Chapter 3: Part 1

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Highlights:

- Fundamental notion of convexity
- Every nonempty closed convex subset C of $\mathcal{H}$: $\underbrace{\text{Chebyshev set}}$

every point in $\mathcal{H}$ has a unique best approximation from C.

3.1. Definitions and Examples:

Definition 3.1. (convex set)

$\underbrace{C}_{\subseteq \mathcal{H}}$: convex $\overset{\text{def}}{\leftrightarrow} \forall_{\alpha \in]0,1[} \ \alpha C + (1-\alpha) C = C$ /* recall the notation : • $C+D = \{x+y \mid x \in C, y \in C\}$

$\underbrace{\lambda}_{\in \mathbb{R}} C = \{\lambda x \mid x \in C\}$ */

$\overset{\text{def}}{\leftrightarrow} \forall_{x \in C} \ \forall_{y \in C} \ ]x,y[\subseteq C$

/* $]x,y[= \{(1-\alpha)x + \alpha y \mid 0 < \alpha < 1\}$ */

Example:

- $C = \mathcal{H}$
- $C = \emptyset$
- C : ball
- C : affine subspace /* C : affine subspace $\overset{\text{def}}{\leftrightarrow} \left(C \neq \emptyset \ \wedge \ \forall_{\lambda \in \mathbb{R}} \ C = \lambda C + (1-\lambda) C\right)$ */
- C : half-space
- $C = \bigcap_{i \in I} C_i$: $(C_i)_{i \in I}$ is a family of convex subsets of $\mathcal{H}$

Definition 3.3. (convex hull of a set C)

[$C \subseteq \mathcal{H}$]

$\operatorname{conv} C$: convex hull of C $\overset{\text{def}}{\leftrightarrow}$ intersection of all the convex subsets of $\mathcal{H}$ containing C

$\updownarrow$

smallest convex subset of $\mathcal{H}$ containing C

$\overline{\operatorname{conv}}\, C$: closed convex hull of C $\overset{\text{def}}{\leftrightarrow}$ smallest closed convex subset of $\mathcal{H}$ containing C

***Proposition 3.4.**

[$C \subseteq \mathcal{H}$,

D : set of all convex combinations of points in C $\overset{\text{def}}{\leftrightarrow}$ $D = \left\{ \sum_{i \in I} \alpha_i x_i \ \middle|\ I : \text{finite}, \{x_i\}_{i \in I} \subseteq C, \{\alpha_i\}_{i \in I} \subseteq]0,1], \sum_{i \in I} \alpha_i = 1 \right\}$

]

$\Rightarrow$

$D = \operatorname{conv} C$

***Proposition 3.5.**

[$\mathcal{H}$: real Hilbert space

$T : \mathcal{H} \to \mathcal{K}$, affine operator

C : convex subset of $\mathcal{H}$

D : " " of $\mathcal{K}$]

$\Rightarrow$

- $T(C)$: convex subset of $\mathcal{H}$
- $T^{-1}(D)$: convex subset of D.

***Proposition 3.6.** /* totally ordered : (i) $\forall_{a \in A} \ a \leq a$, (ii) $\forall_{a,b,c \in A} \ (a \leq b, b \leq c) \Rightarrow (a \leq c)$ (iii) $\forall_{a,b \in A} \ (a \leq b \wedge b \leq a) \Rightarrow b = a$, (iv) $\forall_{a \in A} \forall_{b \in A} \ (a \leq b \vee b \leq a)$ */

[$(C_i)_{i \in I}$: totally ordered finite family of $\underset{|I|}{\overset{}{m}}$ convex subsets of $\mathcal{H}$]

$\Rightarrow$

(i) $\left(\bigtimes_{i \in I} C_i\right)$: convex

(ii) $\forall_{(\alpha_i)_{i \in I} \in \mathbb{R}^m} \ \sum_{i \in I} \alpha_i C_i$: convex

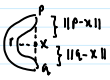
3.2 Best Approximation Property: /* Best approximation and projection are same things */

Definition 3.7.

[C: subset of $\mathcal{H}$

$x \in \mathcal{H}$

$p \in C$]



each of p, q, r are best approximations to x from C

p: best approximation to x onto C $\overset{def}{\Longleftrightarrow}$ $\|x-p\| = \underbrace{d_C(x)}_{\inf d(x,C)}$

(more commonly, projection of x onto C)

C: proximinimal $\overset{def}{\Longleftrightarrow}$ every point in $\mathcal{H}$ has atleast one projection onto C

C: Chebyshev set $\overset{def}{\Longleftrightarrow}$ every point in $\mathcal{H}$ has exactly one projection onto C

P_C: projection operator $\overset{def}{\Longleftrightarrow}$ operator that maps every point in $\mathcal{H}$ to its unique projection onto C

(projector)

Chebyshev set

* Proposition 3.10.

