.15.

⟦ D: nonempty closed convex subset of $\mathcal{H}$,

$T: D \to \mathcal{H}$, nonexpansive ⟧ $\Rightarrow$

Fix T: closed, convex

Corollary 4.16.

Fix T : closed, convex

Corollary 4.16.

[D: nonempty closed convex subset of $\mathcal{H}$,

$T: D \to \mathcal{H}$, firmly nonexpansive] $\Rightarrow$

$$\operatorname{Fix} T = \bigcap_{x \in D} \{ y \in D : \langle y - Tx \mid x - Tx \rangle \leq 0 \}$$

Proof: /* Proposition 4.2. T: firmly nonexpansive $\Leftrightarrow$ $\forall_{\tilde{x} \in D} \ \forall_{\tilde{y} \in D} \ \langle T\tilde{x} - T\tilde{y} \mid \tilde{x} - T\tilde{x} - \tilde{y} + T\tilde{y} \rangle \geq 0$

$$\Rightarrow \forall_{x \in D} \ \forall_{y \in \operatorname{Fix} T} \ \langle Tx - \underbrace{Ty}_{= y} \mid x - Tx - y + \underbrace{Ty}_{= y} \rangle = \langle Tx - y \mid x - Tx \rangle = -\langle y - Tx \mid x - Tx \rangle \geq 0$$

$$\Leftrightarrow \forall_{x \in D} \ \forall_{y \in \operatorname{Fix} T} \ \langle y - Tx \mid x - Tx \rangle \leq 0$$

$$\Leftrightarrow \forall_{y \in \operatorname{Fix} T} \ \forall_{x \in D} \ \langle y - Tx \mid x - Tx \rangle \leq 0 \quad \ldots \text{(i)}$$

$$C = \bigcap_{x \in D} \{ y \in D : \langle y - Tx \mid x - Tx \rangle \leq 0 \}$$

$$= \{ y \in D : \forall_{x \in D} \ \langle y - Tx \mid x - Tx \rangle \leq 0 \}$$

$\therefore y \in C \Leftrightarrow \forall_{x \in D} \ \langle y - Tx \mid x - Tx \rangle \leq 0 \ \ldots \text{(ii)}$

From (i),(ii): $\left(\forall_{y \in \operatorname{Fix} T} \ y \in C \right) \Leftrightarrow \operatorname{Fix} T \subseteq C$

Now let's show $C \subseteq \operatorname{Fix} T$

$x := y \in D$ // $\because C \subseteq D$: by definition

$y \in C \Leftrightarrow \forall_{x \in D} \ \langle y - Tx \mid x - Tx \rangle \leq 0 \Rightarrow \langle y - Ty \mid y - Ty \rangle = \| y - Ty \|^2 = 0 \Leftrightarrow y = Ty \Leftrightarrow y \in \operatorname{Fix} T$

$\therefore \forall_{y \in C} \ y \in \operatorname{Fix} T \Leftrightarrow C \subseteq \operatorname{Fix} T$

$\therefore C = \operatorname{Fix} T$ ■

Theorem 4.17. (demiclosedness principle)

[D: nonempty weakly sequentially closed subset of $\mathcal{H}$,

$T: D \to \mathcal{H}$, nonexpansive

$(x_n)_{n \in \mathbb{N}}$: sequence in D, $x_n \rightharpoonup \underbrace{x}_{\in \mathcal{H}}$, $x_n - Tx_n \to u$] $\Rightarrow$ $x - Tx = u$

Proof:

$(x_n)_{n \in \mathbb{N}} : \subseteq D$, $x_n \rightharpoonup \underbrace{x \in D}$

// $\because D$: weakly sequentially closed

now : $T: D \to \mathcal{H}$, and $x \in D$ $\therefore \operatorname{dom} T = D$, thus Tx: well defined

T: nonexpansive

$\| x - Tx - u \|^2$ /* now: $\| x_n - Tx - u \|^2 = \| \overbrace{x_n - x} + \overbrace{x - Tx - u} \|^2 = \| x_n - x \|^2 + \| x - Tx - u \|^2 + 2 \langle x_n - x \mid x - Tx - u \rangle$ */

$\therefore \| x - Tx - u \|^2 = \| x_n - Tx - u \|^2 - \| x_n - x \|^2 - \| x - Tx - u \|^2 - 2 \langle x_n - x \mid x - Tx - u \rangle$ */

$= \| x_n - Tx - u \|^2 - \| x_n - x \|^2 - 2 \langle x_n - x \mid x - Tx - u \rangle$

// one step at a time trick again

/* $\| x_n - Tx - u \|^2 = \| x_n - Tx_n + \overbrace{Tx_n - Tx - u} \|^2$

$= \| (x_n - Tx_n - u) + (Tx_n - Tx) \|^2$

$= \| x_n - Tx_n - u \|^2 + \| Tx_n - Tx \|^2 + 2 \langle x_n - Tx_n - u \mid Tx_n - Tx \rangle$ */

$= \| x_n - Tx_n - u \|^2 + \underbrace{\| Tx_n - Tx \|^2} + 2 \langle x_n - Tx_n - u \mid Tx_n - Tx \rangle - \| x_n - x \|^2 - 2 \langle x_n - x \mid x - Tx - u \rangle$

$= \|x_n - Tx_n - u\|^2 + \underbrace{\|Tx_n - Tx\|^2}_{\leq \|x_n - x\|^2 \text{ /* T: nonexpansive */}} + 2\langle x_n - Tx_n - u \mid Tx_n - Tx\rangle - \|x_n - x\|^2 - 2\langle x_n - x \mid x - Tx - u\rangle$

$\leq \|x_n - Tx_n - u\|^2 + \cancel{\|x_n - x\|^2} + 2\langle x_n - Tx_n - u \mid Tx_n - Tx\rangle - \cancel{\|x_n - x\|^2} - 2\langle x_n - x \mid x - Tx - u\rangle$

$= \|x_n - Tx_n - u\|^2 + 2\langle x_n - Tx_n - u \mid Tx_n - Tx\rangle - 2\langle x_n - x \mid x - Tx - u\rangle \quad \ldots (4.11)$

Now, given that: $\left.\begin{array}{l} x_n - Tx_n \rightharpoonup u \\ x_n \rightharpoonup x \end{array}\right\} \ominus \Rightarrow (x_n - Tx_n) - x_n \rightharpoonup u - x \Leftrightarrow -Tx_n \rightharpoonup u - x$

$\Leftrightarrow Tx_n \rightharpoonup x - u$

$\Leftrightarrow \boxed{Tx_n - Tx \rightharpoonup x - u - Tx} \quad \ldots (\text{eq } A)$

$\langle x_n - Tx_n - u \mid Tx_n - Tx\rangle \to 0$ /* $a_n \to 0,\ b_n \rightharpoonup 0 \Rightarrow \langle a_n \mid b_n\rangle \to 0$ */

and, $\|x_n - Tx_n - u\|^2 \to 0$

and, $\langle \underbrace{x_n - x}_{\rightharpoonup 0} \mid x - Tx - u\rangle \to 0$

$\|x_n - Tx_n - u\|^2 + 2\langle x_n - Tx_n - u \mid Tx_n - Tx\rangle - 2\langle x_n - x \mid x - Tx - u\rangle \to 0 \quad (4.12)$

From (4.11) and (4.12):

Result 1.2.1. (A very important result) # verified, uses

[$(b_n)_{n \in \mathbb{N}}$: $\subseteq \mathbb{R}$] $(0 \leq a \leq b_n,\ b_n \to 0) \Rightarrow a = 0$

as, $n \to \infty$ $\quad \|x - Tx - u\|^2 \leq 0$

$\Leftrightarrow x - Tx = u \quad \blacksquare$

Corollary 4.18.

