Part 1

7:22 AM

/* required info: T: Gateaux differentiable at x ⇔ $\exists_{DT(x)\in B(\mathcal{H},\mathcal{K})} \forall_{y\in\mathcal{H}}\ DT(x)y = \lim_{\alpha\downarrow 0}\frac{T(x+\alpha y)-T(x)}{\alpha}$ */ Gateaux derivative of T at x

(Fermat's rule for the minimization of two convex functions satisfying constraint qualification)

Theorem 26.2. [$f\in\Gamma_0(\mathcal{H})$, $g\in\Gamma_0(\mathcal{K})$, $L\in B(\mathcal{H},\mathcal{K})$, $\bar{x}\in\mathcal{H}$, one of the following holds:

(a) $0\in\mathrm{sri}(\mathrm{dom}\,g - L(\mathrm{dom}\,f))$

(b) $\mathcal{K}$: finite-dimensional, g: polyhedral, and $\mathrm{dom}\,g\cap\mathrm{ri}\,L(\mathrm{dom}\,f)\neq\emptyset$

(c) $\mathcal{H},\mathcal{K}$: finite-dimensional; f,g: polyhedral, $\mathrm{dom}\,g\cap L(\mathrm{dom}\,f)\neq\emptyset$] The following are equivalent:

(i) $\bar{x}$: solution to $\min_{x\in\mathcal{H}} f(x)+g(Lx)$.

(ii) $\bar{x}\in\mathrm{zer}(\partial f + L^*\circ\partial g\circ L)$

(iii) $\exists_{v\in\partial g(L\bar{x})}\ -L^*v\in\partial f(\bar{x})$

(iv) $\exists_{v\in\partial g(L\bar{x})}\ \forall_{y\in\mathcal{H}}\ \langle \bar{x}-y\mid L^*v\rangle + f(\bar{x})\le f(y)$

Moreover, g: Gateaux differentiable at $L\bar{x}$. Then (i)–(iv) is equivalent to each of the following:

(v) $-L^*(\nabla g(L\bar{x}))\in\partial f(\bar{x})$, (vi) $\forall_{y\in\mathcal{H}}\ \langle\bar{x}-y\mid L^*(\nabla g(L\bar{x}))\rangle + f(\bar{x})\le f(y)$, (vii) $\forall_{\gamma\in\mathbb{R}_{++}}\ \bar{x}=\mathrm{Prox}_{\gamma f}(\bar{x}-\gamma L^*\nabla g(L\bar{x}))$

Proof:

(i)⇔(ii):

$\bar{x}$: solution to $\min_{x\in\mathcal{H}} f(x)+g(Lx)$

⇔ $\bar{x}\in\mathrm{Argmin}(f+g\circ L)$ /* Proposition 26.1: $\mathrm{Argmin}\,f = \mathrm{zer}\,\partial f = \partial f^*(0) = \mathrm{Fix}\,\mathrm{Prox}_f = \mathrm{zer}\,\mathrm{Prox}_{f^*}$ */ $(f\in\Gamma(\mathcal{H}))$

⇔ $\bar{x}\in\mathrm{zer}(\partial(f+g\circ L))$

$\partial f + L^*\circ\partial g\circ L$ /* using

Theorem 16.47: [$\mathcal{K}$: real Hilbert space, $f\in\Gamma_0(\mathcal{H})$, $g\in\Gamma_0(\mathcal{K})$, $L\in B(\mathcal{H},\mathcal{K})$; one of the following holds:

(a) $0\in\mathrm{sri}(\mathrm{dom}\,g - L(\mathrm{dom}\,f))$

(b) $\mathcal{K}$: finite-dimensional; g: polyhedral; $\mathrm{dom}\,g\cap\mathrm{ri}\,L(\mathrm{dom}\,f)\neq\emptyset$

(c) $\mathcal{H},\mathcal{K}$: finite-dimensional; f,g: polyhedral; $\mathrm{dom}\,g\cap L(\mathrm{dom}\,f)\neq\emptyset$] ⇒ $\partial(f+g\circ L) = \partial f + L^*\circ(\partial g)\circ L$ */

⇔ $\bar{x}\in\mathrm{zer}(\partial f + L^*\circ\partial g\circ L)$:)

(ii)⇔(iii):

$\bar{x}\in\mathrm{zer}(\partial f + L^*\circ\partial g\circ L)$

⇔ $(\partial f + L^*\circ\partial g\circ L)(\bar{x}) = \partial f(\bar{x}) + L^*(\underbrace{\partial g(L\bar{x})}_{\exists_{v\in\partial g(L\bar{x})}})\ni 0$ ⇔ $\exists_{v\in\partial g(L\bar{x})}\ \partial f(\bar{x}) + \underbrace{L^*(v)}\ni 0$ ⇔ $\exists_{v\in\partial g(L\bar{x})}\ \partial f(\bar{x})\ni -L^*v$:)

L^*: linear ⇒ singlevalued ⇒ L^*v: single vector

(iii)⇔(iv)

$\exists_{v\in\partial g(L\bar{x})}\ -L^*v\in\partial f(\bar{x})$ /* recall: $u\in\partial f(x) \overset{\text{def}}{\Leftrightarrow} \forall_{y\in\mathcal{H}}\ \langle y-x\mid u\rangle + f(x)\le f(y)$ */

⇔ $\exists_{v\in\partial g(L\bar{x})}\ \forall_{y\in\mathcal{H}}\ \langle y-\bar{x}\mid v\rangle + f(\bar{x})\le f(y)$:)

Now assume. g: Gateaux differentiable at $L\bar{x}$; recall

Proposition 17.26. [$f:\mathcal{H}\to]-\infty,+\infty]$, proper, convex; $x\in\mathrm{dom}\,f$]

(i) f: Gateaux differentiable at x ⇒ $\partial f(x) = \{\nabla f(x)\}$

(ii) ($x\in\mathrm{cont}\,f$; $\partial f(x)=\{u\}$) ⇒ (f: Gateaux differentiable at x; $u=\nabla f(x)$)

