Part 1

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Theorem 24.3. [A, B : maximally monotone from $\mathcal{H}$ to $2^{\mathcal{H}}$,
$\operatorname{cone}(\operatorname{dom} A - \operatorname{dom} B) = \overline{\operatorname{span}}(\operatorname{dom} A - \operatorname{dom} B)$] $\Rightarrow$ $A+B$: maximally monotone

Proof blueprint: Our goal is showing that $0 \in \operatorname{sri}(\operatorname{dom} F_A - \operatorname{dom} F_B)$ which will is the antecedent for the addition of two maximally monotone operators to be maximally monotone

Proof: fact. $A \subseteq B \Rightarrow \operatorname{cone}(A) \subseteq \operatorname{cone}(B)$, also. $\forall_{C:\text{set}}$ $\operatorname{cone}(C) \subseteq \overline{\operatorname{span}}\, C$

/* recall:

Proposition 21.11. (Representing domain of a maximally monotone operator via Fitzpatrick function)
[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone
$Q_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H} : (x,u) \mapsto x$]
$\operatorname{int} \operatorname{dom} A \subseteq \operatorname{int} Q_1(\operatorname{dom} F_A) \subseteq \operatorname{dom} A \subseteq Q_1(\operatorname{dom} F_A) \subseteq \overline{\operatorname{dom}}\, A$
$\operatorname{int} \operatorname{dom} A = \operatorname{int} Q_1(\operatorname{dom} F_A)$
$\overline{\operatorname{dom} A} = \overline{Q_1(\operatorname{dom} F_A)}$
*/

so. $\operatorname{cone}(\operatorname{dom} A - \operatorname{dom} B)$ /* now $\tilde{A} \subseteq \tilde{B}, \tilde{C} \subseteq \tilde{D} \Rightarrow \tilde{A} - \tilde{C} \subseteq \tilde{B} - \tilde{D}$ */

$\subseteq \operatorname{cone}(Q_1(\operatorname{dom} F_A) - Q_1(\operatorname{dom} F_B))$

$\subseteq \overline{\operatorname{span}}(\underbrace{Q_1(\operatorname{dom} F_A)}_{\subseteq \overline{\operatorname{dom}}\, A} - \underbrace{Q_1(\operatorname{dom} F_B)}_{\subseteq \overline{\operatorname{dom}}\, B})$

$\subseteq \overline{\operatorname{span}}(\overline{\operatorname{dom}}\, A - \overline{\operatorname{dom}}\, B)$

$= \overline{\operatorname{span}}(\operatorname{dom} A - \operatorname{dom} B)$ //as if we remove closure from $\overline{\operatorname{dom}}\, A, \overline{\operatorname{dom}}\, B$, the final closure on $\overline{\operatorname{span}}$ will //make the end result same

$= \operatorname{cone}(\operatorname{dom} A - \operatorname{dom} B)$ // given

So, the sandwitched sets collapses !

so, $\operatorname{cone}(Q_1(\operatorname{dom} F_A - \operatorname{dom} F_B)) = \overline{\operatorname{span}}(Q_1(\operatorname{dom} F_A - \operatorname{dom} F_B))$

now recall, $\operatorname{sri} C = \{x \in C \mid \overline{\operatorname{span}}(C - x) = \operatorname{cone}(C - x)\}$

i.e., $x \in C \Leftrightarrow \overline{\operatorname{span}}(C-x) = \operatorname{cone}(C-x)$

$0 \in \operatorname{sri} Q_1(\operatorname{dom} F_A - \operatorname{dom} F_B)$

now recall that

Theorem 24.2. [$Q_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H} : (x,u) \mapsto x$; A, B : maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$, $0 \in \operatorname{sri} Q_1(\operatorname{dom} F_A - \operatorname{dom} F_B)$] $\Rightarrow$ $A+B$: maximally monotone

So, $A+B$: maximally monotone ■

Corollary 24.4. [A, B : maximally monotone, $: \mathcal{H} \to 2^{\mathcal{H}}$; one of the following holds :

(i) $\operatorname{dom} B = \mathcal{H}$

(ii) $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \emptyset$

(iii) $0 \in \operatorname{int}(\operatorname{dom} A - \operatorname{dom} B)$

(iv) $\forall_{x \in \mathcal{H}} \exists_{\varepsilon \in \mathbb{R}_{++}}$ $[0, \varepsilon x] \subseteq \operatorname{dom} A - \operatorname{dom} B$