[D: nonempty closed convex subset of $\mathcal{H}$,

$T: D \to \mathcal{H}$, nonexpansive ; $x \in \mathcal{H}$

$(x_n)_{n \in \mathbb{N}}$: sequence in D, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$] $\Rightarrow$

$x \in \operatorname{Fix} T$

Proof:

D: convex, closed $\Rightarrow$ D: weakly sequentially closed /* for convex set all concepts of closedness are equivalent */

* Theorem 4.17. (demiclosedness principle) * set $u = 0$, rest are same

[D: nonempty weakly sequentially closed subset of $\mathcal{H}$

$T: D \to \mathcal{H}$, nonexpansive

$(x_n)_{n \in \mathbb{N}}$: sequence in D, $x_n \rightharpoonup \underbrace{x}_{\in \mathcal{H}}$, $x_n - Tx_n \to u$] $\Rightarrow \underbrace{x - Tx = u}$

$x - Tx = 0$

$\Leftrightarrow Tx = x$

$\Leftrightarrow x \in \operatorname{Fix} T.$ $\blacksquare$

Theorem 4.19. (Browder-Göhde-Kirk existence theorem)

[D: nonempty bounded closed convex subset of $\mathcal{H}$,

$T: D \to D$, nonexpansive] $\Rightarrow \operatorname{Fix} T \neq \emptyset$

Proof: /* The proof tries to construct a sequence in such a way that Corollary 4.18 can be applied */

D: nonempty bounded closed convex subset of $\mathcal{H}$

$\Rightarrow$ D: weakly sequentially closed /* Theorem 3.32 : for a convex set, all 4 types of closedness (weakly sequentially closed, sequentially closed, closed, weakly closed) collapses */

$\Rightarrow$ D: weakly sequentially compact /* Theorem 3.33. A bounded closed convex subset of $\mathcal{H}$ is weakly compact and weakly sequentially compact */

$\therefore$ D: weakly sequentially closed, and weakly sequentially compact /* $((P \Rightarrow Q) \wedge (P \Rightarrow R)) \Leftrightarrow (P \Rightarrow (Q \wedge R))$

$P \wedge (P \Rightarrow T) \Leftrightarrow (P \wedge T)$ */

define

$x_0 \in D$

$(\alpha_n)_{n\in\mathbb{N}}$: sequence in $]0,1]$, $\alpha_0 = 1$, $\alpha_n \downarrow 0$

$\forall_{n\in\mathbb{N}}\ T_n : D \to D : x \mapsto \alpha_n x_0 + (1-\alpha_n)Tx$

now $T_n(\cdot) = \alpha_n x_0 + (1-\alpha_n)T(\cdot)$: contraction as

$$\|T_n x - T_n y\|^2 = \|\alpha_n x_0 + (1-\alpha_n)Tx - \alpha_n x_0 - (1-\alpha_n)Ty\|^2 = \|(1-\alpha_n)(Tx-Ty)\|^2 = (1-\alpha_n)^2 \underbrace{\|Tx-Ty\|^2}_{\leq \|x-y\|^2 \ [\because T:\text{nonexpansive}]} \leq (1-\alpha_n)^2 \|x-y\|^2$$

$\therefore \|T_n x - T_n y\| \leq (1-\alpha_n)\|x-y\|$

x_n means it is associated with T_n

$\because \forall_n\ (1-\alpha_n) < 1 \therefore T_n$: strict contraction $\Rightarrow \exists_{x_n:\text{unique}}\ T_n x_n = x_n$ /* Banach-Picard theorem says contraction mappings have unique fixed point */

fixed point of T_n

now, $\forall_{n\in\mathbb{N}}\ \|x_n - Tx_n\| = \|T_n x_n - Tx_n\|$ /* $T_n x = \alpha_n x_0 + (1-\alpha_n)Tx$

$\therefore T_n x_n = \alpha_n x_0 + (1-\alpha_n)Tx_n$

$\Leftrightarrow T_n x_n - Tx_n = \alpha_n x_0 - \alpha_n Tx_n$ */

$= \|\alpha_n x_0 - \alpha_n Tx_n\|$

$= \alpha_n \|\underbrace{x_0}_{\in D} - \underbrace{\overbrace{Tx_n}^{\in D}}_{\in D}\|$ /* $\because T: D \to D$ */

$\leq \alpha_n \operatorname{diam}(D)$ /* $\because$ both x_0, Tx_n are in D, so their total distance must be smaller than diam (D) = distance between furthest points in D */

$\therefore \forall_{n\in\mathbb{N}}\ 0 \leq \|x_n - Tx_n\| \leq \alpha_n \operatorname{diam}(D)$

$$0 \leq \underbrace{\underline{\lim}_{n\to\infty} \|x_n - Tx_n\|}_{a} \leq \overline{\lim}_{n\to\infty} \alpha_n \operatorname{diam}(D) = \underbrace{\operatorname{diam}(D)}_{\text{finite as D: bounded}} \overbrace{\lim_{n\to\infty} \alpha_n}^{0} = 0$$ using

*Fact 1.3.1.
(comes handy in dealing with sequences)

- $\forall_{n\in\mathbb{N}}\ a_n \leq a \Rightarrow \underline{\lim}\, a_n \leq a$
- $\forall_{n\in\mathbb{N}}\ b \leq b_n \Rightarrow b \leq \overline{\lim}\, b_n$
- $\forall_{n\in\mathbb{N}}\ a_1 \leq a_n \leq a_2 \Rightarrow$ $a_1 \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq a_2$; $\underline{\lim}\, a_n$ exists in $[a_1, a_2]$; $\overline{\lim}\, a_n$ exists in $[a_1, a_2]$

Result 1.2.1. (A very important result) # verified, uses
$[(b_n)_{n\in\mathbb{N}} : \subseteq \mathbb{R}]\ (0 \leq a \leq b_n,\ b_n \to 0) \Rightarrow a = 0$

$\therefore \lim_{n\to\infty} \|x_n - Tx_n\| = 0$

$\Leftrightarrow \boxed{x_n \to Tx_n}$

Now D: weakly sequentially compact $\overset{\text{def}}{\Leftrightarrow}$ every sequence in D has a weakly convergent subsequence with its weak limit in D

now $(x_n)_{n\in\mathbb{N}}$: sequence in D $\Rightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}} : \text{subsequence of } (x_n)_{n\in\mathbb{N}}}\ \boxed{x_{k_n} \rightharpoonup x \in D}$

Again $(x_n - Tx_n)_{n\in\mathbb{N}}$ converges to zero

$\Rightarrow (x_{k_n} - Tx_{k_n})$ " " 0 /* if a net converges, so does its any subsequence to the same point */

$\Leftrightarrow \boxed{x_{k_n} - Tx_{k_n} \to 0}$

/* recall Corollary 4.18. Corollary 4.18.
[D : nonempty closed convex subset of $\mathcal{H}$
T: $D \to \mathcal{H}$, nonexpansive
$(x_n)_{n\in\mathbb{N}}$: sequence in D, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$] $\Rightarrow x \in \operatorname{Fix} T$ */

$x \in \operatorname{Fix} T$ (proved) ☺

Chapter 4: Part 2

9:07 AM

Proposition 4.21.