(iii)⇔(v):

$\exists_{v\in\partial g(L\bar{x})}\ -L^*v\in\partial f(\bar{x})$ ⇔ $-L^*(\nabla g(L\bar{x}))\in\partial f(\bar{x})$

// $\partial g(L\bar{x}) = \{v\} = \nabla g(L\bar{x})$

(iv)⇔(vi):

$\exists_{v\in\partial g(L\bar{x})}\ \forall_{y\in\mathcal{H}}\ \langle\bar{x}-y\mid L^*v\rangle + f(\bar{x})\le f(y)$ ⇔ $\forall_{y\in\mathcal{H}}\ \langle\bar{x}-y\mid L^*(\nabla g(L\bar{x}))\rangle + f(\bar{x})\le f(y)$

// $\partial g(L\bar{x}) = \{v\} = \nabla g(L\bar{x})$

(v)⇔(vii):

(v): $-L^*(\nabla g(L\bar{x}))\in\partial f(\bar{x})$ // take $\gamma\in\mathbb{R}_{++}$

⇔ $-\gamma L^*(\nabla g(L\bar{x}))\in\gamma\partial f(\bar{x}) = \partial(\gamma f)(\bar{x})$

⇔ $\underbrace{\bar{x}-\gamma L^*(\nabla g(L\bar{x}))}_{\tilde{x}} - \underbrace{\bar{x}}_{\tilde{p}}\in\partial(\underbrace{\gamma f}_{\tilde{f}})(\underbrace{\bar{x}}_{\tilde{p}})$ /* now

Proposition 16.34: [$f\in\Gamma_0(\mathcal{H})$, $x,p\in\mathcal{H}$] $p=\mathrm{Prox}_f\,x \Leftrightarrow x-p\in\partial f(p)$ */

⇔ $\bar{x} = \mathrm{Prox}_{\gamma f}(\bar{x}-\gamma L^*(\nabla g(L\bar{x})))$:)

∎

Proposition 26.5. [C: closed convex subset of $\mathcal{H}$; $f\in\Gamma_0(\mathcal{H})$; $\bar{x}\in\mathcal{H}$; $\gamma\in\mathbb{R}_{++}$; one of the following holds:

(a) $0\in\mathrm{sri}(C-\mathrm{dom}\,f)$

(b) $\mathcal{H}$: finite dimensional, C: polyhedral, $C\cap\mathrm{ri}(\mathrm{dom}\,f)\neq\emptyset$

(c) $\mathcal{H}$: finite dimensional, C: polyhedral set, f: polyhedral function, $C\cap\mathrm{dom}\,f\neq\emptyset$]

The following are equivalent:

(i) $\bar{x}$: a solution to the problem $\min_{x\in C} f(x)$; (ii) $\bar{x}\in\mathrm{zer}(N_C+\partial f)$

(iii) $\bar{x}\in P_C(\mathrm{Fix}(2\mathrm{Prox}_{\gamma f}-\mathrm{Id})\circ(2P_C-\mathrm{Id}))$;

(iv) $\exists_{u\in N_C\bar{x}}\ -u\in\partial f(\bar{x})$; (v) $\exists_{u\in\partial f(\bar{x})}\ -u\in N_C\bar{x}$; (vi) $\bar{x}\in C \wedge \exists_{u\in\partial f(\bar{x})}\ \forall_{y\in C}\ \langle\bar{x}-y\mid u\rangle\le 0$

Moreover, f: Gateaux differentiable at $\bar{x}$, each of the items (i)–(vi) is also equivalent to each of the following:

(vii) $-\nabla f(\bar{x})\in N_C\bar{x}$

(viii) $\bar{x}\in C$ and $\forall_{y\in C}\ \langle\bar{x}-y\mid\nabla f(\bar{x})\rangle\le 0$

(ix) $\bar{x} = P_C(\bar{x}-\gamma\nabla f(\bar{x}))$ // projected gradient method

Proof. Proof follows from the following results: 1) C: nonempty closed convex subset of $\mathcal{H}$ ⇒ $\left|\begin{array}{l}\mathrm{Prox}_{\iota_C} = P_C\\ \partial\iota_C = N_C\end{array}\right.$

(Properties of minimizers for a sum of two Γ_0 functions) /* set $\tilde{g}:=\iota_C$; $\tilde{f}:=f$

Corollary 26.3. [$\tilde{f},\tilde{g}\in\Gamma_0(\mathcal{H})$; $\bar{x}\in\mathcal{H}$; $\gamma\in\mathbb{R}_{++}$; one of the following holds: $\mathrm{dom}\,\tilde{g} = \mathrm{dom}\,\iota_C = C$ // ∵ $\iota_C = \begin{cases}0, & \text{if } x\in C\\ +\infty, & \text{else}\end{cases}$, $\mathrm{dom}\,f = \{x\in\mathcal{H}\mid f(x)<+\infty\}$

(a) $0\in\mathrm{sri}(\mathrm{dom}\,\tilde{g}-\mathrm{dom}\,\tilde{f}) = \mathrm{sri}(C-\mathrm{dom}\,f)$ ✓

(b) $\mathcal{H}$: finite-dimensional, $\tilde{g}$: polyhedral, $\mathrm{dom}\,\tilde{g}\cap\mathrm{ri}\,\mathrm{dom}\,\tilde{f}\neq\emptyset$ ⇔ $C\cap\mathrm{ri}\,\mathrm{dom}\,f\neq\emptyset$; ι_C: polyhedral ⇒ C: polyhedral

(c) $\mathcal{H}$: finite-dimensional; $\tilde{f},\tilde{g}$: polyhedral; $\mathrm{dom}\,\tilde{g}\cap\mathrm{dom}\,\tilde{f}\neq\emptyset$] The following are equivalent: polyhedral function; $\tilde{g}=\iota_C$: polyhedral ⇒ C: polyhedral set