[$\mathcal{H}$: finite dimensional

C: Chebyshev subset of $\mathcal{H}$]

$\Rightarrow$ P_C: continuous

* Proposition 3.12.

(C: nonempty, $\underbrace{\text{weakly closed}}$ subset of $\mathcal{H}$) $\Rightarrow$ C: proximinimal.

/* A subset C of $\mathcal{H}$ is weakly closed $\overset{def}{\Longleftrightarrow}$ weak limit of every weakly convergent net in C is also in C */

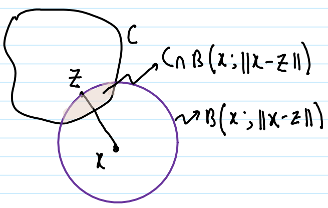
Prop 3.12 start

[Proof:

$x \in \mathcal{H}$,

$z \in C$

$D = C \cap B(x; \|x-z\|)$



$f: \mathcal{H} \to \mathbb{R} : y \mapsto \|x-y\|$

x: cluster point of sequence $(x_n)_{n\in\mathbb{N}}$ $\overset{def}{\Longleftrightarrow}$ $\forall_{V\in\mathcal{V}(x)}\ \forall_{m\in\mathbb{N}}\ \exists_{p\in\mathbb{N}: p\geq n}\ x_p \in V$

C: weakly closed subset of $\mathcal{H}$ /* given */

$B(x;\|x-z\|)$: weakly compact /* weakly compact $\overset{def}{\Longleftrightarrow}$ every sequence has a weak cluster point */

Recall:

Fact 2.27. The closed unit ball $B(0;1)$ of $\mathcal{H}$ is weakly compact

$\Rightarrow B(x;\|x-z\|)$: weakly compact

/*

Lemma 1.12.

C: compact subset $\underbrace{X}_{\text{Hausdorff space}}$ $\Rightarrow$ $\begin{cases} \bullet\ C: \text{closed} \\ \bullet\ \forall_{D:\text{ closed subset of } C}\ D: \text{compact} \end{cases}$ */

so we can extend this result under weak topology

k

So, $B(x;\|x-z\|)$: weakly compact $\Rightarrow$ $B(x;\|x-z\|)$: weakly closed

C: weakly closed

$\therefore$ $C \cap B(x;\|x-z\|)$: (weakly closed, /* arbitrary union and finite intersection of closed (open) sets are closed (open) */

a subset of $B(x;\|x-z\|)$ by definition)

$\Rightarrow$ $C \cap B(x;\|x-z\|)$: weakly compact.

$\Rightarrow$ $C \cap B(x; \|x-z\|)$: weakly compact.

By construction, $z \in C \cap B(x; \|x-z\|)$: nonempty, $\because z \in B(x; \|x-z\|)$

now: $f(y) = \|x-y\|$ is weakly lower semicontinuous /* Lemma 2.35. Norm of $\mathcal{H}$: weakly lower semicontinuous [Norm in H: weakly lower semicontinuous] */

Now we apply Weirstrass theorem:

[$\mathcal{H}^{weak}$: Hausdorff space /* Lemma 2.23. $\mathcal{H}^{weak}$: Hausdorff space see [H^weak is Hausdorff] */

$f : \mathcal{H}^{weak} \to \mathbb{R}$, weakly lower semicontinuous

$C \cap B(x; \|x-z\|)$: weakly compact

$\operatorname{dom} f = \mathcal{H}^{weak} \cap \underbrace{(C \cap B(x; \|x-z\|))}_{\text{because this lies in } \mathcal{H}^{weak}}$: nonempty]

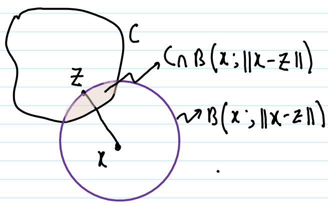
$\Rightarrow$

f achieves infimum over $C \cap B(x; \|x-z\|)$

So, if we define :

$$\begin{pmatrix} \min\limits_{y \in \mathcal{H}^{weak}} & \|x - \overset{\text{any point in } \mathcal{H}}{y}\| \\ \text{s.t.} & y \in C \cap B(x; \|x - \underset{\text{any } z \in C}{z}\|) \end{pmatrix}$$

then we have an $\overset{\text{finite}}{\text{infimum}}$ and a minimizer



lets denote the minimizer by $x^* \in \begin{pmatrix} \operatorname{Argmin}\limits_{y \in \mathcal{H}^{weak}} & \|x-y\| \\ \text{s.t.} & y \in C \cap B(x; \|x-z\|) \end{pmatrix}$ we claim $x \in \begin{pmatrix} \operatorname{Argmin}\limits_{y \in \mathcal{H}^{weak}} & \|x-y\| \\ \text{s.t.} & y \in C \end{pmatrix} = P_C(x)$

