Part 1

1:06 PM

- Haugazeau's Q operator:

$Q: \mathcal{H}^3 \to \mathcal{H}$ /* $\chi = \langle x-y \mid y-z \rangle$; $\mu = \|x-y\|^2$; $\nu = \|y-z\|^2$; $\rho = \mu\nu - \chi^2$ */

$$: (x,y,z) \mapsto \begin{cases} z, & \text{if } \rho = 0,\ \chi \geq 0 \\ x + \left(1+\frac{\chi}{\nu}\right)(z-y), & \text{if } \rho > 0,\ \chi\nu \geq \rho \\ y + \frac{\nu}{\rho}\left(\chi(x-y) + \mu(z-y)\right), & \text{if } \rho > 0,\ \chi\nu < \rho \end{cases}$$

(Strongly convergent proximal point algorithm that finds the zero of a maximally monotone operator at minimal distance from the starting point)

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, $0 \in \operatorname{ran} A$ /* i.e., $\operatorname{zer} A \neq \emptyset$ */;

$x_0 \in \mathcal{H}$; $\forall_{n \in \mathbb{N}}\ x_{n+1} = Q(x_0, x_n, J_A x_n)$] $x_n \to \underbrace{P_{\operatorname{zer} A}}_{\text{this is the zero of } A \text{ that is closest to } x_0} x_0$

Proof:

set $T_1 = J_A$ // A: maximally monotone $\Rightarrow J_A$: firmly nonexpansive (Corollary 23.8)

T_1: firmly nonexpansive

$\Rightarrow \operatorname{zer} A = \operatorname{Fix} T_1 = \operatorname{Fix} J_A \neq \emptyset$

now: from Haugazeau's algorithm:

(Haugazeau's algorithm to find the best approximation to a point from the set of common fixed points of firmly nonexpansive operator) ~ set $I = \{1\}$, $\tilde{i}(\underbrace{n}_{\in\mathbb{N}}) = 1$,

Theorem 29.3. [$(T_i)_{i \in I}$: finite family of firmly nonexpansive operators from $\mathcal{H}$ to $\mathcal{H}$;

$C = \bigcap_{i \in I} \operatorname{Fix} T_i \neq \emptyset$ $= \operatorname{Fix} T_1 = \operatorname{zer} A$; $x_0 \in \mathcal{H}$; $\tilde{i}: \mathbb{N} \to I$: $\exists_{m \in \mathbb{N}_{++}} \forall_{i \in I} \forall_{n \in \mathbb{N}}\ i \in \{\tilde{i}(n), \dots, \tilde{i}(n-m+1)\}$; satisfied

$\forall_{n \in \mathbb{N}}\ x_{n+1} = Q(x_0, x_n, \underbrace{T_{\tilde{i}(n)} x_n}_{T_1 x_n = J_A x_n})$] $x_n \to P_C x_0 = P_{\operatorname{zer} A} x_0$

$\therefore x_n \to P_{\operatorname{zer} A} x_0$

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(Strongly convergent forward-backward algorithm) /* of course can be applied to solving VI problem */

Corollary 29.5. [$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone; $\beta \in \mathbb{R}_{++}$; $B: \mathcal{H} \to \mathcal{H}$, β-cocoercive; $\gamma \in]0, 2\beta[$;

$\operatorname{zer}(A+B) \neq \emptyset$; $x_0 \in \mathcal{H}$; set:

$$\forall_{n \in \mathbb{N}} \left| \begin{array}{l} y_n = x_n - \gamma B x_n \\ z_n = \frac{1}{2}(x_n + J_{\gamma A} y_n) \\ x_{n+1} = Q(x_0, x_n, z_n) \end{array} \right. \quad (29.41)$$

] $\Rightarrow x_n \to \underbrace{P_{\operatorname{zer}(A+B)}}_{\text{zero of } (A+B) \text{ that is closest to } x_0} x_0$

Proof: from (eq: 29.41): $y_n = x_n - \gamma B x_n$

$z_n = \frac{1}{2}(x_n + J_{\gamma A} y_n) = \frac{1}{2}(x_n + \underbrace{J_{\gamma A}(x_n - \gamma B x_n)}_{(J_{\gamma A} \circ (\mathrm{Id} - \gamma B))(x_n)}) = \frac{1}{2}(x_n + (J_{\gamma A} \circ (\mathrm{Id} - \gamma B)) x_n) = \underbrace{\frac{1}{2}(\mathrm{Id} + J_{\gamma A} \circ (\mathrm{Id} - \gamma B))}_{= T_1 \text{ //say}} x_n = T_1 x_n$

now T_1: firmly nonexpansive

/* A: maximally monotone $\Rightarrow J_{\gamma A}$: firmly nonexpansive $\Rightarrow J_{\gamma A}$: nonexpansive