(i) $\bar{x}$: solution to the problem $\min_{x\in\mathcal{H}}\ \tilde{f}(x)+\tilde{g}(x)$; $\min_{x\in\mathcal{H}} f(x)+\iota_C(x) = \min_{x\in C} f(x)$ ✓ (ii) $\bar{x}\in\mathrm{zer}(\partial\tilde{f}+\partial\tilde{g}) = \mathrm{zer}(\partial f+\partial\iota_C) = \mathrm{zer}(\partial f+N_C)$ ✓

(iii) $\bar{x}\in\mathrm{Prox}_{\gamma\tilde{g}}(\mathrm{Fix}(2\mathrm{Prox}_{\gamma\tilde{f}}-\mathrm{Id})\circ(2\mathrm{Prox}_{\gamma\tilde{g}}-\mathrm{Id}))$; $\mathrm{Prox}_{\gamma\iota_C} = P_C$; $\bar{x}\in P_C(\mathrm{Fix}(2\mathrm{Prox}_{\gamma f}-\mathrm{Id})\circ(2\mathrm{Prox}_{\gamma g}-\mathrm{Id}))$ ✓

(iv) $\exists_{u\in\partial\tilde{g}(\bar{x})}\ -u\in\partial\tilde{f}(\bar{x})$ ⇔ $\exists_{u\in N_C\bar{x}}\ -u\in\partial f(\bar{x})$ ✓ /* $\partial\iota_C = N_C$ */

(v) $\exists_{u\in\partial\tilde{g}(\bar{x})}\ \forall_{y\in\mathcal{H}}\ \langle\bar{x}-y\mid u\rangle + \tilde{f}(\bar{x})\le\tilde{f}(y)$ ⇔ $\exists_{u\in N_C\bar{x}}\ \forall_{y\in\mathcal{H}}\ \langle\bar{x}-y\mid u\rangle + f(\bar{x})\le f(y)$

Moreover, if $\tilde{f}$: Gateaux differentiable at $\bar{x}$, each of the items (i)–(v) is also equivalent to each of the following:

(vi) $-\nabla\tilde{f}(\bar{x})\in\partial\tilde{g}(\bar{x})$ ⇔ $-\nabla f(\bar{x})\in\partial\iota_C(\bar{x}) = N_C\bar{x}$

(vii) $\forall_{y\in\mathcal{H}}\ \langle\bar{x}-y\mid\nabla\tilde{f}(\bar{x})\rangle + \tilde{g}(\bar{x})\le\tilde{g}(y)$ ⇔ $\forall_{y\in\mathcal{H}}\ \langle\bar{x}-y\mid\nabla f(\bar{x})\rangle + \iota_C(x)\le\iota_C(y)$ ⇔ $\bar{x}\in C\wedge\forall_{y\in C}\ \langle\bar{x}-y\mid\nabla f(\bar{x})\rangle\le 0$

(viii) $\bar{x} = \mathrm{Prox}_{\gamma\tilde{g}}(\bar{x}-\gamma\nabla\tilde{f}(\bar{x}))$ ⇔ $\bar{x} = P_C(\bar{x}-\gamma\nabla f(\bar{x}))$ /* $\mathrm{Prox}_{\gamma\iota_C} = P_C$ */

∎

Example 26.8. (Stampacchia's theorem which is applied a lot in the study of partial differential equation)

[$F:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$, bilinear form. $\exists_{\alpha,\beta\in\mathbb{R}_{++}}\ \forall_{x\in\mathcal{H}}\ \forall_{y\in\mathcal{H}}\ \left|\begin{array}{l}|F(x,y)|\le\beta\|x\|\,\|y\|\\ F(x,x)\ge\alpha\|x\|^2\end{array}\right\}$ (eq: 26.5)

C: nonempty closed convex subset of $\mathcal{H}$;

$\ell\in B(\mathcal{H},\mathbb{R})$] ⇒

(i) $\exists_{!\bar{x}\in\mathcal{H}}\ \bar{x}\in C\wedge\forall_{y\in C}\ F(\bar{x},y-\bar{x})\ge\ell(y-\bar{x})$ (eq: 26.6)

(ii) [F: symmetric; $\bar{x}\in\mathcal{H}$; $f:\mathcal{H}\to\mathbb{R}: x\mapsto\frac{1}{2}F(x,x)-\ell(x)$] The following are equivalent: (a) $\bar{x}$ solves (eq: 26.6) (b) $\mathrm{Argmin}_C\,f = \{\bar{x}\}$

Proof:

F: bilinear $\overset{\text{def}}{\Leftrightarrow}\left|\begin{array}{l}\forall_{x\in\mathcal{H}}\ (F(x,\cdot)\in B(\mathcal{H},\mathbb{R}))\\ \forall_{y\in\mathcal{H}}\ F(\cdot,y)\in B(\mathcal{H},\mathbb{R})\end{array}\right\}$

set $\tilde{f}:=\ell$

Fact 1.17. (Riesz–Fréchet representation) [$\tilde{f}\in B(\mathcal{H},\mathbb{R})$] $\exists_{!u\in\mathcal{H}}\ \forall_{x\in\mathcal{H}}\ \left|\begin{array}{l}\tilde{f}(x)=\langle x\mid u\rangle\\ \|\tilde{f}\|=\|u\|\end{array}\right.$ $\left\{\begin{array}{l}\exists_{!u}\ \ell=\langle\cdot\mid u\rangle\\ \forall_{x\in\mathcal{H}}\ (\exists_{!v}\ F(x,\cdot)=\langle\cdot\mid v\rangle)\end{array}\right.$ // let us define an onto linear operator $L:\mathcal{H}\to\mathcal{H}$, $\in B(\mathcal{H},\mathcal{H})$ such that $Lx=v$