$\Updownarrow$

$\therefore \forall_{z \in C} \; \forall_{y \in C \cap B(x; \|x-z\|)} \; \|x - x^*\| \leq \|x - y\|$ but per absurdum $\exists_{\tilde{x} \in C} \; \|x - \tilde{x}\| < \|x - x^*\|$

$x^* \in C \cap B(x; \|x-z\|)$

set $z = \tilde{x} \in C$

$\therefore \forall_{y \in C \cap B(x; \|x - \tilde{x}\|)} \; \|x - x^*\| \leq \|x - y\|$

as: $\tilde{x} \in C$, $\tilde{x} \in B(x; \|x - \tilde{x}\|) \Rightarrow \tilde{x} \in C \cap B(x; \|x - \tilde{x}\|)$

$\Rightarrow \|x - x^*\| \leq \|x - \tilde{x}\|$

$\to \|x - x^*\| \leq \|x - \tilde{x}\| < \|x - x^*\| \quad \forall_{x \in \mathcal{H}}$

by setting $x \in \mathcal{H} \setminus C$ we have $\|x - x^*\| \neq 0$

$\|x - x^*\| < \|x - x^*\|$

$\to \quad 1 < 1$: contradiction

$\therefore x^* \in P_C(x)$

$\therefore$ So, C has atleast one projection onto C for every point in $\mathcal{H}$.

■] prop 3.13 end

* Corollary 3.13.

[$\mathcal{H}$: finite dimensional
 C: nonempty subset of $\mathcal{H}$]

C: proximinimal $\Leftrightarrow$ C: closed

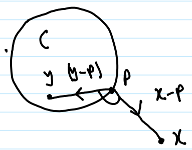
* Every Chebyshev set is proximinimal, but a proximinimal set may not be Chebyshev.

* Theorem 3.14. /* Defining property of projection on a nonempty closed convex set */

C: nonempty closed convex set of $\mathcal{H}$

$\Rightarrow$

- C : Chebyshev set
- $\forall_{x,p\in\mathcal{H}} \left(p = P_C(x) \Leftrightarrow \left(p\in C \wedge \forall_{y\in C} \langle y-p \mid x-p\rangle \leq 0 \right)\right)$



- From 3.14: every nonempty closed convex set is a Chebyshev set
- A Chebyshev set is nonempty and closed

Chebyshev problem: is every Chebyshev set convex ? $\xrightarrow{\text{yes}}$ if $\mathcal{H}$: finite dimensional; $\xrightarrow{\text{do not know}}$ if $\mathcal{H}$: infinite-dimensional : this is still an open problem.

* Proposition 3.17.

[C: nonempty closed convex subset of $\mathcal{H}$
 $x, y \in \mathcal{H}$]

$\Rightarrow$

$P_{y+C}\, x = y + P_C(x-y)$

Proof:

By definition: $P_C(x-y) \in C$

$\Rightarrow y + P_C(x-y) \in y + C$

/* Theorem 3.14.

C: nonempty closed convex set of $\mathcal{H}$

$\Rightarrow$

- C : Chebyshev set
- $\forall_{x,p\in\mathcal{H}} \left(p = P_C(x) \Leftrightarrow \left(p\in C \wedge \forall_{\tilde{y}\in C} \langle \tilde{y}-p \mid x-p\rangle \leq 0 \right)\right)$ */

We check this condition now: for $p = y + P_C(x-y)$, consider any $x \in \mathcal{H}$

$\forall_{\underbrace{\tilde{z}\in y+C}} \langle \tilde{z} - p \mid x-p\rangle = \langle \tilde{z} - y - P_C(x-y) \mid x - y - P_C(x-y)\rangle \overset{?}{\leq} 0$

$\Updownarrow$

$\exists!_{z\in C}\ \tilde{z} = y + z \Leftrightarrow$

$\Leftrightarrow \forall_{z\in C} \langle \cancel{y} + z - \cancel{y} - P_C(x-y) \mid x - y - P_C(x-y)\rangle \overset{?}{\leq} 0$