[$T_1 : \mathcal{H} \to \mathcal{H}$, firmly nonexpansive

$T_2 : \mathcal{H} \to \mathcal{H}$, firmly nonexpansive

$T = T_1(2T_2 - Id) + Id - T_2$] $\Rightarrow$

(i) $2T - Id = (2T_1 - Id)(2T_2 - Id)$

(ii) T: firmly nonexpansive

(iii) $\mathrm{Fix}\, T = \mathrm{Fix}\,(2T_1 - Id)(2T_2 - Id)$

(iv) T_1: projector onto a closed affine subspace $\Rightarrow \mathrm{Fix}\, T = \{x \in \mathcal{H} \mid T_1 x = T_2 x\}$

Proof: /* $(A+B)x = Ax + Bx$, $(A-B)x = Ax - Bx$ */

(i) $T = T_1(2T_2 - Id) + Id - T_2$

$\therefore 2T = 2T_1(2T_2 - Id) + 2Id - 2T_2$

$\Leftrightarrow 2T - Id = 2T_1(2T_2 - Id) + 2Id - 2T_2 - Id = 2T_1(2T_2 - Id) + Id - 2T_2$

$\therefore (2T - Id)x = (2T_1(2T_2 - Id) + Id - 2T_2)x = 2T_1(2T_2 - Id)x + x - 2T_2 x$

$(2T_1 - Id)\underbrace{\overbrace{(2T_2 - Id)x}^{y}}_{(2T_2 x - x)} = (2T_1 y - y) = 2T_1(2T_2 x - x) - (2T_2 x - x) = 2T_1(2T_2 - Id)x + x - 2T_2 x = (2T - Id)x$

$\therefore (2T - Id) = (2T_1 - Id)(2T_2 - Id)$

(ii) T_1 : firmly nonexpansive $\Rightarrow (2T_1 - Id)$: nonexpansive

T_2 : firmly nonexpansive $\Rightarrow (2T_2 - Id)$: nonexpansive

/* Proposition 4.2. (Different representations of firmly nonexpansive operator)

[D: nonempty subset of $\mathcal{H}$,

$T: D \to \mathcal{H}$]

(i) T: firmly nonexpansive $\Leftrightarrow$ (ii) $(I - T)$: firmly nonexpansive $\Leftrightarrow$ (iii) $(2T - I)$: nonexpansive */

$\underbrace{(2T_1 - Id)(2T_2 - Id)}_{(2T - Id)\ [\text{from (i)}]}$: nonexpansive [$\because$ composition of nonexpansive operators is nonexpansive]

$\Leftrightarrow (2T - Id)$: nonexpansive

$\Leftrightarrow$ T: firmly nonexpansive :)

(iii) first note that

$\forall_x \quad x \in \mathrm{Fix}\, T$

$\Leftrightarrow Tx = x$

$\Leftrightarrow 2Tx = 2x$

$\Leftrightarrow 2Tx - x = x$

$\Leftrightarrow (2T - Id)x = x \Leftrightarrow x \in \mathrm{Fix}\,(2T - Id)$ /* this is true for any T */

$\therefore \mathrm{Fix}(T) = \mathrm{Fix}(2T - Id)$

$= \mathrm{Fix}\,((2T_1 - Id)(2T_2 - Id))$ /* from (i) */

$\therefore \mathrm{Fix}\,(T) = \mathrm{Fix}\,((2T_1 - Id)(2T_2 - Id))$ ■

(iv) recall:

Proposition 4.8.

[C: nonempty closed convex set of $\mathcal{H}$] $\Rightarrow P_C$: firmly nonexpansive

* Corollary 3.20.

[C: $\underbrace{\text{closed affine}}_{\forall_{\lambda \in \mathbb{R}} (C = \lambda C + (1-\lambda)C,\ C \neq \emptyset)}$ subspace of $\mathcal{H}$]

(i) [$x, p \in \mathcal{H}$]

$p \in P_C(x) \Leftrightarrow \begin{cases} p \in C \\ \forall_{y \in C} \forall_{z \in C} \langle y - z \mid x - p \rangle = 0 \end{cases}$

(ii) P_C : affine operator /* $P_C x - P_C 0$: linear operator */

T: affine $\overset{\text{def}}{\Leftrightarrow} \forall_{x, y \in \mathcal{X}} \forall_{\lambda \in \mathbb{R}}\ T(\lambda x + (1-\lambda)y) = \lambda T x + (1-\lambda) T y$

let C: the closed $\underbrace{\text{affine subspace}}_{\Rightarrow \text{convex}}$ in given

$T_1 = P_C$

$x \in \mathcal{H}$

P_C: firmly nonexpansive, affine operator.

now suppose, $x \in \mathrm{Fix}\, T$

$\Leftrightarrow x = Tx$

$= (\underbrace{T_1}_{P_C}(2T_2 - Id) + Id - T_2)x$

r_C

$= \left(P_C(2T_2 - \mathrm{Id}) + \mathrm{Id} - T_2\right)x$

$\Leftrightarrow \cancel{x} = P_C(2T_2 - \mathrm{Id})x + \cancel{x} - T_2 x$

$\Leftrightarrow T_2 x = P_C(2T_2 - \mathrm{Id})x = P_C(2T_2 x - x) = P_C(2T_2 x + (1-2)x)$ // affine /* now $T: \text{affine} \overset{\text{def}}{\Leftrightarrow} \forall_{x,y \in \mathcal{X}} \forall_{\lambda \in \mathbb{R}} \; T(\lambda x + (1-\lambda)y) = \lambda Tx + (1-\lambda)Ty \Leftrightarrow x \mapsto Tx - T0 : \text{linear}$ */

$= \underbrace{2 P_C(\underbrace{T_2 x}_{\in \mathcal{H}}) + (1-2) P_C(\underbrace{x}_{\in \mathcal{H}})}_{\in C}$ /* $\because T_2 : \mathcal{H} \to \mathcal{H}$ */

$\in C$ /* C: affine, $P_C(T_2)x \in C$, $P_C(x) \in C$

$\therefore$ affine combination $2(P_C T_2 x) + (1-2) P_C(x) \in C$ */

$\Leftrightarrow T_2 x = 2P_C(T_2 x) + (1-2) P_C(x) \in C$ /* so applying $P_C(\cdot)$ on $T_2 x$ will give $T_2 x$ */

$\Leftrightarrow P_C(T_2 x) = T_2 x = 2P_C(T_2 x) + (1-2) P_C(x)$

$\Leftrightarrow P_C(x) = P_C(T_2 x) = T_2 x$ /* projection on a closed convex set is unique */

$\therefore T_2 x = \underbrace{P_C}_{T_1}(x) \Leftrightarrow T_1 x = T_2 x$

so we have shown:

$\operatorname{Fix} T = \{x \in \mathcal{H} \mid T_1 x = T_2 x\}$ ■

$*$ 4.22.

[T_1: projector onto a linear subspace of $\mathcal{H}$

$T_2 : \mathcal{H} \to \mathcal{H}$, firmly nonexpansive

$T = T_1 T_2 + (\mathrm{Id} - T_1)(\mathrm{Id} - T_2)$] $\Rightarrow$

- T: firmly nonexpansive
- $\operatorname{Fix} T = \{x \in \mathcal{H} : T_1 x = T_2 x\}$

4.4. Averaged Nonexpansive Operators:

- Definition. 4.23. (Averaged nonexpansive operator)

[D: nonempty subset of $\mathcal{H}$,

$T: D \to \mathcal{H}$,

$\alpha \in \,]0,1[$]

T: α-averaged $\overset{\text{def}}{\Leftrightarrow} T = (1-\alpha)\mathrm{Id} + \alpha R$

- T: firmly nonexpansive $\Leftrightarrow T: \frac{1}{2}$ averaged

Proof: $T: \frac{1}{2}$ averaged $\Leftrightarrow \exists_{R:\text{nonexpansive}} \; T = \frac{1}{2}\mathrm{Id} + \frac{1}{2}R$

now R: nonexpansive $\Leftrightarrow \mathrm{Id} + R - \mathrm{Id}$: nonexpansive $\Leftrightarrow 2\overbrace{\left(\frac{1}{2}\mathrm{Id} + \frac{1}{2}R\right)}^{T} - \mathrm{Id}$: nonexpansive $\Leftrightarrow 2T - \mathrm{Id}$: nonexpansive $\Leftrightarrow T$: firmly nonexpansive ■

$*$ Proposition 4.25.