⇒ $\forall_{x\in\mathcal{H}}\ F(x,\cdot)=\langle\cdot\mid Lx\rangle$

/* now, Example 20.15. [$A:\mathcal{H}\to\mathcal{H}$, linear] A: monotone ⇔ $\forall_{x\in\mathcal{H}}\ \langle x\mid Ax\rangle\ge 0$ */

given: $\forall_{x\in\mathcal{H}}\ \forall_{y\in\mathcal{H}}\ |F(x,y)|\le\beta\|x\|\,\|y\|\ \wedge\ F(x,x)\ge\alpha\|x\|^2$ ⇒ $F(x,x)=\langle x\mid Lx\rangle\ge\alpha\|x\|^2\ge 0$ ⇒ L: monotone /* from Example 20.29. [$A\in\underbrace{B(\mathcal{H})}_{\text{space of linear continuous operators from }\mathcal{H}\text{ to }\mathcal{H}}$, monotone] A: maximally monotone */

$\forall_{y\in\mathcal{H}}\ |F(x,y)| = |\langle y\mid Lx\rangle|\le\beta\|x\|\,\|y\|$

⇓

L: maximally monotone

/* now, from Example 22.7. [$A:\mathcal{H}\to\mathcal{H}$, linear] A: strongly monotone with constant $\beta\in\mathbb{R}_{++}$ ⇔ $\forall_{x\in\mathcal{H}}\ \langle x\mid Ax\rangle\ge\beta\|x\|^2$ */

L: strongly monotone with constant α

/* now recall that:

Proposition 20.22. (Maximally monotonicity preserving operations) [$A:\mathcal{H}\to2^{\mathcal{H}}$, maximally monotone; $z\in\mathcal{H}$; $\gamma\in\mathbb{R}_{++}$] $\left|\begin{array}{l}\bullet\ A^{-1}\text{: maximally monotone}\\ \bullet\ \gamma A(\cdot+z)\text{: maximally monotone}\end{array}\right.$ */

$B = L(\cdot)-u$: maximally monotone; $\mathrm{dom}\,B=\mathcal{H}$ /* as $B:\mathcal{H}\to\mathcal{H}$, for a single valued operator $S:A\to B$, $\mathrm{dom}\,S=A$ */ Also $\boxed{B\text{: strongly monotone with constant }\alpha}$, because $\forall_{(x,Lx-u)\in\mathrm{gra}\,B}\ \forall_{(y,Ly-u)\in\mathrm{gra}\,B}\ \langle x-y\mid Lx-u-Ly+u\rangle = \underbrace{\langle x-y\mid Lx-Ly\rangle\ge\alpha\|x-y\|^2}_{\because\ L\text{: strongly monotone with constant }\alpha}$

the given equations we want to solve from (eq: 26.6): $\left|\begin{array}{l}\bar{x}\in C\\ \forall_{y\in C}\ F(\bar{x},y-\bar{x})\ge\ell(y-\bar{x})\end{array}\right\}$ now we will show that $\bar{x}\in\mathrm{zer}(N_C+B)$ will solve this.

$(N_C+B)\bar{x}\ni 0$

⇔ $N_C\bar{x}+\underbrace{B\bar{x}}_{\text{single vector as }B\text{: affine}}\ni 0$

⇔ $N_C\bar{x}+(L\bar{x}-u)\ni 0$

⇔ $N_C\bar{x}\ni u-L\bar{x}$ // $N_C\bar{x}\neq\emptyset$ as $u-L\bar{x}$ is in it, now recall

⇔ $\{v\in\mathcal{H}\mid\forall_{y\in C}\ \langle y-\bar{x}\mid v\rangle\le 0\}\ni u-L\bar{x}$, $\bar{x}\in C$

⇔ $\exists_{v\in\mathcal{H}}\ v=u-L\bar{x}\ \wedge\ \forall_{y\in C}\ \langle y-\bar{x}\mid u-L\bar{x}\rangle\le 0\ \wedge\ \bar{x}\in C$

$\Leftrightarrow N_C\bar{x} + (L\bar{x} - u) \ni 0$

$\Leftrightarrow N_C\bar{x} \ni u - L\bar{x}$ // $N_C\bar{x} \neq \emptyset$ as $u - L\bar{x}$ is in it. now recall

$\Leftrightarrow \{v \in \mathcal{H} \mid \forall_{y \in C} \langle y - \bar{x} \mid v \rangle \leq 0\} \ni u - L\bar{x}$, $\bar{x} \in C$

$\Leftrightarrow \exists_{v \in \mathcal{H}} \; v = u - L\bar{x} \;\wedge\; \forall_{y \in C} \langle y - \bar{x} \mid u - Lx \rangle \leq 0 \;\wedge\; \bar{x} \in C$

$\Leftrightarrow \forall_{y \in C} \underbrace{\langle y - \bar{x} \mid u \rangle}_{\ell(y - \bar{x})} - \underbrace{\langle y - \bar{x} \mid Lx \rangle}_{F(\bar{x}, y - \bar{x})} \leq 0 \;\wedge\; \bar{x} \in C$

$\Leftrightarrow \forall_{y \in C} F(\bar{x}, y - \bar{x}) \geq F(\bar{x}, y - \bar{x}) \;\wedge\; \bar{x} \in C$

$$\boxed{\therefore \bar{x} \in \operatorname{zer}(N_C + B) \Leftrightarrow \left(\forall_{y \in C} F(\bar{x}, y - \bar{x}) \geq F(\bar{x}, y - \bar{x}) \;\wedge\; \bar{x} \in C\right)}$$

Now: recall:

// set $A := N_C$

$\therefore$ $\boxed{N_C + B : \text{maximally monotone}}$

$\boxed{N_C + B : \text{strongly monotone}}$ /* $\because B$: strongly monotone, and adding a monotone and strongly monotone operator creates a maximally monotone operator in their intersected domain */

/* now [illegible] set $\tilde{A} := N_C + B$ */

$\therefore$ $\boxed{\operatorname{zer}(N_C + B) : \text{singleton}}$

$\exists_{!x \in \mathcal{H}} \left(\forall_{y \in C} F(\bar{x}, y - \bar{x}) \geq F(\bar{x}, y - \bar{x}) \;\wedge\; \bar{x} \in C\right)$ ↓

(b) given F : symmetric,

$f : \mathcal{H} \to \mathbb{R},\; x \mapsto \frac{1}{2}F(x, x) - \ell(x)$

now $\left|\begin{array}{l} f \in \Gamma_0(\mathcal{H}) \text{ // need to figure out how} \\ \operatorname{dom} f = \mathcal{H} \end{array}\right.$

Example 2.47: $Df(x) = F(x, \cdot) - \ell$

$\Downarrow$

result follows from (i) and proposition 26.5.