$\Leftrightarrow \forall_{z\in C} \langle z - P_C(x-y) \mid (x-y) - P_C(x-y)\rangle \overset{\checkmark}{\leq} 0$

$_{z \in C}$

$\Leftrightarrow \forall_{z \in C} \quad \langle z - P_C(x-y) \mid (x-y) - P_C(x-y) \rangle \leq 0$

/* now as $x, y \in \mathcal{H} \to x - y \in \mathcal{H}$

as $P_C(\tilde{x})$ is a projection onto C for any $\tilde{x} \in \mathcal{H}$

$\forall_{\tilde{x} \in \mathcal{H}} \ \forall_{\hat{y} \in C} \ \langle \hat{y} - P_C(\tilde{x}) \mid \tilde{x} - P_C(\tilde{x}) \rangle \leq 0$

$\tilde{x} := x - y \in \mathcal{H},\ \hat{y} = z \in C \Rightarrow \langle z - P_C(x-y) \mid (x-y) - P_C(\tilde{x}) \rangle \leq 0$ */

$\therefore \forall_{\tilde{z} \in y + C} \quad \langle \tilde{z} - (y + P_C(x-y)) \mid x - (y + P_C(x-y)) \rangle \leq 0$

So both of the defining properties for projection are satisfied:

$\therefore P_{y+C}\, x = y + P_C(x-y)$ ■

Proposition 3.18.

$((C_n)_{n \in \mathbb{N}}$: sequence of nonempty, bounded, closed convex subsets of $\mathcal{H}$,

$\forall_{n \in \mathbb{N}} \ C_{n+1} \subseteq C_n)$

$\Rightarrow \bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$

Proposition 3.19.

C : (nonempty closed convex subset of $\mathcal{H}$,

$x \in \mathcal{H})$

$\Rightarrow \forall_{\lambda \in \mathbb{R}_+} \ P_C(P_C x + \lambda(x - P_C x)) = P_C(x)$

* Additional properties for projections onto affine subspaces:

Corollary 3.20.

C: closed affine subspace of $\mathcal{H}$

$\Rightarrow$

(i) $[\![x, p \in \mathcal{H}]\!]$

$p = P_C(x) \Leftrightarrow (p \in C \wedge \forall_{y \in C} \forall_{z \in C} \ \langle y - z \mid x - p \rangle = 0)$

(ii) P_C : affine operator

* Projections onto linear subspaces.

Corollary 3.22. /* A very important application of this corollary is least squares solution to linear equations */

[V: closed linear subspace of $\mathcal{H}$

$x \in \mathcal{H}$]

$\Rightarrow$

(i) $P_V(x) : \in V, \ x - P_V(x) \perp V$

(ii) $\|P_V(x)\|^2 = \langle P_V(x) \mid x \rangle$

(iii) • $V \neq \{0\} \Rightarrow (P_V \in \mathcal{B}(\mathcal{H}), \|P_V\| = 1)$

• $V = \{0\} \Rightarrow \|P_V\| = 0$

(iv) $V^{\perp\perp} = V$

(v) $P_{V^\perp} = I - P_V$

(vi) $P_V^* = P_V$

(vii) $\|x\|^2 = \|P_V x\|^2 + \|P_{V^\perp} x\|^2 = d_V^2(x) + d_{V^\perp}^2(x)$

Proposition 3.23.

[C: nonempty subset of $\mathcal{H}$

$V = \overline{\operatorname{span}} C$

$\underbrace{\Pi_C}_{\text{set valued projector onto } C} : \mathcal{H} \to 2^C,\ x \mapsto \{p \in C \mid \|x-p\| = d_C(x)\}$]

- $\Pi_C = \Pi_C \circ P_V$
- C: proximinimal subset of $\mathcal{H} \Leftrightarrow C$: proximinimal subset of V.

*Notion of least-squares to linear equations

Definition 3.24. (Least squares solution)

[$\mathcal{K}$: real Hilbert space

$T: \in \mathcal{B}(\mathcal{H}, \mathcal{K})$

$y \in \mathcal{K}$

$x \in \mathcal{H}$]

x: least-squares solution to $T\underbrace{z}_{\text{variable}} = \underbrace{y}_{\text{fixed}}$

$\overset{\text{def}}{\Leftrightarrow} \|Tx - y\| = \min_{z \in \mathcal{H}} \|Tz - y\|$

Proposition 3.25. (Charecterization of least squares solution on closed range linear operator)

[$\mathcal{K}$: real Hilbert space

$\underbrace{T}_{\in \mathcal{B}(\mathcal{H},\mathcal{K})}$: (ran T : closed)

$y \in \mathcal{K}$] $\Rightarrow$

- $Tz = y$ has atleast one least squares solution
- $\forall_{x \in \mathcal{H}} \begin{cases} \text{(i) } x: \text{least squares solution} \Leftrightarrow \\ \text{(ii) } Tx = P_{\operatorname{ran} T}\, y \Leftrightarrow \\ \text{(iii) } T^* T x = T^* y \text{ (normal equation)} \end{cases}$

Definition 3.26.