(v) $\operatorname{dom} A, \operatorname{dom} B$: convex, $0 \in \operatorname{sri}(\operatorname{dom} A - \operatorname{dom} B)$] $\Rightarrow$ $A+B$: maximally monotone

Proof:

$\operatorname{dom} B = \mathcal{H}$ // by (i)

$\Leftrightarrow \forall_{x \in \mathcal{H}}$ $Bx \neq \emptyset$

$\Rightarrow \operatorname{int} \operatorname{dom} B = \mathcal{H}$ /* as $\mathcal{H}$ is both open and closed

then $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B = \operatorname{dom} A \cap \mathcal{H} \neq \emptyset$ /* $\operatorname{dom} A \neq \emptyset$ */
/* this is (ii) */

$\operatorname{int} \operatorname{dom} B = \mathcal{H} \Rightarrow \operatorname{dom} A \subseteq \operatorname{dom} B$

$C := \operatorname{dom} A - \operatorname{dom} B = \{x - y \mid Ax \neq \emptyset, y \in \mathcal{H}\}$, note that no matter what $0 \in C$ as $y \in \mathcal{H}$

note that $C := \mathcal{H}$ as $\forall_{z \in \mathcal{H}}$ setting $\underset{\in \mathcal{H}}{y = -z + x}$, will give $x - y = x - (-z + x) = z$

so. $\operatorname{int} C = \mathcal{H} \to 0 \in \operatorname{int} C = \operatorname{int}(\operatorname{dom} A - \operatorname{dom} B)$ /* this is (iii) */

as $\operatorname{dom} A - \operatorname{dom} B = \mathcal{H} \Rightarrow \forall_{x \in \mathcal{H}} \forall_{\varepsilon \in \mathbb{R}_{++}}$ $[0, \varepsilon x] \subseteq \operatorname{dom} A - \operatorname{dom} B$ /* this is (iv) */

clearly (iv) $\Rightarrow$ $\operatorname{cone}(\operatorname{dom} A - \operatorname{dom} B) = \overline{\operatorname{span}}(\operatorname{dom} A - \operatorname{dom} B)$ $\Rightarrow$ $A+B$: maximally monotone

/* using Theorem 24.3. [A, B : maximally monotone, $: \mathcal{H} \to 2^{\mathcal{H}}$; $\operatorname{cone}(\operatorname{dom} A - \operatorname{dom} B) = \overline{\operatorname{span}}(\operatorname{dom} A - \operatorname{dom} B)$] $A+B$: maximally monotone */

now consider (v) : $\operatorname{dom} A, \operatorname{dom} B$: convex, $0 \in \operatorname{sri}(\operatorname{dom} A - \operatorname{dom} B)$ /* recall: $\operatorname{sri} C = \{x \in C \mid \operatorname{cone}(C - x) = \overline{\operatorname{span}}(C - x)\}$ */

$\Leftrightarrow \operatorname{cone}(\operatorname{dom} A - \operatorname{dom} B) = \overline{\operatorname{span}}(\operatorname{dom} A - \operatorname{dom} B)$

$\Rightarrow A+B$: maximally monotone ■

/* A : uniformly monotone with modulus $\phi \overset{\text{def}}{\Leftrightarrow} \phi$: increasing, vanishes only at 0, $\forall_{(x,u),(y,v) \in \operatorname{gra} A}$ $\langle x-y \mid u-v \rangle \geq \phi(\|x-y\|)$ */

Example 24.11 [$A : \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, uniformly monotone with $\underbrace{\text{supercoercive}}_{\Downarrow}$ modulus ϕ] A : 3^* monotone $\overset{\text{def}}{\Leftrightarrow}$ $\operatorname{dom} A \times \operatorname{ran} A \subseteq \operatorname{dom} F_A$

/* $\lim_{t \to +\infty} \frac{\phi(t)}{t} = +\infty$ */

Proof: set. $(x,u) \in \operatorname{gra} A$, $w \in \mathcal{H}$,

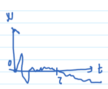
$\underset{\text{fixed for fixed } w,u}{\gamma := \|w - u\|}$, $\psi : \mathbb{R}_+ \to [-\infty, +\infty[: t \mapsto \gamma t - \phi(t)$