[D: nonempty subset of $\mathcal{H}$

$T: D \to \mathcal{H}$, nonexpansive

$\alpha \in \,]0,1[$]

(i) T: α-averaged $\Leftrightarrow$

(ii) $\left(1 - \frac{1}{\alpha}\right)\mathrm{Id} + \frac{1}{\alpha}T$: nonexpansive $\Leftrightarrow$

(iii)

$\forall_{x \in D} \forall_{y \in D} \quad \|Tx - Ty\|^2 \leq \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2 \Leftrightarrow$

(iv)

$\forall_{x \in D} \forall_{y \in D} \quad \|Tx - Ty\|^2 + (1-2\alpha)\|x-y\|^2 \leq 2(1-\alpha)\langle x-y \mid Tx-Ty \rangle$

this is a very important theorem, especially in splitting algorithm.

Proof: $\forall_{x,y \in D}$

(i) $\Leftrightarrow$ (ii) proof

T: α-averaged $\Leftrightarrow \exists_{R:\text{nonexpansive}} \; \underbrace{T = (1-\alpha)\mathrm{Id} + \alpha R}$

$\Updownarrow$

$\alpha R = T - (1-\alpha)\mathrm{Id} \Leftrightarrow R = \frac{1}{\alpha}T - \left(\frac{1}{\alpha} - 1\right)\mathrm{Id} = \left(1 - \frac{1}{\alpha}\right)\mathrm{Id} + \frac{1}{\alpha}T$: nonexpansive

$\therefore T$: α-averaged $\Leftrightarrow \left(1 - \frac{1}{\alpha}\right)\mathrm{Id} + \frac{1}{\alpha}T$: nonexpansive

$\therefore$ (i) $\Leftrightarrow$ (ii) ■

$$\alpha R = T-(1-\alpha)Id \Leftrightarrow R = \frac{1}{\alpha}T - \left(\frac{1}{\alpha}-1\right)Id = \left(1-\frac{1}{\alpha}\right)Id + \frac{1}{\alpha}T : \text{nonexpansive}$$

$\therefore T : \alpha\text{-averaged} \Leftrightarrow \left(1-\frac{1}{\alpha}\right)Id + \frac{1}{\alpha}T : \text{nonexpansive}$

$\therefore$ (i) $\Leftrightarrow$ (ii) ■

Proof (ii) $\Leftrightarrow$ (iii)

let $\lambda = \frac{1}{\alpha}$

then

$R = (1-\lambda)Id + \lambda T.$

$\Leftrightarrow \lambda T = R - (1-\lambda)Id$

$\Leftrightarrow T = \frac{1}{\lambda}R - \left(\frac{1}{\lambda}-1\right)Id$

$\therefore T = \underbrace{\frac{1}{\lambda}}_{\alpha}R + \Big(1-\underbrace{\frac{1}{\lambda}}_{\alpha}\Big)Id = (1-\alpha)Id + \alpha R$

Consider the identity: $\forall_{x,y\in D}$

$\|Rx-Ry\|^2 = \|((1-\lambda)Id+\lambda T)x - ((1-\lambda)Id+\lambda T)y\|^2$

$= \|(1-\lambda)x+\lambda Tx-(1-\lambda)y+\lambda Ty\|^2 = \|(1-\lambda)(x-y)+\lambda(Tx-Ty)\|^2$

$= (1-\lambda)\|x-y\|^2 + \lambda\|Tx-Ty\|^2 - \lambda(1-\lambda)\|\underbrace{x-y-Tx+Ty}_{(Id-T)x-(Id-T)y}\|^2$ /* $\because$ $\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2$ */
Corollary 2.14

$= (1-\lambda)\|x-y\|^2 + \lambda\|Tx-Ty\|^2 - \lambda(1-\lambda)\|(Id-T)x-(Id-T)y\|^2$

$\Leftrightarrow \quad \|Rx-Ry\|^2 = (1-\lambda)\|x-y\|^2 + \lambda\|Tx-Ty\|^2 - \lambda(1-\lambda)\|(Id-T)x-(Id-T)y\|^2$

$\Leftrightarrow \underbrace{\lambda}_{\left(\frac{1}{\alpha}\right)}\|x-y\|^2 - \underbrace{\lambda}_{\left(\frac{1}{\alpha}\right)}\|Tx-Ty\|^2 + \underbrace{\lambda(1-\lambda)}_{\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right) = \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right) = -\frac{1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)}\|(Id-T)x-(Id-T)y\|^2 = \|x-y\|^2 - \|Rx-Ry\|^2$

$\Leftrightarrow \frac{1}{\alpha}\|x-y\|^2 - \frac{1}{\alpha}\|Tx-Ty\|^2 - \frac{1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)\|(Id-T)x-(Id-T)y\|^2 = \|x-y\|^2 - \|Rx-Ry\|^2$

$\Leftrightarrow \|x-y\|^2 - \|Tx-Ty\|^2 - \left(\frac{1-\alpha}{\alpha}\right)\|(Id-T)x-(Id-T)y\|^2 = \alpha\left(\|x-y\|^2 - \|Rx-Ry\|^2\right)$

now $\underbrace{R}_{\left(1-\frac{1}{\alpha}\right)Id+\frac{1}{\alpha}T}$: nonexpansive $\Leftrightarrow \|Rx-Ry\|^2 \leq \|x-y\|^2$

$\Leftrightarrow \alpha\left(\|x-y\|^2 - \|Rx-Ry\|^2\right) \geq 0$ [$\because \alpha \in]0,1[$ by given]

$\Leftrightarrow \|x-y\|^2 - \|Tx-Ty\|^2 - \left(\frac{1-\alpha}{\alpha}\right)\|(Id-T)x-(Id-T)y\|^2 \geq 0 \quad \forall_{x,y\in D}$

$\therefore \left(1-\frac{1}{\alpha}\right)Id + \frac{1}{\alpha}T$: nonexpansive $\Leftrightarrow \forall_{x\in D}\forall_{y\in D} \quad \|Tx-Ty\|^2 \leq \|x-y\|^2 - \left(\frac{1-\alpha}{\alpha}\right)\|(Id-T)x-(Id-T)y\|^2$

$\therefore$ (ii) $\Leftrightarrow$ (iii) ■

(iii) $\Leftrightarrow$ (iv) proof:

(iii) : $\forall_{x\in D}\forall_{y\in D} \quad \|Tx-Ty\|^2 \leq \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|\underbrace{(Id-T)x-(Id-T)y}\|^2$

/* $\|x-Tx-y+Ty\|^2 = \|(x-y)-(Tx-Ty)\|^2$

$= \|x-y\|^2 + \|Tx-Ty\|^2 - 2\langle x-y \mid Tx-Ty\rangle$ */

$\Leftrightarrow \forall_{x\in D}\forall_{y\in D} \quad \|Tx-Ty\|^2 \leq \|x-y\|^2 - \frac{1-\alpha}{\alpha}\left(\|x-y\|^2 + \|Tx-Ty\|^2 - 2\langle x-y \mid Tx-Ty\rangle\right)$

$= \underbrace{\|x-y\|^2 - \frac{1-\alpha}{\alpha}\|x-y\|^2}_{} - \frac{1-\alpha}{\alpha}\|Tx-Ty\|^2 + 2\frac{1-\alpha}{\alpha}\langle x-y \mid Tx-Ty\rangle$

$\left(1-\frac{1-\alpha}{\alpha}\right)\|x-y\|^2 = \left(\frac{\alpha-1+\alpha}{\alpha}\right)\|x-y\|^2 = -\frac{1}{\alpha}(1-2\alpha)\|x-y\|^2$: take to L.H.S

$\Leftrightarrow \underbrace{\left(1+\frac{1-\alpha}{\alpha}\right)}_{\left(\frac{1}{\alpha}\right)}\|Tx-Ty\|^2 + \frac{1}{\alpha}(1-2\alpha)\|x-y\|^2 \leq \frac{2(1-\alpha)}{\alpha}\langle x-y \mid Tx-Ty\rangle$

$\Leftrightarrow \frac{1}{\alpha}\|Tx-Ty\|^2 + \frac{1}{\alpha}(1-2\alpha)\|x-y\|^2 \leq \frac{2(1-\alpha)}{\alpha}\langle x-y \mid Tx-Ty\rangle$

$\Leftrightarrow \forall_{x\in D}\forall_{y\in D} \quad \|Tx-Ty\|^2 + (1-2\alpha)\|x-y\|^2 \leq 2(1-\alpha)\langle x-y \mid Tx-Ty\rangle$

$\Leftrightarrow$ (iv)

$\therefore$ (iii') $\Leftrightarrow$ (iv) ∎

* Two implications of Proposition 4.25.