[$\mathcal{K}$: real Hilbert space

$T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, ran$(T)$: closed

$\forall_{y \in \mathcal{K}}\ C_y = \{x \in \mathcal{H} \mid T^* T x = T^* y\}$ /*set of least squares solution*/

]

$T^\dagger$: generalized inverse (Moore-Penrose inverse) $\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \cdot\ T^\dagger : \mathcal{K} \to \mathcal{H} \\ \cdot\ T^\dagger y = P_{C_y}(0) \end{cases}$

*Proposition 3.28.

[$\mathcal{K}$: real Hilbert space

$T: \in \mathcal{B}(\mathcal{H},\mathcal{K})$, ran$(T)$: closed] $\Rightarrow$

(i) $\forall_{y \in \mathcal{K}}\ \{x \in \mathcal{H} \mid T^* T x = T^* y\} \cap (\ker T)^\perp = \{T^\dagger y\}$

(ii) $P_{\operatorname{ran} T} = T T^\dagger$

(iii) $P_{\ker T} = I - T^* T^{*\dagger}$

(iv) $T^\dagger \in \mathcal{B}(\mathcal{K}, \mathcal{H})$

(v) $\operatorname{ran} T^\dagger = \operatorname{ran} T^*$

(vi) $P_{\operatorname{ran} T^\dagger} = T^\dagger T$

*Proposition 3.29. (Moore-Desoer-Whalen)

[$\mathcal{K}$: real Hilbert space,

$T: \in \mathcal{B}(\mathcal{H},\mathcal{K})$, ran$(T)$: closed,

$\tilde{T}: \in \mathcal{B}(\mathcal{K},\mathcal{H})$, ran $\tilde{T}$: closed]

$\tilde{T} = T^\dagger \Leftrightarrow T\tilde{T} = P_{\operatorname{ran} T} \wedge \tilde{T} T = P_{\operatorname{ran} \tilde{T}} \Leftrightarrow \tilde{T} T|_{(\ker T)^\perp} = I \wedge \tilde{T}|_{(\operatorname{ran} T)^\perp} = 0$

* Corollary 3.30.

[$\mathcal{K}$: real Hilbert space,

$T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, ran T : closed] $\Rightarrow$

(i) $T^{\dagger\dagger} = T$

(ii) $T^{\dagger *} = T^{*\dagger}$

Chapter 3: Part 2

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Hilbert space: metric space ⇒ for strong topology closedness and sequential closedness are same

Theorem 3.32.

[C: convex subset of $\mathcal{H}$]

(i) C: weakly sequentially closed /* C: weakly sequentially closed $\overset{\text{def}}{\Leftrightarrow}$ weak limit of every weakly convergent sequence $\underset{\text{in } C}{\underbrace{}}$ in C is also in C */

⇔ (ii) C: sequentially closed /* C: sequentially closed $\overset{\text{def}}{\Leftrightarrow}$ limit of every convergent sequence $(x_n)_{n\in\mathbb{N}}$ is in C */

⇔ (iii) C: closed /* C: closed $\overset{\text{def}}{\Leftrightarrow}$ (strong) limit of every $\overset{\text{(strong)}}{\text{convergent}}$ net that lies in C is also in C */

⇔ (iv) C: weakly closed /* C: weakly closed $\overset{\text{def}}{\Leftrightarrow}$ weak limit of every weakly convergent net that lies in C is also in C */

Proof: Suppose C: nonempty, else it is trivial.