$\Leftrightarrow \psi(t) = \gamma t - \phi(t) = t\left(\gamma - \frac{\phi(t)}{t}\right) \Rightarrow \psi(0) = \gamma 0 - \underbrace{\phi(0)}_{0} = 0$ /* as ϕ : vanishes only at zero */

$\Rightarrow \lim_{t \to +\infty} \psi(t) = \lim_{t \to +\infty} t\left(\gamma - \frac{\phi(t)}{t}\right)$ /* recall that. if $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ exists $\Rightarrow \lim_{x \to a} f(x) g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$ */

now. as $\lim_{t \to \infty} \frac{\phi(t)}{t} = +\infty$, $\lim_{t \to +\infty} t = +\infty$. $\Rightarrow$ $\lim_{t \to +\infty} \psi(t) = +\infty(\gamma - \infty) = -\infty$

$\Rightarrow$ we can find a τ : $\forall_t$ $t > \tau \Rightarrow \psi(t) \leq 0 = \psi(0)$, so. the graph will look somewhat like this :



thus from the figure, $\sup_{t \in \mathbb{R}_{++}} \psi(t) = \sup_{t \in [0,\tau]} \psi(t) \leq \gamma\tau - \phi(\tau)$

$\Rightarrow \sup_{t \in \mathbb{R}_{++}} \psi(t) = \sup_{t \in [0,\tau]} \psi(t) \leq \gamma\tau - \phi(\tau)$ /* now $\phi(0) = 0$. and increasing $\therefore \forall_{\tau \geq 0}$ $\phi(\tau) \geq 0$, so removing $-\phi(\tau)$ will give a larger number */

$\leq \gamma\tau < +\infty$ ... (eq:1)

* Cauchy-Schwarz inequality: $|\langle x|y \rangle| \leq \|x\| \|y\|$, $\max\{\langle x|y\rangle, -\langle x|y\rangle\}$
Flavor 1: $\langle x|y \rangle \leq \|x\| \|y\|$
Flavor 2: $-\langle x|y \rangle \leq \|x\| \|y\| \Leftrightarrow \langle x|y\rangle \geq -\|x\| \|y\|$

so. $\underbrace{F_A(x,w) - \langle x|w \rangle}_{\text{def}: \langle x|w\rangle - \inf \langle x-y | w-v \rangle} = \langle x|w \rangle - \inf_{(y,v) \in \operatorname{gra} A} \langle x-y \mid w-v \rangle - \langle x|w \rangle = -\inf_{(y,v) \in \operatorname{gra} A} \langle x-y \mid w-v \rangle = \sup_{(y,v) \in \operatorname{gra} A} -\langle x-y \mid w-v \rangle = \sup_{(y,v) \in \operatorname{gra} A} -\langle x-y \mid \overbrace{(w-u)+(u-v)}^{w-u+u-v} \rangle = \sup_{(y,v) \in \operatorname{gra} A} \left(-\langle x-y \mid w-u \rangle - \langle x-y \mid u-v \rangle\right)$ /* now using Cauchy-Schwarz flavor 2: $-\langle x-y \mid w-u \rangle \leq \|x-y\| \overbrace{\|w-u\|}^{\gamma} = \gamma\|x-y\|$

and as A : uniformly monotone with modulus $\phi \overset{\text{def}}{\Leftrightarrow} \phi$: increasing, vanishes only at 0, $\forall_{(x,u),(y,v) \in \operatorname{gra} A}$ $\langle x-y \mid u-v \rangle \geq \phi(\|x-y\|)$

$(x,u), (y,v) \in \operatorname{gra} A \Rightarrow \langle x-y \mid u-v \rangle \geq \phi(\|x-y\|) \Leftrightarrow -\langle x-y \mid u-v \rangle \leq -\phi(\|x-y\|)$ */

// watch out: trick move!