- Averaged operators are $\underbrace{\text{strictly quasi-nonexpansive}}_{\forall_{x \in D \setminus \mathrm{Fix}\,T}\ \forall_{y \in \mathrm{Fix}\,T}\ \|Tx - \underbrace{Ty}_{y}\| < \|x-y\|}$

- $\underbrace{T}_{D \to \mathcal{H}} : \underbrace{\alpha}_{\alpha \in]0,\frac{1}{2}]}$-averaged $\Rightarrow$ T: firmly nonexpansive

* Proposition 4.28.

[D: nonempty subset of $\mathcal{H}$

$T: D \to \mathcal{H}$

$\alpha \in]0,1]$,

$\lambda =]0, \frac{1}{\alpha}[$]

T: α-averaged $\leftrightarrow$ $(1-\lambda)\mathrm{Id} + \lambda T$: $\lambda\alpha$ averaged

* Corollary 4.29.

[D: nonempty subset of $\mathcal{H}$

$T: D \to \mathcal{H}$

$\lambda \in]0,2[$]

T: firmly nonexpansive $\leftrightarrow$ $(1-\lambda)\mathrm{Id} + \lambda T$: $\frac{\lambda}{2}$-averaged

Proof:

/* Proposition 4.28 : $\underbrace{T}_{: D \to \mathcal{H},\ D \neq \emptyset} : \underbrace{\alpha}_{\in]0,1[}$ - averaged $\leftrightarrow$ $(1-\underset{\overset{\parallel}{]0,\frac{1}{\alpha}[}}{\lambda})\mathrm{Id} + \lambda T : \lambda\alpha$ averaged */

set $\alpha = \frac{1}{2} \in]0,1[$

$\therefore$ T: $\frac{1}{2}$ averaged $\leftrightarrow$ $(1-\underset{\in]0,\frac{1}{\alpha}[=]0,2[}{\lambda})\mathrm{Id} + \lambda T : (\lambda\alpha = \frac{\lambda}{2})$-averaged [Proved]

* Proposition 4.30.

[D: nonempty subset of $\mathcal{H}$;

$\underbrace{(T_i)_{i \in I}}_{: D \to \mathcal{H}}$: (• finite family of nonexpansive operators ;

• $\forall_{i \in I}$ T_i : $\underbrace{\alpha_i}_{\alpha_i \in]0,1]}$-averaged)

$\underbrace{(\omega_i)_{i \in I}}_{\in]0,1]}$: $\sum_{i \in I} \omega_i = 1$]

$\sum_{i \in I} \omega_i T_i$: $\left(\max_{i \in I} \alpha_i\right)$-averaged

Proof: set $T = \sum_{i \in I} \omega_i T_i$

$x, y \in D$

/*

Proposition 4.25. (different faces of an α-averaged operator) ***

[D: nonempty subset of $\mathcal{H}$

$T: D \to \mathcal{H}$, nonexpansive

$\alpha \in]0,1[$]

(i) T: α-averaged $\Leftrightarrow$

(ii) $(1-\frac{1}{\alpha})\mathrm{Id} + (\frac{1}{\alpha})T$: nonexpansive $\Leftrightarrow$

(iii) $\forall_{x \in D}\ \forall_{y \in D}$ $\|Tx - Ty\|^2 \leq \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2$ $\Leftrightarrow$

(iv) $\forall_{x \in D}\ \forall_{y \in D}$ $\|Tx - Ty\|^2 + (1-2\alpha)\|x-y\|^2 \leq 2(1-\alpha)\langle x-y \mid Tx - Ty\rangle$ */

as each T_i : α_i averaged

$\leftrightarrow \forall_{x,y \in D}$ $\|T_i x - T_i y\|^2 \leq \|x-y\|^2 - \frac{1-\alpha_i}{\alpha_i}\|(\mathrm{Id}-T_i)x - (\mathrm{Id}-T_i)y\|^2$

$\leftrightarrow \forall_{x,y \in D}$ $\|T_i x - T_i y\|^2 + \frac{1-\alpha_i}{\alpha_i}\|(\mathrm{Id}-T_i)x - (\mathrm{Id}-T_i)y\|^2 \leq \|x-y\|^2$ (1)

now: $\forall_{x,y \in D}$ $\|Tx - Ty\|^2 + \frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2$ /* want to show this $\leq \|x-y\|^2$

$= \|\sum_{i \in I} \omega_i T_i x - \sum_{i \in I} \omega_i T_i y\|^2 + \frac{1-\alpha}{\alpha}\|(\mathrm{Id} - \sum_{i \in I} \omega_i T_i)x - (\mathrm{Id} - \sum_{i \in I} \omega_i T_i)y\|^2$ /* as $\sum_{i \in I} \omega_i = 1 \Rightarrow (\sum_{i \in I} \omega_i \mathrm{Id})x = \sum_{i \in I} \omega_i(\mathrm{Id}\,x) = \sum_{i \in I} \omega_i x = x\sum_{i \in I} \omega_i = x = \mathrm{Id}\,x$ $\therefore \mathrm{Id} = \sum_{i \in I} \omega_i \mathrm{Id}$ */

now: $\forall_{x,y\in D}$ $\|Tx-Ty\|^2+\frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2$ /* want to show this $\leqslant \|x-y\|^2$

$$= \Big\|\sum_{i\in I}\omega_i T_i x-\sum_{i\in I}\omega_i T_i y\Big\|^2+\frac{1-\alpha}{\alpha}\Big\|\Big(\mathrm{Id}-\sum_{i\in I}\omega_i T_i\Big)x-\Big(\mathrm{Id}-\sum_{i\in I}\omega_i T_i\Big)y\Big\|^2$$

/* as $\sum_{i\in I}\omega_i=1 \Rightarrow \Big(\sum_{i\in I}\omega_i \mathrm{Id}\Big)x=\sum_{i\in I}\omega_i(\mathrm{Id}\,x)=\sum_{i\in I}\omega_i x = x\sum_{i\in I}\omega_i = x=\mathrm{Id}\,x \ \therefore \mathrm{Id}=\sum_{i\in I}\omega_i \mathrm{Id}$ */

$$= \underbrace{\Big\|\sum_{i\in I}\omega_i T_i x-\sum_{i\in I}\omega_i T_i y\Big\|^2}_{\|\sum_{i\in I}\omega_i(T_ix-T_iy)\|^2}+\frac{1-\alpha}{\alpha}\Big\|\underbrace{\underbrace{\Big(\sum_{i\in I}\omega_i \mathrm{Id}-\sum_{i\in I}\omega_i T_i\Big)x}_{\sum_{i\in I}\omega_i(\mathrm{Id}-T_i)x}-\underbrace{\Big(\sum_{i\in I}\omega_i \mathrm{Id}-\sum_{i\in I}\omega_i T_i\Big)y}_{\sum_{i\in I}\omega_i(\mathrm{Id}-T_i)y}}_{\sum_{i\in I}\omega_i((\mathrm{Id}-T_i)x-(\mathrm{Id}-T_i)y)}\Big\|^2$$