(i) ⇒ (ii):

/*
(one characterization of strong convergence) *
Corollary 2.42. [$(x_n)_{n\in\mathbb{N}}: \subseteq \mathcal{H};\ x\in\mathcal{H}$] $x_n \to x \Leftrightarrow (x_n \rightharpoonup x \wedge \|x_n\| \to \|x\|)$ */

C: weakly sequentially closed

$\Leftrightarrow \forall_{(x_n)_{n\in\mathbb{N}} \subseteq C:\, x_n \rightharpoonup x}\ x \in C$ // so $\|x\|$: finite ⇒ $\|x_n\| \to \|x\|$ // as in norms $\in\mathbb{R}$: finite dimensional weak convergence = strong convergence

$\therefore (x_n \rightharpoonup x \wedge \|x_n\| \to \|x\|)$

$\Leftrightarrow x_n \to x$

$\therefore \forall_{(x_n)_{n\in\mathbb{N}} \subseteq C:\, x_n \to x}\ x\in C \Leftrightarrow C$: sequentially closed. ☺

(ii) ⇔ (iii)

In Hilbert space closedness ⇔ sequential closedness
continuity ⇔ sequential continuity
lower semicontinuity ⇔ sequential lower semicontinuity.
compactness ⇔ sequential compactness

So, C: sequentially closed ⇔ C: closed ☺

/*
/* Charecterization of projection on closed convex nonempty set */ Theorem 3.14. ★★★
(C: nonempty closed convex subset of $\mathcal{H}$) ⇒ $\begin{cases} \bullet\ C\text{: Chebyshev set, i.e., every point in } \mathcal{H} \text{ has exactly one projection on } C \\ \bullet\ \forall_{x\in\mathcal{H}}\ (p = P_C(x) \Leftrightarrow (p\in C, \forall_{y\in C} \langle y-p \mid x-p\rangle \leq 0)) \end{cases}$
*/

(iii) ⇒ (iv)

C: closed $\overset{\text{def}}{\Leftrightarrow} \forall_{(\tilde{x}_a)_{a\in A}: \subseteq C,\ \tilde{x}_a \to \tilde{x}}\ \tilde{x} \in C$: given

? C: weakly closed $\overset{\text{def}}{\Leftrightarrow} \forall_{\underbrace{(x_a)_{a\in A} \subseteq C:\ x_a \rightharpoonup x}_{\text{given}}} \underbrace{x\in C \Leftrightarrow x = P_C x}_{\text{goal}}$

$P_C x$ will satisfy

$\forall_{z\in C} \langle z - P_C x \mid x - P_C x\rangle \leq 0$

as $(x_a)_{a\in A} \subseteq C$ ⇒ $\forall_{a\in A} \langle x_a - P_C x \mid x - P_C x\rangle \leq 0$ ⇒ $\overline{\lim} \langle x_a - P_C x \mid x - P_C x\rangle \leq 0$

/* using
*Fact 1.3.1.
(comes handy in dealing with sequences)
• $\forall_{n\in\mathbb{N}}\ a_n \leq a \Rightarrow \overline{\lim}\, a_n \leq a$
• $\forall_{n\in\mathbb{N}}\ b \leq b_n \Rightarrow b \leq \underline{\lim}\, b_n$
• $\forall_{n\in\mathbb{N}}\ a_1 \leq a_n \leq a_2 \Rightarrow$ $a_1 \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq a_2$
$\underline{\lim}\, a_n$ exists in finite norm
$\overline{\lim}\, a_n$ exists in finite norm
*/

now $x_a \rightharpoonup x$, $P_C x \to P_C x$ ⇒ $(x_a - P_C x) \rightharpoonup (x - P_C x)$ and $x - P_C x \to x - P_C x$ } $\langle x_a - P_C x \mid x - P_C x\rangle \to \langle x - P_C x \mid x - P_C x\rangle = \|x - P_C x\|^2$

using

$\therefore \lim \langle x - P_C x \mid x - P_C x\rangle = \overline{\lim} \langle x - P_C x \mid x - P_C x\rangle = \|x - P_C x\|^2$

/* if we have one weakly convergent bounded net, and one strongly convergent net with same index set, then the inner product "net" strongly converges */
** |
Lemma 2.36.
[$(x_a)_{a\in A}$: net in $\mathcal{H}$, bounded, $x_a \rightharpoonup \underset{\in\mathcal{H}}{x}$
$(u_a)_{a\in A}$: net in $\mathcal{H}$, $u_a \to \underset{\in\mathcal{H}}{u}$] ⇒ $\langle x_a \mid u_a\rangle \to \langle x \mid u\rangle$

$0 \leq \|x - P_C x\|^2 \leq 0 \Leftrightarrow \|x - P_C x\|^2 = 0 \Leftrightarrow x = P_C x \Leftrightarrow x \in C$ [goal achieved]

$\therefore C$: weakly closed. ☺

(iv) ⇒ (i):

any net is a sequence

$\therefore C$: weakly closed ⇒ C: weakly sequentially closed.

■

Theorem 3.33.