$= \sup \left(\|x-y\| \left(\|v-u\| + \|u-w\|\right)\right)$

$\text{Flavor 2}: -\langle x|y\rangle \leq \|x\|\|y\| \Leftrightarrow \langle x|y\rangle \geq -\|x\|\|y\|$

$\leq \gamma\tau < +\infty \quad \ldots \text{(eq:1)}$

$w-u+u-v$

So. $F_A(x,w)-\langle x|w\rangle = \langle x|w\rangle - \inf_{(y,v)\in \text{gra}A} \langle x-y|w-v\rangle - \langle x|w\rangle = -\inf_{(y,v)\in \text{gra}A} \langle x-y|w-v\rangle = \sup_{(y,v)\in \text{gra}A} -\langle x-y|w-v\rangle = \sup_{(y,v)\in \text{gra}A} -\langle x-y|(w-u)+(u-v)\rangle = \sup_{(y,v)\in \text{gra}A}\left(-\langle x-y|w-u\rangle - \langle x-y|u-v\rangle\right)$ /* now using Cauchy-Schwarz flavor 2: $-\langle x-y|w-u\rangle \leq \|x-y\|\underbrace{\|w-u\|}_{\gamma} = \gamma\|x-y\|$

$\overset{\text{def}}{=} \langle x|w\rangle - \inf_{(y,v)\in \text{gra}A} \langle x-y|w-v\rangle$

$= \sup_{(y,v)\in \text{gra}A} \left(\|x-y\|\left(\|v-u\|+\|u-w\|\right)\right)$

// watch out: trick move!

and as A: uniformly monotone with modulus $\phi \overset{\text{def}}{\Leftrightarrow} \phi$: increasing, vanishes only at 0, $\forall_{(x,u),(y,v)\in \text{gra}A} \langle x-y|u-v\rangle \geq \phi(\|x-y\|)$

$(x,u),(y,v)\in \text{gra}A \Rightarrow \langle x-y|u-v\rangle \geq \phi(\|x-y\|) \Leftrightarrow -\langle x-y|u-v\rangle \leq -\phi(\|x-y\|)$ */

$\leq \sup_{(y,v)\in \text{gra}A} \left(\underbrace{\gamma\|x-y\| - \phi(\|x-y\|)}_{\psi(\|x-y\|)}\right) = \sup_{y\in \text{dom}A} \psi(\|x-y\|) = \sup_{p=\|x-y\|\in \mathbb{R}_{++}} \psi(p) < +\infty$ [from (eq:1)]

$(y,v)\in \text{gra}A \Rightarrow y\in \text{dom}A$ /* as v absent in objective */

x: fixed

$\therefore F_A(x,w)-\langle x|w\rangle = \text{finite}$

$\Rightarrow F_A(x,w) = \text{finite} + \underbrace{\langle x|w\rangle}_{\text{finite}} = \text{finite} < +\infty \Leftrightarrow (x,w)\in \text{dom}F_A$ /* for a function f, $\text{dom} f = \{x\in\mathcal{H} \mid f(x)<+\infty\}$ */

So. $\forall_{x\in \text{dom}A} \forall_{w\in\mathcal{H}} (x,w)\in \text{dom}F_A$

$\Leftrightarrow \forall_{(x,w)\in \text{dom}A\times\mathcal{H}} (x,w)\in \text{dom}F_A \Leftrightarrow \text{dom}A\times\mathcal{H} \subseteq \text{dom}F_A$

$\therefore \text{dom}A\times\text{ran}A \subseteq \text{dom}A\times\mathcal{H} \subseteq \text{dom}F_A \Rightarrow \text{dom}A\times\text{ran}A \subseteq \text{dom}F_A$

Definition 24.7 (3* monotone operator) [$A:\mathcal{H}\to 2^{\mathcal{H}}$, monotone] A: 3* monotone $\overset{\text{def}}{\Leftrightarrow} \text{dom}A\times\text{ran}A\subseteq\text{dom}F_A$

A: 3* monotone ■

Proposition 24.15.

[$A:\mathcal{H}\to 2^{\mathcal{H}}$, monotone; $\gamma\in\mathbb{R}_{++}$]

(i) A: 3*-monotone $\Leftrightarrow A^{-1}$: 3*-monotone

(ii) A: 3*-monotone $\Leftrightarrow \gamma A$: 3*-monotone

Proof.