■

(KKT sufficient conditions)

Proposition 26.18: [$f \in \Gamma_0(\mathcal{H})$; $m \in \mathbb{N}_{++}$; $I = \{1, \dots, m\}$; $(g_i)_{i \in I} : \subseteq \Gamma_0(\mathcal{H})$, Slater condition: $\begin{cases} \forall_{i \in I} \operatorname{lev}_{\leq 0} g_i \subseteq \operatorname{int} \operatorname{dom} g_i \\ \operatorname{dom} f \cap \left(\bigcap_{i \in I} \operatorname{lev}_{<0} g_i\right) \neq \emptyset \end{cases}$ [eq: 26.44]

is satisfied ; $\bar{x} \in \mathcal{H}$; Problem in consideration: $\min_{x \in \mathcal{H}} f(x)$ s.t. $g_1(x) \leq 0, \dots, g_m(x) \leq 0$ (26.45)] $\Rightarrow$

- $\bar{x}$: solution to (26.45) $\Leftrightarrow$

$\left(\exists_{(\bar{\nu}_i)_{i \in I} \in \mathbb{R}_+^m} \exists_{(u_i)_{i \in I} \in \times_{i \in I} \partial g_i(\bar{x})} -\sum_{i \in I} \bar{\nu}_i u_i \in \partial f(\bar{x}) \;\wedge\right.$

$\forall_{i \in I} \; g_i(\bar{x}) \leq 0 \;\wedge$

$\left.\forall_{i \in I} \; \bar{\nu}_i g_i(\bar{x}) = 0\right)$

in which case $(\bar{\nu}_i)_{i \in I}$: Lagrange multipliers associated with $\bar{x}$, and $\bar{x}$: solves $\min_{x \in \mathcal{H}} f(x) + \sum_{i \in I} \bar{\nu}_i g_i(x)$

becomes

- Moreover

$(f, (g_i)_{i \in I}$: Gateaux differentiable at $\bar{x}) \Rightarrow \exists_{(\bar{\nu}_i)_{i \in I} \in \mathbb{R}_+^m} \left| \begin{array}{l} \nabla f(\bar{x}) = -\sum_{i \in I} \bar{\nu}_i \nabla g_i(\bar{x}) \\ \forall_{i \in I} \; g_i(\bar{x}) \leq 0 \\ \forall_{i \in I} \; \bar{\nu}_i g_i(\bar{x}) = 0 \end{array}\right.$

Proof:

Define : $\forall_{i \in I} \; C_i = \operatorname{lev}_{\leq 0} g_i$ and $C = \bigcap_{i \in I} C_i$ // so this is the constraint set

$= \{x \in \mathcal{H} \mid g_i(x) \leq 0\}$

now: recall

Corollary 8.30: set $\tilde{f} := g_i$

[$\tilde{f} : \mathcal{H} \to]-\infty, +\infty]$, proper, convex ✓ // as $g_i \in \Gamma_0(\mathcal{H})$

one of the following holds

- $\tilde{f}$: bounded above on some neighborhood
- $\tilde{f}$: lower semicontinuous ✓ // as $g_i \in \Gamma_0(\mathcal{H})$
- $\mathcal{H}$: finite-dimensional] $\Rightarrow$ $\operatorname{cont} \tilde{f} = \operatorname{int} \operatorname{dom} \tilde{f}$ /* $\operatorname{cont} f$: domain of continuity of a function f */

$\therefore \operatorname{cont} g_i = \operatorname{int} \operatorname{dom} g_i = \operatorname{int}\left(\{x \in \mathcal{H} \mid g_i(x) < +\infty\}\right)$ $\quad\therefore g_i$: continuous on $\operatorname{int} \operatorname{dom} g_i$ and all subsets of it

$\Rightarrow g_i$: upper-semicontinuous on all subsets of $\operatorname{int} \operatorname{dom} g_i$

now clearly $\operatorname{lev}_{<0} g_i = \{x \in \mathcal{H} \mid g_i(x) < 0\} \subseteq \operatorname{int} \operatorname{dom} g_i$

$\therefore \boxed{g_i : \text{upper-semicontinuous on } \operatorname{lev}_{<0} g_i}$

given Slater's condition holds $\quad \forall_{i \in I} \operatorname{lev}_{\leq 0} g_i \subseteq \operatorname{int} \operatorname{dom} g_i$

$\operatorname{dom} f \cap \left(\bigcap_{i \in I} \operatorname{lev}_{<0} g_i\right) \neq \emptyset \Rightarrow \bigcap_{i \in I} \operatorname{lev}_{<0} g_i \neq \emptyset \Rightarrow \boxed{\forall_{i \in I} \operatorname{lev}_{<0} g_i \neq \emptyset}$

Corollary 8.38: set $\tilde{f} := g_i$

[$\tilde{f} : \mathcal{H} \to]-\infty, +\infty]$, convex ✓, $\operatorname{lev}_{<0} \tilde{f} \neq \emptyset$ ✓

one of the following holds:

(i) $\tilde{f}$: upper semi-continuous on $\operatorname{lev}_{<0} \tilde{f}$ ✓

(ii) $\tilde{f}$: lower semicontinuous, $\operatorname{dom} \tilde{f}$: open

(iii) $\mathcal{H}$: finite-dimensional, $\operatorname{dom} f$: open] $\Rightarrow$ $\boxed{\operatorname{int} \operatorname{lev}_{\leq 0} \tilde{f} = \operatorname{lev}_{<0} \tilde{f}}$ $\Rightarrow \operatorname{int} \operatorname{lev}_{\leq 0} g_i = \operatorname{lev}_{<0} g_i$ // i.e. $\operatorname{int}\{x \mid g_i(x) \leq 0\} = \{x \mid g_i(x) < 0\}$

now, $\operatorname{int} C = \bigcap_{i \in I} \operatorname{int} C_i = \bigcap_{i \in I} \operatorname{int} \operatorname{lev}_{\leq 0} g_i = \bigcap_{i \in I} \operatorname{lev}_{<0} g_i$

$\Downarrow$ // $\operatorname{int} \bigcap_i C_i = \bigcap_i \operatorname{int} C_i$

$\left.\begin{array}{l} \operatorname{int} C \cap \operatorname{dom} f \neq \emptyset \Rightarrow C \cap \operatorname{dom} f \neq \emptyset \\ \text{now } C : \text{convex} \\ \operatorname{dom} f : \text{convex } // \because f \in \Gamma_0(\mathcal{H}) \end{array}\right\} \Rightarrow 0 \in \operatorname{int}(C - \operatorname{dom} f)$ // using Result 3.1.m. [A, B : convex, $A \cap B \neq \emptyset$] $0 \in \operatorname{int}(A - B)$]

$\subseteq \operatorname{sri}(C - \operatorname{dom} f)$ /* inclusion formula for different types of interiors $\operatorname{int} C \subseteq \operatorname{core} C \subseteq \operatorname{sri} C \subseteq \operatorname{ri} C \subseteq \dots C$ */

now we use Proposition 26.5.

Proposition 26.5: [C : closed convex subset of $\mathcal{H}$; $f \in \Gamma_0(\mathcal{H})$; $\bar{x} \in \mathcal{H}$; $\gamma \in \mathbb{R}_{++}$; one of the following holds: ...]

$$\boxed{\bar{x} \in \operatorname{Argmin}_C f \Leftrightarrow \exists_{u \in N_C \bar{x}} -u \in \partial f(\bar{x})}$$ (eq: 26.50) [eq:26.50]

suppose, $\bar{x} \in C$ // as $\bar{x} \in \operatorname{Argmin}_C f$

we want to calculate $N_C \bar{x}$

define,

$\left| \begin{array}{l} I_+ = \{i \in I \mid g_i(\bar{x}) = 0\} \\ I_- = \{i \in I \mid g_i(\bar{x}) < 0\} \end{array}\right. \Rightarrow \boxed{I = I_+ \cup I_-}$

now $\bigcap_{i \in I} \operatorname{int} \operatorname{dom} \iota_{C_i}$ /* $\operatorname{dom} \iota_{C_i} = C_i$ */

$= \bigcap_{i \in I} \operatorname{int} C_i = \bigcap_{i \in I} \operatorname{lev}_{<0} g_i \neq \emptyset$ ... (eq: 26.52) [eq: 26.52]

/* $\forall_C \operatorname{int} C \subseteq \operatorname{core} C$ */

now $\bar{x} \in C = \bigcap_{i \in I} \underbrace{\operatorname{lev}_{\leq 0} g_i}_{\subseteq \operatorname{int} \operatorname{dom} g_i \text{ // from [eq: 26.44]}} \subseteq \bigcap_{i \in I} \operatorname{int} \operatorname{dom} g_i \Rightarrow \boxed{\forall_{i \in I} \bar{x} \in \operatorname{int} \operatorname{dom} g_i} \Rightarrow \bar{x} \in \operatorname{core} \operatorname{dom} g_i$

$\tilde{C} := \operatorname{dom} g_i$: convex

Proposition 6.47: /* Relationship between tangent and normal cone */

$\therefore \boxed{N_{\operatorname{dom} g_i} \bar{x} = \{0\}}$ ... (eq: 26.53)

Now:

$N_C \bar{x}$ /* $C = \bigcap_{i \in I} \overbrace{C_i}^{\operatorname{lev}_{\leq 0} g_i}$ ∧ [illegible] */

$= \partial \iota_C \bar{x} = \partial \iota_{\bigcap_{i \in I} C_i}(\bar{x})$ /* By definition, $\iota_{\bigcap_{i \in I} C_i} = \sum_{i \in I} \iota_{C_i}$ */

$= \partial\left(\sum_{i \in I} \partial \iota_{C_i}\right)(\bar{x})$ /* Theorem 16.47: [m ; ... ; $I = \{1, \dots, m\}$; now set : $\tilde{f}_i := \partial_{C_i} \in \Gamma_0(\mathcal{H})$

Now:

$N_C\bar{x}$ /* $C = \bigcap_{i\in I} \widetilde{C_i}$ ($\widetilde{C_i} = \mathrm{lev}_{\le 0} g_i$) $\wedge$ [illegible] */

$= \partial\iota_C \bar{x} = \partial\iota_{\bigcap_{i\in I} C_i}(\bar{x})$ /* By definition, $\iota_{\bigcap_{i\in I} C_i} = \sum_{i\in I}\iota_{C_i}$ */