[C: bounded, closed, convex subset of $\mathcal{H}$] ⇒

C: weakly compact, weakly sequentially compact /* C: weakly compact $\overset{\text{def}}{\Leftrightarrow}$ Every net in C has a weak cluster point in C

C: weakly sequentially compact $\overset{\text{def}}{\Leftrightarrow}$ Every sequence in C has a weak sequential cluster point in C */

Proof:

C: weakly compact, weakly sequentially compact /* C: weakly compact $\overset{\text{def}}{\Leftrightarrow}$ Every net in C has a weak cluster point in C

C: weakly sequentially compact $\overset{\text{def}}{\Leftrightarrow}$ Every sequence in C has a weak sequential cluster point in C */

Proof:

C: closed, convex /* now: Theorem 3.32. /* this is a very important theorem which says that for a convex set all the different types of closedness coincides */

[C: convex subset of $\mathcal{H}$]

C: weakly sequentially closed $\Leftrightarrow$ C: sequentially closed $\Leftrightarrow$ C: closed $\Leftrightarrow$ C: weakly closed */

$\Rightarrow$ C: weakly closed

C: bounded : given

/* now, Fact 2.29. [$C \subseteq \mathcal{H}$] C: weakly compact $\Leftrightarrow$ C: weakly closed, bounded */

C: weakly compact /* now using (Eberlein-Šmulian) theorem: Fact 2.30 (Eberlein-Šmulian) **

[C: subset of $\mathcal{H}$]

C: weakly compact $\Leftrightarrow$ C: weakly sequentially compact */

$\Updownarrow$

C: weakly sequentially compact.

■

Proposition 3.35.

C: convex subset of $\mathcal{H}$ $\Rightarrow$

$\forall_{x \in \operatorname{int} C} \ \forall_{y \in \bar{C}} \ [x,y[\subseteq C$ /* $[x,y[= \{(1-\alpha)x+\alpha y \mid 0 \leq \alpha < 1\}$ */

Proposition 3.36.

[C: convex subset of $\mathcal{H}$] $\Rightarrow$

(i) $\bar{C}$: convex

(ii) int C: convex

(iii) $\operatorname{int} C \neq \emptyset \Rightarrow \begin{cases} \bullet \ \operatorname{int} C = \operatorname{int} \bar{C} \\ \bullet \ \bar{C} = \overline{\operatorname{int} C} \end{cases}$

3.4. Seperation

Definition 3.37. (seperated sets)

[C, D : subsets of $\mathcal{H}$]

C, D: seperated $\overset{\text{def}}{\Leftrightarrow} \exists_{u \in \mathcal{H} \setminus \{0\}} \ \sup \langle C|u \rangle \leq \inf \langle D|u \rangle$ /* $\langle C|u\rangle = \{\langle x|u\rangle \mid x \in C\}$

$\langle D|u\rangle = \{\langle y|u\rangle \mid y \in C\}$ */

C, D: strongly seperated $\overset{\text{def}}{\Leftrightarrow} \exists_{u \in \mathcal{H} \setminus \{0\}} \ \sup \langle C|u\rangle < \inf \langle D|u \rangle$

$\underbrace{x}_{\text{a point in } \mathcal{H}}$: seperated from D $\overset{\text{def}}{\Leftrightarrow} \exists_{u \in \mathcal{H} \setminus \{0\}} \ \sup \langle x|u\rangle = \langle x|u\rangle \leq \inf \langle D|u\rangle$

x : strongly seperated from D $\overset{\text{def}}{\Leftrightarrow} \exists_{u \in \mathcal{H} \setminus \{0\}} \ \langle x|u\rangle < \inf \langle D|u\rangle$

Theorem 3.38.

[C: nonempty closed convex subset of $\mathcal{H}$

$x \in \mathcal{H} \setminus C$]

x : strongly seperated from C

/* Two corollaries for strong seperateness of two sets */

Corollary 3.39.

[C, D : nonempty subsets of $\mathcal{H}$, $C \cap D = \emptyset$

C−D: closed, convex] /* $C-D = \{x-y \mid x \in C, y \in D\}$ */

C, D: strongly seperated

Corollary 3.40.