/* now,

* Proposition 8.9. [$f:\mathcal{H}\to[-\infty,+\infty]$]

f: convex $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{(\alpha_i)_{i\in I}:\, \subseteq]0,1[,\ \sum_{i\in I}\alpha_i=1}$ $\forall_{(x_i)_{i\in I}\subseteq \mathrm{dom}\, f}$ $f\big(\sum_{i\in I}\alpha_i x_i\big)\leqslant\sum_{i\in I}\alpha_i f(x_i)$

finite index set

as $\|\cdot\|^2$: convex $\Rightarrow$ norm_squared $\Big(\sum_{i\in I}\omega_i(T_ix-T_iy)\Big)\leqslant\sum_{i\in I}\omega_i$ norm_squared (T_ix-T_iy)

$$=\sum_{i\in I}\omega_i\|T_ix-T_iy\|^2$$

similarly, $\Big\|\sum_{i\in I}\omega_i\big((\mathrm{Id}-T_i)x-(\mathrm{Id}-T_i)y\big)\Big\|^2\leqslant\sum_{i\in I}\omega_i\|(\mathrm{Id}-T_i)x-(\mathrm{Id}-T_i)y\|^2$ */

$$\leqslant \sum_{i\in I}\omega_i\|T_ix-T_iy\|^2+\frac{1-\alpha}{\alpha}\underbrace{\sum_{i\in I}\omega_i\|(\mathrm{Id}-T_i)x-(\mathrm{Id}-T_i)y\|^2}$$

/* now $\min_{i\in I}\Big(\frac{1-\alpha_i}{\alpha_i}=\frac{1}{\alpha_i}-1\Big)=\frac{1}{\max_{i\in I}\alpha_i}-1=\frac{1}{\alpha}-1=\frac{1-\alpha}{\alpha}$

$\therefore \forall_{i\in I}$ $\frac{1-\alpha}{\alpha}\leqslant\frac{1-\alpha_i}{\alpha_i}$ */

$$\sum_{i\in I}\frac{1-\alpha}{\alpha}\omega_i\|(\mathrm{Id}-T_i)x-(\mathrm{Id}-T_i)y\|^2$$

$$\leqslant\sum_{i\in I}\frac{1-\alpha_i}{\alpha_i}\omega_i\|(\mathrm{Id}-T_i)x-(\mathrm{Id}-T_i)y\|^2$$

$$\leqslant\sum_{i\in I}\omega_i\|T_ix-T_iy\|^2+\sum_{i\in I}\frac{1-\alpha_i}{\alpha_i}\omega_i\|(\mathrm{Id}-T_i)x-(\mathrm{Id}-T_i)y\|^2$$

$$=\sum_{i\in I}\omega_i\underbrace{\Big(\|T_ix-T_iy\|^2+\frac{1-\alpha_i}{\alpha_i}\|(\mathrm{Id}-T_i)x-(\mathrm{Id}-T_i)y\|^2\Big)}_{\leqslant\|x-y\|^2}$$

$\leqslant \|x-y\|^2$ /* from (1) */

$\therefore \forall_{x,y\in D}$ $\|Tx-Ty\|^2+\frac{(1-\alpha)}{\alpha}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2\leqslant\|x-y\|^2 \leftrightarrow T:\alpha$-averaged

/* using

Proposition 4.25. (different faces of an α-averaged operator) ***

[D: nonempty subset of $\mathcal{H}$

$T:D\to\mathcal{H}$, nonexpansive

$\alpha\in]0,1[$]

(i) $T:\alpha$-averaged $\Leftrightarrow$

(ii) $\left(1-\frac{1}{\alpha}\right)\mathrm{Id}+\left(\frac{1}{\alpha}\right)T$: nonexpansive $\Leftrightarrow$

(iii) $\forall_{x\in D}\forall_{y\in D}$ $\|Tx-Ty\|^2\leqslant\|x-y\|^2-\frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2 \Leftrightarrow$

(iv) $\forall_{x\in D}\forall_{y\in D}$ $\|Tx-Ty\|^2+(1-2\alpha)\|x-y\|^2\leqslant 2(1-\alpha)\langle x-y\mid Tx-Ty\rangle$ */

■

* **Proposition 4.32.** (composition of averaged operators)

[D: nonempty subset of $\mathcal{H}$

$I=\{1,\dots,m\}$ — m: strictly positive integer

$\underbrace{(T_i)_{i\in I}}_{:D\to D}$: $\big(\forall_{i\in I}\ T_i:\underbrace{\alpha_i}_{\in]0,1[}\text{-averaged}\big)$]

$\left(T=T_1\cdots T_m,\ \alpha=\frac{m}{m+\frac{1}{\max_{i\in I}\alpha_i}}\right)\Rightarrow T:\alpha$-averaged

Proof: $\forall_{i\in I}$ $\kappa_i=\alpha_i/(1-\alpha_i)$

$\kappa=\max_{i\in I}\kappa_i$

$\forall x,y\in D$

now

$\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2/m$

$=\|(\mathrm{Id}-T_1T_2\cdots T_m)x-(\mathrm{Id}-T_1T_2\cdots T_m)y\|^2/m$ /* use one step at a time trick */

$=\|x-y-(T_1T_2\cdots T_mx-T_1\cdots T_m)y\|^2$

$$=\|\underbrace{(x-y)-(T_mx-T_my)}+\underbrace{(T_mx-T_my)-(T_{m-1}T_mx-T_{m-1}T_my)}+(T_{m-1}T_mx-T_{m-1}T_my)-\ \cdots\ -(T_2\cdots T_mx-T_2\cdots T_my)+(T_2\cdots T_mx-T_2\cdots T_my)$$

$$-(T_1\cdots T_mx-T_1\cdots T_my)\|^2/m$$

$$=\|\overbrace{(\mathrm{Id}-T_m)x-(\mathrm{Id}-T_m)y}^{a_1}+\overbrace{(\mathrm{Id}-T_{m-1})T_mx-(\mathrm{Id}-T_{m-1})T_my}^{a_2}+\cdots+\overbrace{(\mathrm{Id}-T_1)T_2\cdots T_mx-(\mathrm{Id}-T_1)T_2\cdots T_my}^{a_m}\|^2/m$$

$$-\left(T_1\cdots T_m x - T_1\cdots T_m y\right)\|^2/m$$

$$= \| \underbrace{(\mathrm{Id}-T_m)x - (\mathrm{Id}-T_m)y}_{a_1} + \underbrace{(\mathrm{Id}-T_{m-1})T_m x - (\mathrm{Id}-T_{m-1})T_m y}_{a_2} + \dots + \underbrace{(\mathrm{Id}-T_1)T_2\cdots T_m x - (\mathrm{Id}-T_1)T_2\cdots T_m y}_{a_m} \|^2/m$$

/* $\|a_1+a_2+\dots+a_m\|^2 \frac{1}{m} = \|\underbrace{\frac{1}{m}a_1+\frac{1}{m}a_2+\dots+\frac{1}{m}a_m\|^2}\cdot m$

Result 2.13.1. [$(x_i)_{i\in I}$: finite families in $\mathcal{H}$; $(\alpha_i)_{i\in I}$: $\subseteq \mathbb{R}$, $\sum_{i\in I}\alpha_i=1$] $\|\sum_{i\in I}\alpha_i x_i\|^2 \leq \sum_{i\in I}\alpha_i\|x_i\|^2$