(i) /* Proposition 20.47. (vi). $F_A(x,u) = F_{A^{-1}}(u,x)$, now $F_A^{\intercal}(u,x) = F_A(x,u)$ $\therefore F_A^{\intercal}(u,x) = F_{A^{-1}}(u,x)\ \forall_{(u,x)} \Leftrightarrow F_A^{\intercal} = F_{A^{-1}} \Leftrightarrow \text{dom}F_A^{\intercal} = \text{dom}F_{A^{-1}}$

A: 3* monotone $\Leftrightarrow$

$\text{dom}A\times\text{ran}A \subseteq \text{dom}F_A = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_A(x,u)<+\infty\}$ /* now, $\underbrace{\text{dom}F_A^{\intercal}}_{:(x,u)\mapsto(u,x)} = \{(u,x) \mid F_A^{\intercal}(x,u) = F_A(x,u)<+\infty\}$, so if we swap the coordinates of the vectors in $\text{dom}F_A$ we get $\text{dom}F_A^{\intercal}$ */

$\Leftrightarrow \underbrace{\text{ran}A}_{=\text{dom}A^{-1}} \times \underbrace{\text{dom}A}_{=\text{ran}A^{-1}} \subseteq \text{dom}F_A^{\intercal}$ /* ran A^{-1} = dom A; dom A^{-1} = ran A page 2 */

$\Leftrightarrow \text{dom}A^{-1}\times\text{ran}A^{-1} \subseteq \text{dom}F_A^{\intercal} = \text{dom}F_{A^{-1}}$

$\therefore \text{dom}A^{-1}\times\text{ran}A^{-1} \subseteq \text{dom}F_{A^{-1}}$

$\overset{\text{def}}{\Leftrightarrow} A^{-1}$: 3* monotone ☺

(ii)

A: 3* monotone

$\Leftrightarrow \text{dom}A\times\text{ran}A \subseteq \text{dom}F_A$ /* $\gamma\,\text{ran}A = \gamma\{u\in\mathcal{H} \mid \exists_{x\in\mathcal{H}} Ax\ni u\}$;

$\Leftrightarrow \text{dom}A\times\gamma\,\text{ran}A \subseteq (\text{Id}\times\gamma\text{Id})(\text{dom}F_A)$ /* now, according to Proposition 20.47. (vii)

$\Leftrightarrow \text{dom}\gamma A\times\text{ran}\gamma A \subseteq \text{dom}F_{\gamma A}$

$\Leftrightarrow \gamma A$: 3* monotone

$u = \frac{\tilde{u}}{\gamma} \therefore \tilde{u} = u\gamma$

[A: monotone, gra $A\neq\varnothing$] $F_{\gamma A}(x,\tilde{u}) = \gamma F_A\left(x,\frac{\tilde{u}}{\gamma}\right) \Leftrightarrow F_A(x,u) = \frac{1}{\gamma}F_{\gamma A}(x,u\gamma)$

$\text{dom}F_A = \{(x,u) \mid F_A(x,u)<+\infty\}$

$\Rightarrow (\text{Id}\times\gamma\text{Id})(\text{dom}F_A) = \{(x,\gamma u) \mid \underbrace{F_A(x,u)<+\infty}_{\Leftrightarrow}\} = \{(x,\overbrace{\gamma u}^{\tilde{u}}) \mid F_{\gamma A}(x,u\gamma)<+\infty\} = \{(x,\tilde{u}) \mid F_{\gamma A}(x,\tilde{u})<+\infty\} = \text{dom}F_{\gamma A}$

$\frac{1}{\gamma}F_{\gamma A}(x,u\gamma)<+\infty \Leftrightarrow F_{\gamma A}(x,u\gamma)<+\infty$

$\therefore (\text{Id}\times\gamma\text{Id})(\text{dom}F_A) = \text{dom}F_{\gamma A}$

also, $\text{dom}A = \{x\in\mathcal{H} \mid Ax\neq\varnothing\}$

now $x\in\text{dom}A \Leftrightarrow Ax\neq\varnothing \Leftrightarrow \gamma Ax\neq\varnothing \Leftrightarrow x\in\text{dom}\gamma A$ $\therefore \text{dom}A = \text{dom}\gamma A$ */

$\gamma\,\text{ran}A = \gamma\{y \mid \exists_{x\in\mathcal{H}} Ax\ni y\} = \{\gamma y \mid \exists_{x\in\mathcal{H}} Ax\ni y\} = \{v \mid v=\gamma y \wedge \exists_{x\in\mathcal{H}} Ax\ni \underset{=\frac{v}{\gamma}}{y}\} = \{v \mid \exists_{x\in\mathcal{H}} Ax\ni\frac{v}{\gamma}\}$

$= \{v \mid \exists_{x\in\mathcal{H}} \gamma Ax\ni v\}$

$= \text{ran}\gamma A$

$\therefore \gamma\,\text{ran}A = \text{ran}\gamma A$ */

■

Proposition 24.17.