[C, D: nonempty closed convex subsets of $\mathcal{H}$,

$C \cap D = \emptyset$

D: bounded

[C, D : nonempty closed convex subsets of $\mathcal{H}$,
$C \cap D = \emptyset$
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C, D : strongly seperated

Proof:

sketch we show $C-D$: closed and convex $\overset{\text{Corollary 3.39}}{\Rightarrow}$ C, D strongly seperated

now C, D : nonempty closed convex

/* Proposition 3.6 (ii) : [$(C_i)_{i\in I}$: totally ordered finite family of m convex sets, $\subseteq \mathcal{H}$]

$\Rightarrow \forall_{(\alpha_i)_{i\in I} : \forall_{i\in I} \alpha_i \in \mathbb{R}} \; \sum_{i\in I} \alpha_i C_i$: convex */

so for $\underset{I=\{1,2\}}{\alpha = (1,-1)}$, $C_1 = C$, $C_2 = D$ we have $\sum_{i\in I} \alpha_i C_i = C_1 - C_2 = C - D$: convex ✓

now we show $C-D$: closed $\Leftrightarrow$ $C-D$: weakly sequentially closed /* means weak limit of every convergent sequence in $C-D$ is also in $C-D$ */

[as we have shown $C-D$: convex, which we can
use as a given now ($\because$ $(P \land P \Rightarrow Q) \Leftrightarrow (P \land Q)$),
and for a convex set all types of closedness coincides */

take a convergent sequence $(x_n - y_n)_{n\in\mathbb{N}}$: $(x_n)_{n\in\mathbb{N}} \subseteq C$, $(y_n)_{n\in\mathbb{N}} \subseteq C$, $x_n - y_n \to \underbrace{z}_{\in \mathcal{H}}$

/* we can always construct
such a sequence, the question
is whether the limit is also in
$C-D$ */

given, D : bounded, closed, convex, $\subseteq \mathcal{H}$ $\Rightarrow$ $\begin{cases} \bullet\ D : \text{weakly compact} \\ \bullet\ D : \text{weakly sequentially compact} \end{cases}$ using

Theorem 3.33.
[C : bounded, closed, convex subset of $\mathcal{H}$]
C : weakly compact, weakly sequentially compact /* C : weakly compact $\overset{\text{def}}{\Leftrightarrow}$ Every net in has a weak cluster point in C
C : weakly sequentially compact $\overset{\text{def}}{\Leftrightarrow}$ Every sequence in C has a weak sequential cluster point in C
x : (strong) cluster point of $(x_a)_{a\in A}$ $\overset{\text{def}}{\Leftrightarrow}$ $(x_a)_{a\in A}$ has a subnet that (strongly) converges to $x \in \mathcal{H}$
x : weak cluster point of $(x_a)_{a\in A}$ $\overset{\text{def}}{\Leftrightarrow}$ $(x_a)_{a\in A}$ has a subnet that weakly converges to $x \in \mathcal{H}$ */

$\therefore$ D : weakly sequentially compact

$\Leftrightarrow \exists_{(y_{k_n})_{n\in\mathbb{N}} : \text{subsequence of } (y_n)_{n\in\mathbb{N}}}$ $(y_{k_n} \rightharpoonup y \land y \in D)$

as $x_n - y_n \to z$

$\Rightarrow$ any subsequence $x_{k_n} - y_{k_n} \to z$ /* as • Hilbert space is a Hausdorff space
• In a Hausdorff space every subnet of a convergent net converges to the same point */

$(z_{k_n})_{n\in\mathbb{N}} = (x_{k_n} - y_{k_n})_{n\in\mathbb{N}}$

$(x_{k_n})_{n\in\mathbb{N}} \subseteq C$ $\quad (y_{k_n})_{n\in\mathbb{N}} \subseteq D$

$\Downarrow$

$x_{k_n} \rightharpoonup z+y$ $\Rightarrow$ $\overbrace{(x_{k_n})_{n\in\mathbb{N}}}^{\subseteq C}$: weakly convergent to $z+y$ using /* $(a_n \to a,\ b_n \to b) \Rightarrow a_n + b_n \to a + b$ */

But $\underbrace{C}_{\text{convex}}$: closed $\Rightarrow$ C : $\underbrace{\text{weakly sequentially closed}}_{\text{weak limit of every weakly convergent sequence in } C \text{ is also in } C.}$

/* Theorem 3.32. /* this is a very important theorem which says that for a convex set all the different types of closedness coincides */
[C : convex subset of $\mathcal{H}$]
C : weakly sequentially closed $\Leftrightarrow$ C : sequentially closed $\Leftrightarrow$ C : closed $\Leftrightarrow$ C : weakly closed */

$z + \underbrace{y}_{\in D} \in C$

$\therefore$ $z \in C - D \Rightarrow$ closed

$\therefore$ (C, D : nonempty, $\subseteq \mathcal{H}$,
$C \cap D = \emptyset$
$C-D$: closed, convex)

$\overset{\text{Corollary 3.39}}{\Rightarrow}$

(C, D) : strongly seperated ∎