B: β-cocoercive $\Rightarrow (\mathrm{Id} - \gamma B)$: nonexpansive /* as

Proposition 4.33.
[D: nonempty subset of $\mathcal{H}$
$\beta \in \mathbb{R}_{++}$
$T: D \to \mathcal{H}$, β-cocoercive
$\gamma \in]0, 2\beta[$]
$(\mathrm{Id} - \gamma T)$: $\frac{\gamma}{2\beta}$-averaged

and averaged operators are nonexpansive */

$\therefore J_{\gamma A} \circ (\mathrm{Id} - \gamma B)$: nonexpansive

now set

$2\tilde{T} - \mathrm{Id} = J_{\gamma A} \circ (\mathrm{Id} - \gamma B)$: nonexpansive

$\Leftrightarrow T_1 = \tilde{T} = \frac{1}{2}(\mathrm{Id} + J_{\gamma A} \circ (\mathrm{Id} - \gamma B))$

: firmly nonexpansive

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this is a very important theorem ★★★★★

Proposition 4.4 (Different representations of firmly nonexpansive operator)
[D: nonempty subset of $\mathcal{H}$,
$T: D \to \mathcal{H}$]
(i) T: firmly nonexpansive $\Leftrightarrow$ (ii) $(\mathrm{Id} - T)$: firmly nonexpansive $\Leftrightarrow$ (iii) $(2T - \mathrm{Id})$: nonexpansive
$\Leftrightarrow$ (iv) $\forall_{x \in D} \forall_{y \in D}\ \|Tx - Ty\|^2 \leq \langle x - y \mid Tx - Ty \rangle$
$\Leftrightarrow$ (v) $\forall_{x \in D} \forall_{y \in D}\ \langle Tx - Ty \mid (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \rangle \geq 0$
$\Leftrightarrow$ (vi) $\forall_{x \in D} \forall_{y \in D} \forall_{\alpha \in [0,1]}\ \|Tx - Ty\| \leq \|\alpha(x - y) + (1 - \alpha)(Tx - Ty)\|$

$x_{n+1} = Q(x_0, x_n, z_n) = Q(x_0, x_n, T_1 x_n)$

Proposition 25.1. [$A: \mathcal{H} \to 2^{\mathcal{H}}$, $B: \mathcal{H} \to 2^{\mathcal{H}}$, $\gamma \in \mathbb{R}_{++}$] $\Rightarrow$

(i) $\operatorname{zer}(A+B) = \operatorname{dom}(A \cap (-B))$

(ii) A, B: monotone $\Rightarrow \operatorname{zer}(A+B) = J_{\gamma B}(\operatorname{Fix} R_{\gamma A} R_{\gamma B})$ // Note: very general, can it be used in nonconvex setup?

(iii) (C: closed affine subspace of $\mathcal{H}$; $V := C - C$; $A = N_C$) $\Rightarrow$

$\operatorname{zer}(A+B) = \{x \in C \mid V^\perp \cap Bx \neq \emptyset\}$

(iv) (A: monotone, $\underset{B:\ \beta\text{-cocoercive}}{B: \text{almost single-valued}}$) $\Rightarrow \operatorname{zer}(A+B) = \operatorname{Fix} \underbrace{J_{\gamma A} \circ (\mathrm{Id} - \gamma B)}_{2T_1 - \mathrm{Id} \text{ // in our case}}$

$\therefore \operatorname{zer}(A+B) = \operatorname{Fix}(2T_1 - \mathrm{Id}) = \operatorname{Fix} T_1$

/* $x \in \operatorname{Fix} T \Leftrightarrow Tx \ni x$

$\Leftrightarrow 2Tx \ni 2x$

$\Leftrightarrow 2Tx - \mathrm{Id}x \ni 2x - x = x$

$\Leftrightarrow (2T - \mathrm{Id})x \ni x \Leftrightarrow x \in \operatorname{Fix}(2T - \mathrm{Id})$ */

(Haugazeau's algorithm to find the best approximation to a point from the set of common fixed points of firmly nonexpansive operator) set: $I = \{1\}$, $\tilde{i}: n \mapsto 1$,

Theorem 29.3. [$(\underset{= T_1}{T_i})_{i \in I}$: finite family of firmly nonexpansive operators from $\mathcal{H}$ to $\mathcal{H}$;

$C = \bigcap_{i \in I} \operatorname{Fix} T_i \neq \emptyset$ $= \operatorname{Fix} T = \operatorname{zer}(A+B)$; $x_0 \in \mathcal{H}$; $\tilde{i}: \mathbb{N} \to I$: $\exists_{\underset{m\in\mathbb{N}_{++}}{1=m}} \forall_{i \in I = \{1\}} \forall_{n \in \mathbb{N}}\ i \in \{\underbrace{\tilde{i}(n)}_{1}, \dots, \tilde{i}(n-m+1)\}$; ✓

$\forall_{n \in \mathbb{N}}\ x_{n+1} = Q(x_0, x_n, T_{\tilde{i}(n)} x_n)$] $x_n \to P_C x_0$

$x_n \to P_C x_0 = P_{\operatorname{zer}(A+B)} x_0$

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