Part 1

7:34 AM

Proposition 28.7.

$[(C_n)_{n\in\mathbb{N}}$: sequence of nonempty closed convex subsets of $\mathcal{H}$;

$\forall_{n\in\mathbb{N}}\ C_n \subseteq C_{n+1}$; $C = \overline{\bigcup_{n\in\mathbb{N}} C_n}$; $x\in\mathcal{H}]\ P_{C_n}x \to P_C x$

Proof: $\forall_{\tilde{n}\in\mathbb{N}}\ \bigcup_{n\in\tilde{N}} C_n$: convex, nonempty /* using (.) */ $\Rightarrow \bigcup_{n\in\mathbb{N}} C_n$: convex, nonempty

/* Proposition 3.36.(i) : $[C$: convex, $\subseteq\mathcal{H}]\ \overline{C}$: convex */

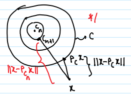
$\therefore C = \overline{\bigcup_{n\in\mathbb{N}} C_n}$: convex, nonempty /* A consequence of Theorem 1.52 · $[C$: convex subset of $\mathcal{H}]\ x \in \overline{C} \Leftrightarrow \exists_{(x_n)_{n\in\mathbb{N}}\text{ : sequence in } C}\ x_n \to x$ */

$P_C x \in C \leftrightarrow \exists_{(\tilde{y}_n)_{n\in\mathbb{N}}\subseteq C}\ \tilde{y}_n \to P_C x$

now as $C = \overline{\bigcup_{n\in\mathbb{N}} C_n}$ we can cherrypick components of $(\tilde{y}_n)_{n\in\mathbb{N}}$ and create a subsequence $(y_n)_{n\in\mathbb{N}} : y_n \in C_n$

$\therefore \exists_{(y_n)_{n\in\mathbb{N}}\subseteq C}\ (y_n \to P_C x,\ y_n \in C_n)$

$\forall_{n\in\mathbb{N}}\ \|x - P_C x\| \leqslant \|x - P_{C_n} x\|$ /* consider the picture:



*/

$\leqslant \|x - y_n\|$ /* as by definition: $\|x - P_{C_n}x\| \leqslant \|x - y\|\ \forall_{y\in C_n}$ $\overset{y:=y_n\in C}{\Rightarrow} \|x - P_{C_n}x\| \leqslant \|x - y_n\|$ */

$= \|x - P_C x + P_C x - y_n\|$

$\leqslant \|x - P_C x\| + \|y_n - P_C x\|$

sub this from all sides

$\Rightarrow \forall_{n\in\mathbb{N}}\ \|x - P_C x\| \leqslant \|x - P_{C_n}x\| \leqslant \|x - P_C x\| + \|y_n - P_C x\|$

$\Rightarrow \forall_{n\in\mathbb{N}}\ 0 \leqslant \|x - P_{C_n}x\| - \|x - P_C x\| \leqslant \|y_n - P_C x\|$

as $n\to\infty$ $\quad 0 \leqslant \|x - P_{C_n}x\| - \|x - P_C x\| \leqslant \underset{\downarrow\ 0}{\|y_n - P_C x\|}$

$\Rightarrow \|x - P_{C_n}x\| - \|x - P_C x\| \to 0$

$\therefore \boxed{\|x - P_{C_n}x\| \to \|x - P_C x\|} \Rightarrow \forall_{(P_{C_{k_n}}x)}\ \|x - P_{C_{k_n}}x\| \to \|x - P_C x\|$

$\Rightarrow \boxed{\underbrace{(P_{C_n}x)_{n\in\mathbb{N}}}_{\subseteq C} : \text{bounded}}$ /* $P_{C_n}x \in C_n \subseteq C_{n+1} \subseteq \ldots C$ */

/* now Lemma 2.37 · [illegible] */

$\underbrace{(P_{C_n}x)_{n\in\mathbb{N}}}_{\subseteq C}$: possesses a weak sequential cluster point z (say) $\Leftrightarrow \exists_{(P_{C_{k_n}}x)_{n\in\mathbb{N}}\subseteq C}\ P_{C_{k_n}} \rightharpoonup z \Leftrightarrow \boxed{P_{C_{k_n}} - x \rightharpoonup z - x} \Rightarrow \|z - x\| \leqslant \underline{\lim}\ \|P_{C_{k_n}} - x\|$