[A,B: monotone operators, $\mathcal{H}\to 2^{\mathcal{H}}$]

$\forall_{x\in\mathcal{H}} \forall_{u\in\mathcal{H}}\ F_{A+B}(x,u) \leq \left(F_A(x,\cdot) \,\square\, F_B(x,\cdot)\right)(u)$

Proof:

Fix $(x,u)\in\mathcal{H}\times\mathcal{H}$

$\exists (u_1,u_2)\in\mathcal{H}\times\mathcal{H}: u=u_1+u_2$

take, $(y,v_1)\in\text{gra}A \Leftrightarrow v_1\in Ay$, $(y,v_2)\in\text{gra}B \Leftrightarrow v_2\in By$ $\Rightarrow v_1+v_2\in Ay+By=(A+B)y \Leftrightarrow (y,v_1+v_2)\in\text{gra}(A+B)$

$F_{A+B}(x,u) = F_{A+B}(x,u_1+u_2) = \sup_{(y,v)\in\text{gra}(A+B)} \left(\langle x|v\rangle + \langle y|u\rangle - \langle y|v\rangle\right)$ //now, $(y,v_1+v_2)\in\text{gra}(A+B)$

$\geq \langle x|v_1+v_2\rangle + \langle y|u\rangle - \langle y|v_1+v_2\rangle$

$= \langle x|v_1+v_2\rangle + \langle y|u\rangle - \langle y|v_1+v_2\rangle$ // $u=u_1+u_2$

$= \langle x|v_1\rangle + \langle x|v_2\rangle + \langle y|u_1\rangle + \langle y|u_2\rangle - \langle y|v_1\rangle - \langle y|v_2\rangle$

$= \left(\langle x|v_1\rangle + \langle y|u_1\rangle - \langle y|v_1\rangle\right) + \left(\langle x|v_2\rangle + \langle y|u_2\rangle - \langle y|v_2\rangle\right)$ /* recall, $F_A(x,u) = \sup_{(y,v)\in\text{gra}A}\left(\langle x|v\rangle + \langle y|u\rangle - \langle y|v\rangle\right)$ */

$\leq \underbrace{\sup_{(y,v_1)\in\text{gra}A}\left(\langle x|v_1\rangle + \langle y|u_1\rangle - \langle y|v_1\rangle\right)}_{F_A(x,u_1)} + \underbrace{\sup_{(y,v_2)\in\text{gra}A}\left(\langle x|v_2\rangle + \langle y|u_2\rangle - \langle y|v_2\rangle\right)}_{F_B(x,u_2)}$

$= F_A(x,u_1) + F_B(x,u_2)$

I donot follow how $F_{A+B}(x,u) \leq F_A(x,u_1) + F_B(x,u_2)$? ক্যামনে কী?

Theorem 24.20. (Brézis-Haraux)

[A,B: monotone, $\mathcal{H}\to 2^{\mathcal{H}}$, $A+B$: maximally monotone]

one of the following holds

(i) A, B : 3* monotone

(ii) $\operatorname{dom} A \subseteq \operatorname{dom} B$, B : 3* monotone] $\Rightarrow$

- $\overline{\operatorname{ran}}(A+B) = \overline{\operatorname{ran} A + \operatorname{ran} B}$
- $\operatorname{int} \operatorname{ran}(A+B) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B)$