$= \partial\left(\sum_{i\in I}\iota_{C_i}\right)(\bar{x})$ /* Theorem 16.?? [m: ..., I = {1, …, m} ; $(\tilde{f}_i)_{i\in I} \in \Gamma_0(\mathcal{H})$, one of the following holds:
(i) ...
(ii) ... : closed linear subspace
(iii) ... : linear subspaces, ... : closed
(iv) $\mathrm{dom}\,\tilde{f}_m \cap \left(\bigcap_{i=1}^{m-1}\mathrm{int\,dom}\,\tilde{f}_i\right) \neq \emptyset$ ✓ /* ∵ [eq: 26.52] $\bigcap_{i\in I}\mathrm{int\,dom}\,\underbrace{\tilde{f}_i}_{\iota_{C_i}} = \bigcap_{i\in I}\mathrm{int\,dom}\,\iota_{C_i} \neq \emptyset \Rightarrow \underbrace{\mathrm{int\,dom}\,\iota_{C_m}}_{\subseteq\,\mathrm{dom}\,\iota_{C_m}} \cap \left(\bigcap_{i\in I\setminus\{m\}}\mathrm{int\,dom}\,\iota_{C_m}\right) \neq \emptyset$
$\Rightarrow \mathrm{dom}\,\iota_{C_m} \cap \left(\bigcap_{i\in I\setminus\{m\}}\mathrm{int\,dom}\,\iota_{C_m}\right) \neq \emptyset$ */
(v) $\mathcal{H}$: finite-dimensional, $\bigcap_{i\in I}\mathrm{ri\,dom}\,\tilde{f}_i \neq \emptyset$] $\partial\left(\sum_{i\in I}\tilde{f}_i\right) = \sum_{i\in I}\partial\tilde{f}_i$

now set. $\tilde{f}_i := \iota_{C_i} \in \Gamma_0(\mathcal{H})$
$\underbrace{C_i}_{\mathrm{lev}_{\le 0} g_i}$: $\neq\emptyset$, closed, convex

$\therefore \partial\sum_{i\in I}\iota_{C_i} = \sum_{i\in I}\partial\iota_{C_i}$ */

$= \sum_{i\in I}\underbrace{\partial\iota_{C_i}(\bar{x})}_{N_{C_i}}$

$= \sum_{i\in I = I_+\cup I_-} N_{C_i}(\bar{x})$

$= \sum_{i\in I_+} N_{C_i}(\bar{x}) + \sum_{i\in I_-} N_{C_i}(\bar{x})$

$= \sum_{i\in I_+}\bigcup_{v_i\in\mathbb{R}_+}\left(v_i\,\partial g_i(\bar{x})\right)$

$= \sum_{i\in I_+}\bigcup_{v_i\in\mathbb{R}_+} v_i\,\partial g_i(\bar{x})$

/* set: $\tilde{g} := g_i$, $\tilde{C} = \mathrm{lev}_{\le 0}\tilde{g} = \mathrm{lev}_{\le 0} g_i = C_i$ //by our def
$\tilde{x} := \bar{x}$

Lemma 26.17. [$\tilde{g}: \mathcal{H}\to\,]-\infty,+\infty]$: convex, $\mathrm{lev}_{<0}\tilde{g} \neq \emptyset$; $\tilde{C} := \mathrm{lev}_{\le 0}\tilde{g}$; $x\in\tilde{C}$]

$$N_{\tilde{C}}\tilde{x} = \begin{cases} N_{\mathrm{dom}\,\tilde{g}}\tilde{x} \cup \mathrm{cone}\,\partial\tilde{g}(\tilde{x}), & \text{if } \tilde{g}(\tilde{x}) = 0 \\ N_{\mathrm{dom}\,\tilde{g}}\tilde{x}, & \text{if } \tilde{g}(\tilde{x}) < 0 \end{cases}$$

$$N_{C_i}\bar{x} = \begin{cases} \underbrace{N_{\mathrm{dom}\,g_i}\bar{x}}_{=\{0\}} \cup \mathrm{cone}\,\partial g_i(\bar{x}), & \text{if } g_i(\bar{x}) = 0 \Leftrightarrow i\in I_+ \text{ //by def} \\ \underbrace{N_{\mathrm{dom}\,g_i}\bar{x}}_{=\{0\}}, & \text{if } g_i(\bar{x}) < 0 \Leftrightarrow i\in I_- \text{ // " "} \end{cases}$$

$\therefore \forall_{i\in I_+}$ $N_{C_i}\bar{x} = \{0\}\cup\mathrm{cone}\,\partial g_i(\bar{x}) = \mathrm{cone}(\underbrace{\partial g_i(\bar{x})}_{\text{closed, convex as } g_i\in\Gamma_0(\mathcal{H})\text{, proposition 16.5}}) = \mathbb{R}_{++}\,\partial g_i(\bar{x})\cup\{0\}$ /* $\forall_C$ $\mathrm{cone}\,C = \mathbb{R}_{++}C$ */ $= \bigcup_{v_i\in\mathbb{R}_{++}} v_i\,\partial g_i(\bar{x}) \cup \{0\} = \bigcup_{v_i\in\mathbb{R}_+} v_i\,\partial g_i(\bar{x})$

this 'i' is to keep track of associated g_i

/* By def: [$\Lambda\subseteq\mathbb{R}$] $\Lambda C = \bigcup_{\lambda\in\Lambda}\lambda C$ */

$\forall_{i\in I_-}$ $N_{C_i}\bar{x} = \{0\}$ */

$\therefore N_C\bar{x} = \sum_{i\in I_+}\bigcup_{v_i\in\mathbb{R}_+} v_i\underbrace{\partial g_i(\bar{x})}_{\{u_i \mid u_i\in\partial g_i(\bar{x})\}} = \sum_{i\in I_+}\{v_i u_i \mid v_i\in\mathbb{R}_+,\ u_i\in\partial g_i(\bar{x})\} = \left\{\sum_{i\in I_+} v_i u_i \,\middle|\, \forall_{i\in I} v_i\in\mathbb{R}_+,\ u_i\in\partial g_i(\bar{x})\right\}$ /* makes sense, visualize small examples */

from, [eq:26.50] : $\bar{x}\in\mathrm{Argmin}_C f \Leftrightarrow \exists_{u\in N_C\bar{x}}\ -u\in\partial f(\bar{x}) \Leftrightarrow \exists_{u = \sum_{i\in I_+} v_i u_i\ :\ \forall_{i\in I_+} v_i\in\mathbb{R}_+,\ u_i\in\partial g_i(\bar{x})}\ -u\in\partial f(\bar{x})$

/* now we want to extend the index set to $I = I_+\cup I_-$, so that still holds
now we design $(v_i)_{i\in I_-}$, $(u_i)_{i\in I_-}$ as follows:

$(u_i)_{i\in I_-} \in \bigtimes_{i\in I_-}\partial g_i(\bar{x})$
now we want $\forall_{i\in I_-}$ $v_i g_i(\bar{x}) = 0 \Rightarrow (v_i)_{i\in I_-} = (0)_{i\in I_-}$ //so, $\sum_{i\in I_-} v_i u_i = 0$ is maintained.
($g_i(\bar{x}) < 0$)

again $\forall_{i\in I_+ \Leftrightarrow g_i(\bar{x}) = 0}$ $v_i g_i(\bar{x}) = 0$ } $\boxed{\forall_{i\in I}\ v_i g_i(\bar{x}) = 0}$

also, as $\bar{x}\in\mathrm{Argmin}_C f \Rightarrow \bar{x}\in C \Leftrightarrow \boxed{\forall_{i\in I}\ g_i(\bar{x})\le 0}$ */

$\Leftrightarrow \exists_{u = \sum_{i\in I} v_i u_i\ :\ \forall_{i\in I} v_i\in\mathbb{R}_+,\ u_i\in\partial g_i(\bar{x})}\ -u\in\partial f(\bar{x}) \wedge \forall_{i\in I}\begin{cases} g_i(\bar{x})\le 0 \\ v_i g_i(\bar{x}) = 0 \end{cases}$

$\therefore$ we have arrived at the key claim:

$$\bar{x}:\text{solves}\begin{pmatrix} \min_{x\in\mathcal{H}} & f(x) \\ \text{s.t.} & g_1(x)\le 0 \\ & \vdots \\ & g_m(x)\le 0 \end{pmatrix} \Leftrightarrow \exists_{(\bar{v}_i)_{i\in I}\in\mathbb{R}^m_+}\ \exists_{(u_i)_{i\in I}\in\bigtimes_{i\in I}\partial g_i(\bar{x})}\ -\sum_{i\in I}\bar{v}_i u_i\in\partial f(\bar{x}) \wedge \forall_{i\in I}\begin{cases} g_i(\bar{x})\le 0 \\ \bar{v}_i g_i(\bar{x}) = 0 \end{cases}$$

naturally this means: $\sum_{i\in I}\bar{v}_i\underbrace{u_i}_{\in\partial g_i(\bar{x})} + \partial f(\bar{x}) \ni 0$

$\Leftrightarrow \sum_{i\in I}\bar{v}_i\,\partial g_i(\bar{x}) + \partial f(\bar{x}) \Leftrightarrow \partial\left(\sum_{i\in I}\bar{v}_i g_i(\bar{x}) + f(\bar{x})\right)\ni 0$

$\Leftrightarrow \bar{x}$: solves $\min_{x\in\mathcal{H}}\ f(x) + \sum\bar{v}_i g_i(x)$

clearly $\bar{v}_i$: are the associated Lagrange multipliers.

using

set $K = \mathbb{R}^m$, $K = \mathbb{R}^m_-$; $R: x\mapsto (g_i(\ldots$

Proposition 19.25. [$f\in\Gamma_0(\mathcal{H})$, K: nonempty ...
$F: \mathcal{H}\times K\to\,]-\infty,+\infty]: (x,y)\mapsto\begin{cases} f(x), & \text{if } Rx\in y+K \ldots \\ +\infty, & \text{if } Rx\notin y+K \ldots \end{cases}$
(i) $F\in\Gamma_0(\mathcal{H}\times K)$
(ii) Primal problem: min. $f(x)$ s.t. $Rx\in K$, $x\in\mathcal{H}$
(iii) Dual problem: min. $\varphi(v)$ s.t. $v\in K^{\ominus}$ where ...
(iv) The Lagrangian:
$\mathcal{L}: \mathcal{H}\times K\to[-\infty,+\infty]: (x,v)\mapsto\begin{cases} -\infty \\ f(x)+\langle Rx \mid v\rangle \\ +\infty \end{cases}$
(v) $\underbrace{(\bar{x},\bar{v})}_{\in\mathcal{H}\times K}$: saddle point of $\mathcal{L}$ ⇔
$\bar{x}\in\mathrm{dom}\,f$
$R\bar{x}\in K$
$\bar{v}\in K^{\ominus}$
$f(\bar{x}) = \inf_{x\in\mathcal{H}}\left(f(x)+\langle Rx\mid\bar{v}\rangle\right)$
in which case: $\langle R\bar{x}\mid\bar{v}\rangle = 0$ & $\bar{x}$: primal ...
Remark 19.16. $\underbrace{(\bar{x},\bar{v})}_{\in(\mathcal{H}\times K)}$: saddle point of $\mathcal{L}$ ...

The claims for Gateaux differentiability follows from

Proposition 17.26. [$f: \mathcal{H}\to\,]-\infty,+\infty]$, proper, convex ; $x\in\mathrm{dom}\,f$]
(i) f : Gateaux differentiable at x ⇒ $\partial f(x) = \{\nabla f(x)\}$
(ii) ($x\in\mathrm{cont}\,f$; $\partial f(x) = \{u\}$) ⇒ (f : Gateaux differentiable at x ; $u = \nabla f(x)$)

$x))_{i \in I}$

closed convex cone in $\mathcal{K}$; B : continuous, convex with K, $K \cap B(\mathrm{dom} f) \neq \emptyset$;

$f + \iota$

$) := \sup_{x \in \mathcal{H}} \left(\langle Bx \mid v \rangle - f(x) \right)$

, if $x \in \mathrm{dom} f$, $v \notin K^{\ominus}$

, if $x \in \mathrm{dom} f$, $v \in K^{\ominus}$

, if $x \notin \mathrm{dom} f$

l solution

$\Leftrightarrow$ $\bar{v}$: Lagrange multiplier to $\bar{x}$

$\bar{x}$: solves $\min_{x \in \mathcal{H}} f(x) + \langle Bx \mid v \rangle$