/* Lemma 2.55 · /* norm of $\mathcal{H}$: weakly lower semicontinuous */ $\forall_{(x_a)_{a\in A}\subseteq\mathcal{H}}\ \forall_{x\in\mathcal{H}}\ x_a \rightharpoonup x \Rightarrow \|x\| \leqslant \underline{\lim}\ \|x_a\|$ */

now C : convex, closed /* now Theorem 3.32 · [illegible] */

$\Rightarrow C$: sequentially closed

$\boxed{z \in C}$

now: By def

$\forall_{\tilde{x}\in C}\ \|x - P_C x\| \leqslant \|x - \tilde{x}\|$

$\overset{\tilde{x}:=z\in C}{\Rightarrow} \|x - P_C x\| \leqslant \|x - z\| \leqslant \underline{\lim}\ \|x - P_{C_{k_n}}x\| = \lim \|x - P_{C_{k_n}}x\| = \|x - P_C x\|$

$\Rightarrow \boxed{\|x - z\| = \|x - P_C x\|}$

now By def: $\|x - P_C x\| = \min_{\tilde{z}\in C} \|x - \tilde{z}\|$ and $P_C x$: unique as C : closed, convex

$\Rightarrow \boxed{P_C x = z}$ But z : sequential cluster point of an arbitrary subsequence of $(P_{C_n}x)_{n\in\mathbb{N}}$

$\Rightarrow \boxed{z = P_C x : \text{only weak sequential cluster point of } (P_{C_n}x)_{n\in\mathbb{N}}}$

$\therefore P_{C_n}x \rightharpoonup z = P_C x$ /* using Lemma 2.38 · [illegible] */

$\Rightarrow \boxed{x - P_{C_n}x \rightharpoonup x - P_C x}$ /* using (one characterization of strong convergence) Corollary 2.52 · $[(x_n)_{n\in\mathbb{N}} : \subseteq \mathcal{H};\ x\in\mathcal{H}]\ x_n \to x \Leftrightarrow (x_n \rightharpoonup x \wedge \|x_n\| \to \|x\|)$ */

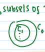
$\therefore x - P_{C_n}x \to x - P_C x$

$\Rightarrow \boxed{P_{C_n}x \to P_C x}$ ∎

Proposition 28.8.

$[(C_n)_{n\in\mathbb{N}}$: sequence of nonempty, closed, convex, subsets of $\mathcal{H}$;

$C = \bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$; $\forall_{n\in\mathbb{N}}\ C_{n+1} \subseteq C_n$; $x\in\mathcal{H}]\ P_{C_n}x \to P_C x$



seems very intuitive

Proof: $\forall_{n\in\mathbb{N}}\ P_n := P_{C_n}x$

$\forall_{n\in\mathbb{N}}\ \|x - P_n\| \leqslant \|x - P_{n+1}\| \leqslant \|x - P_C x\|$ /*

*/

$\Rightarrow (P_n)_{n\in\mathbb{N}}$: bounded

Theorem 5.11 · [illegible]

now $(\|x - P_n\|)_{n\in\mathbb{N}}$: monotonic, bounded $\Rightarrow (\|x - P_n\|)_{n\in\mathbb{N}}$: converges

$\forall_{m,n\in\mathbb{N} : m<n}\ \frac{1}{2}P_m + \frac{1}{2}P_n \in C_m$ /* $n \geqslant m \Rightarrow C_n \subseteq C_m$ (given) by def: $P_m \in C_m$, $P_n \in C_n \subseteq C_m$ now C_m : convex $\Rightarrow \frac{1}{2}P_m + \frac{1}{2}P_n \in C_m$ */

now:

$\|P_n - P_m\|^2$

$= 2(\|P_n - x\|^2 + \|P_m - x\|^2) - 4\|x - \frac{P_n + P_m}{2}\|^2$ /* Lemma 2.12 · [illegible] */

/* now: $P_m = P_{C_m}x$:

$\forall_{y\in C_m}\ \|x - P_m\| \leqslant \|x - y\|$ /* By def */

$\overset{y:=(P_n+P_m)/2\in C_m}{\Rightarrow} \|x - P_m\| \leqslant \|x - \frac{P_n + P_m}{2}\|$

$\leftrightarrow \|x - P_m\|^2 \leqslant \|x - \frac{P_n + P_m}{2}\|^2$

$\leftrightarrow -4\|x - P_m\|^2 \geqslant -4\|x - \frac{P_n + P_m}{2}\|^2$ */

$\leqslant 2\|P_n - x\|^2 + 2\|P_m - x\|^2 - 4\|x - P_m\|^2$

$= 2(\|P_n - x\|^2 - \|P_m - x\|^2)$ /* now as $(\|P_n - x\|)_{n\in\mathbb{N}}$ converges $\Rightarrow$ as $n,m\to\infty$, $\|P_n - x\|^2 - \|P_m - x\|^2 \to 0$ */