/* $\leq \sum_{i=1}^{m}\frac{1}{m}\|a_i\|^2$ */

$\leq \sum_{i=1}^{m}\|a_i\|^2$ */

$$\leq \|(\mathrm{Id}-T_m)x-(\mathrm{Id}-T_m)y\|^2 + \|(\mathrm{Id}-T_{m-1})T_m x-(\mathrm{Id}-T_{m-1})T_m y\|^2 + \dots + \|(\mathrm{Id}-T_1)T_2\cdots T_m x-(\mathrm{Id}-T_1)T_2\cdots T_m y\|^2$$ /*

now

> Proposition 4.25. (different faces of an α-averaged operator) ★★★
>
> [D: nonempty subset of $\mathcal{H}$
>
> $T: D\to\mathcal{H}$, nonexpansive
>
> $\alpha\in\,]0,1[$]
>
> (i) T: α-averaged $\Leftrightarrow$
>
> (ii) $\left(1-\frac{1}{\alpha}\right)\mathrm{Id}+\left(\frac{1}{\alpha}\right)T$: nonexpansive $\Leftrightarrow$
>
> (iii) $\forall_{x,y\in D}\; 0\leq\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2 \leq \frac{\alpha}{1-\alpha}\left(\|x-y\|^2-\|Tx-Ty\|^2\right)$
>
> $\forall_{x\in D}\forall_{y\in D}\; \|Tx-Ty\|^2 \leq \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2 \Leftrightarrow$
>
> (iv) $\forall_{x\in D}\forall_{y\in D}\; \|Tx-Ty\|^2+(1-2\alpha)\|x-y\|^2 \leq 2(1-\alpha)\langle x-y \mid Tx-Ty\rangle$.

*/

$$\leq \underbrace{\frac{\alpha_m}{1-\alpha_m}}_{\kappa_m}\left(\|x-y\|^2-\|T_m x-T_m y\|^2\right) + \underbrace{\frac{\alpha_{m-1}}{1-\alpha_{m-1}}}_{\kappa_{m-1}}\left(\|T_m x-T_m y\|^2-\|T_{m-1}T_m x-T_{m-1}T_m y\|^2\right)+\dots+\overbrace{\frac{\alpha_1}{1-\alpha_1}}^{\kappa_1}\left(\|T_2\cdots T_m x-T_2\cdots T_m x\|^2 - \|T_1\cdots T_m x-T_1\cdots T_m y\|^2\right)$$

each of these are positive

$$= \kappa_m\overbrace{\left(\|x-y\|^2-\|T_m x-T_m y\|^2\right)}+\kappa_{m-1}\left(\|T_m x-T_m y\|^2-\|T_{m-1}T_m x-T_{m-1}T_m y\|^2\right)+\dots+\kappa_1\left(\|T_2\cdots T_m x-T_2\cdots T_m x\|^2 - \|T_1\cdots T_m x-T_1\cdots T_m y\|^2\right)$$

/* now, $\kappa = \max_{i\in I}\kappa_i$ */

$$\leq \kappa\left(\|x-y\|^2-\|T_m x-T_m y\|^2\right)+\kappa\left(\|T_m x-T_m y\|^2-\|T_{m-1}T_m x-T_{m-1}T_m y\|^2\right)+\dots+\kappa\left(\|T_2\cdots T_m x-T_2\cdots T_m x\|^2 - \|\underbrace{T_1\cdots T_m}_{T} x-\underbrace{T_1\cdots T_m}_{T} y\|^2\right)$$

$$= \kappa\left(\|x-y\|^2-\|Tx-Ty\|^2\right)$$

$$\therefore \frac{1}{\kappa m}\left(\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2\right) \leq \kappa\left(\|x-y\|^2-\|Tx-Ty\|^2\right)$$

$\Leftrightarrow \|Tx-Ty\|^2 \leq \|x-y\|^2 - \frac{1}{\kappa m}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2$ /* now set $\frac{1-\beta}{\beta}=\frac{1}{\kappa m} \Leftrightarrow \kappa m(1-\beta)=\beta \Leftrightarrow \kappa m = \beta(1+\kappa m)$

$\Leftrightarrow \beta = \frac{\kappa m}{1+\kappa m}$ */

$\therefore \|Tx-Ty\|^2 \leq \|x-y\|^2 - \frac{1-\beta}{\beta}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2$ where $\beta = \frac{\kappa m}{1+\kappa m} = \alpha$ //given

$\Leftrightarrow T$: α-averaged. ■

*Proposition 4.33.

[D: nonempty subset of $\mathcal{H}$

$\beta\in\mathbb{R}_{++}$

$T: D\to\mathcal{H}$, β-cocoercive

$\gamma\in\,]0,2\beta[$]

$(\mathrm{Id}-\gamma T)$: $\frac{\gamma}{2\beta}$ averaged

Proof:

$\beta\in\mathbb{R}_{++}$

T: β-cocoercive $\Leftrightarrow$ βT: firmly nonexpansive

$\Leftrightarrow$ βT: $\frac{1}{2}$averaged

$\Leftrightarrow \exists_{\underbrace{R:D\to\mathcal{H}}_{\because T:D\to\mathcal{H}}}\left(R\text{: nonexpansive} \wedge \underbrace{\beta T = \frac{1}{2}\mathrm{Id}+\frac{1}{2}R}_{\Updownarrow\; T=\frac{1}{2\beta}(\mathrm{Id}+R)}\right)$

$\mathrm{Id}-\gamma T$ { $\gamma\in\,]0,2\beta[$ }

$= \mathrm{Id}-\gamma\cdot\frac{1}{2\beta}(\mathrm{Id}+R)$

$= \left(1-\frac{\gamma}{2\beta}\right)\mathrm{Id} + \left(\frac{\gamma}{2\beta}\right)\underbrace{(-R)}_{\text{nonexpansive}}$

/* as $\gamma\in\,]0,2\beta[$

$\frac{\gamma}{2\beta}\in\,]0,1[$ */

/* R: nonexpansive $\Leftrightarrow$ $\underbrace{\|Rx-Ry\|^2}\leq\|x-y\|^2$

$= \|-(Rx-Ry)\|^2 = \|(-R)x-(-R)y\|^2$

$\Leftrightarrow \|(-R)x-(-R)y\|^2 \leq \|x-y\|^2$

$\therefore (-R)$: nonexpansive */

$\therefore \mathrm{Id}-\gamma T$: $\frac{\gamma}{2\beta}$ averaged ■

Chapter 4: Part 3

9:24 PM

* **Proposition 4.34.**

[D: nonempty subset of $\mathcal{H}$

$(T_i)_{i\in I}$: finite family of quasinonexpansive operators $:D\to\mathcal{H}$, $\bigcap_{i\in I} \operatorname{Fix} T_i \neq \emptyset$

$(\omega_i)_{i\in I}$: strictly positive real numbers, $\sum_{i\in I} \omega_i = 1$]

$\Rightarrow \operatorname{Fix} \sum_{i\in I} \omega_i T_i = \bigcap_{i\in I} \operatorname{Fix} T_i$

Proof: set $T = \sum_{i\in I} \omega_i T_i$

first, $\bigcap_{i\in I} \operatorname{Fix} T_i \subseteq \operatorname{Fix} T$, as

$$\forall_x \left(x \in \bigcap_{i\in I} \operatorname{Fix} T_i \leftrightarrow \forall_{i\in I}\ \underbrace{x \in \operatorname{Fix} T_i}_{T_i x = x} \Rightarrow \forall_{i\in I}\ \omega_i T_i x = \omega_i x \Rightarrow \overbrace{\sum_{i\in I} \omega_i T_i}^{T} x = \sum_{i\in I} \omega_i x = x \overbrace{\sum_{i\in I} \omega_i}^{1} = x \leftrightarrow Tx = x \leftrightarrow x \in \operatorname{Fix} T \right)$$

$\leftrightarrow \bigcap_{i\in I} \operatorname{Fix} T_i \subseteq \operatorname{Fix} T$