$\therefore$ as $m,n \to +\infty$, $\|P_n - P_m\|^2 \to 0$

$\Leftrightarrow (P_n)_{n\in\mathbb{N}}$: Cauchy sequence in $\mathcal{H}$ /* $\mathcal{H}$: complete metric space */

$\therefore \exists_{p\in\mathcal{H}}\ P_n \to p$

$\forall_{n\in\mathbb{N}}\ \underset{\downarrow\ p}{(P_k)_{k\geqslant n}}$ lies in $C_n \Rightarrow p \in C_n$ /* C_n : closed, so the limit point of any convergent sequence will lie in C_n */ /* as $(P_k)_{k\geqslant n}$: subsequence of $(P_n)_{n\in\mathbb{N}}$ */

now, as $C = \bigcap_{n\in\mathbb{N}} C_n \Rightarrow p \in C$

$\therefore$ By definition $\|x - P_C x\| \leqslant \|x - \tilde{p}\|\ \forall_{\tilde{p}\in C} \overset{\tilde{p}:=p}{\Rightarrow} \|x - P_C x\| \leqslant \|x - p\| = \lim \|x - P_n\| \leqslant \|x - P_C x\|$

$\Rightarrow \|x - P_C x\| = \|x - p\|$, But $P_C x$: unique minimizer of $\min_{y\in C} \|x - y\|$

$\therefore p = P_C x$

$\therefore P_{C_n}x \to p = P_C x$

∎

Proposition 28.28 · $[f : \mathcal{H} \to \mathbb{R}$, convex, continuous

$C = \operatorname{epi} f$

$(z,\zeta) \in (\mathcal{H}\times\mathbb{R})\setminus C] \Rightarrow$

- $z \in x + (f(x) - \zeta)\,\partial f(x)$: has unique solution (say $\bar{x}$)
- $P_C(z,\zeta) = (\bar{x}, f(\bar{x}))$

$(z - x, \zeta - \xi)$ /* as, $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2$ */

Proof:

$(z,\zeta) \in (\mathcal{H}\times\mathbb{R})\setminus C \,] \Rightarrow$

- $z \in x + (f(x)-\zeta)\,\partial f(x)$: has unique solution (say $\bar{x}$)
- $P_C(z,\zeta) = (\bar{x}, f(\bar{x}))$

Proof:

$$(\bar{x},\bar{\xi}) = P_C(z,\zeta) \overset{\text{def}}{\Leftrightarrow} (\bar{x},\bar{\xi}) = \begin{pmatrix} \operatorname{argmin} & \frac{1}{2}\|\underbrace{(z,\zeta)-(x,\xi)}_{(z-x,\,\zeta-\xi)}\|^2 = \frac{1}{2}\|x-z\|^2 + \frac{1}{2}\|\zeta-\xi\|^2 = \frac{1}{2}\|x-z\|^2 + \frac{1}{2}|\xi-\zeta|^2 \\ \text{s.t} & (x,\xi)\in C = \operatorname{epi} f \Leftrightarrow f(x)\leq \xi \end{pmatrix}$$

/* as, $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2$ */

Lagrangian: $L((x,\xi),\nu) = \frac{1}{2}\|x-z\|^2 + \frac{1}{2}|\xi-\zeta|^2 + \nu(f(x)-\xi)$

if we apply KKT condition: then

primal feasibility: $f(\bar{x}) \leq \bar{\xi}$

dual feasibility: $\bar{\nu} \geq 0$

complementary slackness: $\bar{\nu}(f(\bar{x})-\bar{\xi}) = 0$

minimizer of the Lagrangian: $\partial_{(\bar{x},\bar{\xi})} L((\bar{x},\bar{\xi}),\bar{\nu}) \ni 0$

$$\Leftrightarrow \begin{bmatrix} \frac{\partial}{\partial \bar{x}} \\ \frac{\partial}{\partial \bar{\xi}} \end{bmatrix} \left(\frac{1}{2}\|\bar{x}-z\|^2 + \frac{1}{2}|\bar{\xi}-\zeta|^2 + \bar{\nu}(f(\bar{x})-\bar{\xi}) \right) = \begin{bmatrix} \bar{x}-z + \bar{\nu}\,\partial f(\bar{x}) \\ \bar{\xi}-\zeta-\bar{\nu} \end{bmatrix} \ni \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Updownarrow$

$\boxed{\bar{x}-z+\bar{\nu}\,\partial f(\bar{x}) \ni 0,\ \bar{\xi}-\zeta-\bar{\nu}=0}$

assume, $\bar{\nu}=0 \Rightarrow \bar{x}=z,\ \bar{\xi}=\zeta \Leftrightarrow \underbrace{(\bar{x},\bar{\xi})}_{\in C} = \underbrace{(z,\zeta)}_{\notin C} \Rightarrow$ contradiction $\Rightarrow \bar{\nu}\neq 0 \Rightarrow \bar{\nu}>0 \Rightarrow$ then using complementary slackness: $f(\bar{x})-\bar{\xi}=0 \Leftrightarrow \boxed{\bar{\xi}=f(\bar{x})}$ $\therefore \bar{\nu} = f(\bar{x})-\zeta$

$z-\bar{x} \in \bar{\nu}\,\partial f(\bar{x}) = (f(\bar{x})-\zeta)\,\partial f(x)$

$\Leftrightarrow \boxed{z \in \bar{x} + (f(\bar{x})-\zeta)\,\partial f(\bar{x})}$ now $\bar{x}$: unique as the Lagrangian is strongly convex in x

$\therefore P_C(z,\zeta) = (\bar{x}, f(\bar{x}))$: ($\bar{x}$: solves $z \in x + (f(x)-\zeta)\,\partial f(x)$ uniquely) ∎

Proposition 28.30. [$f: \mathcal{H}\to\mathbb{R}$, convex, continuous ; $z\in\mathcal{H}$; $\zeta \in\,]\inf f(\mathcal{H}), f(z)[$; $C = \operatorname{lev}_{\leq\zeta} f$] $\Rightarrow$

- $f(\operatorname{Prox}_{\underbrace{\nu}_{\text{variable}} f} z) = \zeta$: has atleast one solution in $\mathbb{R}_{++}$ say $\bar{\nu}$
- $P_C z = \operatorname{Prox}_{\bar{\nu} f} z$

Proof: Take $z\notin C$, as otherwise it is just $P_C z = z$, trivial. say $\bar{x} = P_C z$

By definition, $P_C z = \begin{pmatrix} \operatorname{argmin}_{x\in\mathcal{H}} & \frac{1}{2}\|x-z\|^2 \\ \text{s.t.} & x\in C = \operatorname{lev}_{\leq\zeta} f \Leftrightarrow f(x)\leq\zeta \end{pmatrix}$ Lagrangian:

$L(x,\nu) = \frac{1}{2}\|x-z\|^2 + \nu(f(x)-\zeta)$ // convex and proper

KKT conditions: for primal-dual pair $(\bar{x},\bar{\nu}) = (P_C z, \bar{\nu})$

Primal feasibility: $f(\bar{x})\leq\zeta$

Dual feasibility: $\bar{\nu}\geq 0$

Complementary slackness: $\bar{\nu}(f(\bar{x})-\zeta) = 0$

Vanishing gradient at the Lagrangian: $\nabla_{\bar{x}} L(\bar{x},\bar{\nu}) = 0$

$\Updownarrow$

$(\bar{x}-z) + \bar{\nu}\,\partial f(\bar{x}) \ni 0$ // there is atleast one solution

suppose $\bar{\nu}=0 \Rightarrow \underbrace{\bar{x}}_{= P_C z \in C} = \underbrace{z}_{\notin C} \Rightarrow$ contradiction $\therefore \bar{\nu}>0 \Rightarrow f(\bar{x}) = f(P_C z) = \zeta$

$\bar{x} - z + \bar{\nu}\,\partial f(\bar{x}) \ni 0$

$\Leftrightarrow \bar{\nu}\,\partial f(\bar{x}) \ni z-\bar{x} \Leftrightarrow \partial(\bar{\nu} f(\bar{x})) \ni z-\bar{x}$ // $x-p\in\partial f(p) \Leftrightarrow p = \operatorname{Prox}_f x$

$\Leftrightarrow \boxed{\bar{x} = \operatorname{Prox}_{\bar{\nu} f} z = P_C z}$ //the primal variable has been written in terms of the dual // variable

$f(\bar{x}) = f(P_C z) = f(\operatorname{Prox}_{\bar{\nu} f} z) = \zeta$ $\therefore f(\operatorname{Prox}_{\underbrace{\nu}_{\text{variable}} f} z) = \zeta$. must have atleast one solution in $\mathbb{R}_{++}$, and if $\bar{\nu}$ is such a solution then: $\boxed{P_C z = \operatorname{Prox}_{\bar{\nu} f} z}$ ∎