Now show $\operatorname{Fix} T \subseteq \bigcap_{i\in I} \operatorname{Fix} T_i \leftrightarrow \forall_{\underbrace{x \in \operatorname{Fix} T}_{\text{given}}} \overbrace{\left(x \in \bigcap_{i\in I} \operatorname{Fix} T_i \leftrightarrow \forall_{i\in I}\ x \in \operatorname{Fix} T_i \leftrightarrow \forall_{i\in I}\ T_i x = x \right)}^{\text{goal}}$

now given, $\exists\, y \in \bigcap_{i\in I} \operatorname{Fix} T_i \leftrightarrow \forall_{i\in I}\ y \in \operatorname{Fix} T_i$

/* $\because \neq \emptyset$ */

now, T_i: quasinonexpansive $\forall_{i\in I}$

$\leftrightarrow \forall_{i\in I}\ \forall_{\tilde{x}\in D}\ \forall_{\tilde{y} \in \operatorname{Fix} T_i}\ \|T_i \tilde{x} - T_i \tilde{y}\| = \|T_i \tilde{x} - \tilde{y}\| \leq \|\tilde{x} - \tilde{y}\|$

set $\tilde{x} := x \in \operatorname{Fix} T \subseteq D$, $\tilde{y} := y$

$\Rightarrow \forall_{i\in I}\ \|T_i x - y\|^2 \leq \|x - y\|^2$

// a variant of "one step at a time" rule again

/* now $\|T_i x - y\|^2 = \|\underbrace{T_i x - x}+\underbrace{x - y}\|^2 = \|T_i x - x\|^2 + \|x-y\|^2 + 2\langle T_i x - x \mid x - y\rangle$ */

$\forall_{i\in I}\ \|T_i x - x\|^2 + \cancel{\|x - y\|^2} + 2\langle T_i x - x \mid x - y\rangle \leq \cancel{\|x-y\|^2}$

$\leftrightarrow \forall_{i\in I}\ 2\langle T_i x - x \mid x - y\rangle \leq -\|T_i x - x\|^2$

mult. both sides by ω_i $\leftrightarrow$ $\forall_i\ 2\langle \omega_i (T_i x - x) \mid x - y\rangle \leq -\omega_i \|T_i x - x\|^2$

$$\Rightarrow \sum_{i\in I} 2\langle \omega_i (T_i x - x) \mid x - y\rangle = 2\left(\langle \omega_1 T_1 x - \omega_1 x \mid x - y\rangle + \dots + \langle \omega_m T_m x - \omega_m x \mid x - y\rangle \right) \leq -\sum_{i\in I} \omega_i \|T_i x - x\|^2$$

$$\leftrightarrow 2\langle (\omega_1 T_1 x + \dots + \omega_m T_m x) - \underbrace{(\omega_1 + \dots + \omega_m)}_{1} x \mid x - y\rangle \leq -\sum_{i\in I} \omega_i \|T_i x - x\|^2$$

$$\leftrightarrow 2\langle \underbrace{\underbrace{(\omega_1 T_1 + \dots + \omega_m T_m)}_{\sum_{i\in I}\omega_i T_i = T} x - x}_{Tx - x = 0\ /*\ \because x \in \operatorname{Fix} T\ */} \mid x - y\rangle \leq -\sum_{i\in I} \omega_i \|T_i x - x\|^2$$

$$\leftrightarrow 0 \leq -\sum_{i\in I} \omega_i \|T_i x - x\|^2 \leftrightarrow \sum_{i\in I} \omega_i \|T_i x - x\|^2 \leq 0 \leftrightarrow \forall_{i\in I}\ \|T_i x - x\|^2 = 0 \leftrightarrow \forall_{i\in I}\ x = T_i x$$

$\therefore$ goal achieved. :)

■

* **Proposition 4.35.**

[D: nonempty subset of $\mathcal{H}$

T_1: quasinonexpansive operator $:D\to D$
T_2: " " $:D\to D$
} one of them is strictly quasinonexpansive, $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 \neq \emptyset$

] $\Rightarrow$

- $\operatorname{Fix} T_1 T_2 = \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$
- $T_1 T_2$: quasinonexpansive
- (T_1, T_2): strictly quasinonexpansive $\Rightarrow$ $T_1 T_2$: strictly quasinonexpansive

* Corollary 4.36.

strictly quasinonexpansive, $: D \to D$, $\bigcap_{i \in I} \operatorname{Fix} T_i \neq \emptyset$ (nonempty) /* $I = \{1, \ldots, m\}$ */

- $\underbrace{T = T_1 T_2 \cdots T_m}$: strictly quasinonexpansive $\wedge$ $\operatorname{Fix} T = \bigcap_{i \in I} \operatorname{Fix} T_i$

Proof: strong induction on m. /* Strong induction:

To prove a goal of the form $\forall n \in \mathbb{N} P(n)$:
Prove that $\forall n[(\forall k < n P(k)) \to P(n)]$, where both n and k range over the natural numbers in this statement. Of course, the most direct way to prove this is to let n be an arbitrary natural number, assume that $\forall k < n P(k)$, and then prove $P(n)$.

*/

for $m = 1 \Rightarrow T = T_1$: strictly quasinonexpansive (by given)

for $m = 1, 2 \Rightarrow T = T_1 T_2$: strictly quasinonexpansive /* from

* Proposition 4.35. (Common fixed points of composition of two quasinonexpansive operators */
[D : nonempty subset of $\mathcal{H}$
T_1 : quasinonexpansive operator, $: D \to D$ } one of them strictly quasinonexpansive
T_2 : " " , $: D \to D$ } $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 \neq \emptyset$
] $\Rightarrow$
(i) $\operatorname{Fix} T_1 T_2 = \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$
(ii) $T_1 T_2$: quasinonexpansive
(iii) (T_1, T_2) : strictly quasinonexpansive $\Rightarrow T_1 T_2$: strictly quasinonexpansive

*/

now assume:

$\forall_{i \in \{1, 2, \ldots, m\}}$ $T = T_1 T_2 \cdots T_i$: strictly quasinonexpansive

and $\operatorname{Fix} T = \bigcap_{i \in \{1, \ldots, m\}} \operatorname{Fix} T_i$

now consider. $(\underbrace{T_i}_{D \to D})_{i \in \{1, \ldots, m+1\}}$: strictly quasinonexpansive, $\bigcap_{i=1}^{m+1} \operatorname{Fix} T_i \neq \emptyset$

set, $\underbrace{R_1 = T_1 \cdots T_m}$, $R_2 = T_{m+1}$: strictly quasinonexpansive

: strictly quasinonexpansive by base assumption (induction hypothesis)

and $\operatorname{Fix} R_1 = \bigcap_{i=1}^{m} \operatorname{Fix} T_i$, /* by induction hypothesis */ 1

$\operatorname{Fix} R_2 = \operatorname{Fix} T_{m+1}$

/* using */

$R_1 R_2 = T_1 T_2 \cdots T_{m+1}$: strictly quasinonexpansive, and

$\operatorname{Fix} \underbrace{T_1 T_2 \cdots}_{R_1} \underbrace{T_{m+1}}_{R_2} = \operatorname{Fix} R_1 \cap \operatorname{Fix} R_2 = \bigcap_{i=1}^{m+1} \operatorname{Fix} T_i$:) ✓

■

Corollary 4.37

averaged nonexpansive, $: D \to D$, $\bigcap_{i \in I} \operatorname{Fix} T_i \neq \emptyset$ (nonempty)

$\underbrace{T = T_1 T_2 \cdots T_m}$ $\Rightarrow$ $\operatorname{Fix} T = \bigcap_{i \in I} \operatorname{Fix} T_i$

Proof: Because averaged operators are strictly quasinonexpansive

apply 4.36.

■