Proof: From Simon's theorm

Theorem 24.17 (Simons)
[A, B : monotone, $\mathcal{H} \to 2^{\mathcal{H}}$, $A+B$: maximally monotone ;
$\forall_{u \in \operatorname{ran} A} \forall_{v \in \operatorname{ran} B} \exists_{x \in \mathcal{H}} \; (x,u) \in \operatorname{dom} F_A \wedge (x,v) \in \operatorname{dom} F_B$]
$\Rightarrow \overline{\operatorname{ran}}(A+B) = \overline{\operatorname{ran} A + \operatorname{ran} B}$, $\operatorname{int} \operatorname{ran}(A+B) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B)$

so if we can prove $\forall_{u \in \operatorname{ran} A} \forall_{v \in \operatorname{ran} B} \exists_{x \in \mathcal{H}} \quad (x,u) \in \operatorname{dom} F_A \wedge (x,v) \in \operatorname{dom} F_B$

then Simon's theorem will take us to the goal.

suppose antecedent (i) holds: i.e., A, B : 3* monotone

$\overset{\text{def}}{\leftrightarrow} \operatorname{dom} A \times \operatorname{ran} A \subseteq \operatorname{dom} F_A$, $\operatorname{dom} B \times \operatorname{ran} B \subseteq \operatorname{dom} F_B$

take any (u,v): $u \in \operatorname{ran} A$, $v \in \operatorname{ran} B$

$A+B$: maximally monotone $\Rightarrow \operatorname{gra}(A+B)$: nonempty $\Rightarrow \operatorname{dom}(A+B)$: nonempty

(maximally monotone operator) Definition 20.20 [$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone] A : maximally monotone $\overset{\text{def}}{\leftrightarrow}$ ... (a/o by the stand definition gra A : nonempty ...)

$\leftrightarrow \exists_{\tilde{x} \in \mathcal{H}} (A+B)\tilde{x} = A\tilde{x} + B\tilde{x} \neq \emptyset$

$\Rightarrow \tilde{x} \in \operatorname{dom} A \cap \operatorname{dom} B$

$\therefore \forall_{u \in \operatorname{ran} A} \forall_{v \in \operatorname{ran} A} \exists_{x := \tilde{x} \in \operatorname{dom} A \cap \operatorname{dom} B} \quad \underbrace{(\tilde{x},u) \in \operatorname{dom} A \times \operatorname{ran} A}_{\subseteq \operatorname{dom} F_A}, \quad \underbrace{(\tilde{x},v) \in \operatorname{dom} B \times \operatorname{ran} B}_{\subseteq \operatorname{dom} F_B}$

... (Simon's antecedent)
$\Downarrow$
goal :)

now suppose antecedent (ii) holds:

$\forall u \in \operatorname{ran} A$
$\exists_{\tilde{x}} \; A\tilde{x} \ni u \Rightarrow \tilde{x} \in \operatorname{dom} A$

$\therefore \forall_{u \in \operatorname{ran} A} \exists_{\tilde{x} \in \operatorname{dom} A} \; \underbrace{(\tilde{x},u) \in \operatorname{gra} A}_{\subseteq \operatorname{dom} F_A}$

Proposition 20.47 (Fitzpatrick function of monotone operator) ★★★★★
[$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone, $\operatorname{gra} A \neq \emptyset$ ($u \in \mathcal{H} \times \mathcal{H}$)] // This proposition charecterizes points in the graph of // a monotone operator via Fitzpatrick function. Nice!
(i) $(x,u) \in \operatorname{gra} A \Rightarrow F_A(x,u) = \langle x | u \rangle$ /* i.e., $(x,u) \in \operatorname{gra} A \Rightarrow (x,u) \in \operatorname{dom} F_A$ */ $\therefore \operatorname{gra} A \subseteq \operatorname{dom} F_A$

$\therefore \forall_{u \in \operatorname{ran} A} \exists_{\tilde{x} \in \operatorname{dom} A} \; (\tilde{x},u) \in \operatorname{dom} F_A$

now. $\tilde{x} \in \operatorname{dom} A \subseteq \operatorname{dom} B$ // given

$\therefore \forall_{v \in \operatorname{ran} B} \; (\tilde{x},v) \in \operatorname{dom} B \times \operatorname{ran} B \subseteq \operatorname{dom} F_B$ // as B : 3* monotone

$\forall_{u \in \operatorname{ran} A} \forall_{v \in \operatorname{ran} B} \exists_{\tilde{x} \in \mathcal{H}} \quad (\tilde{x},u) \in \operatorname{dom} F_A \wedge (\tilde{x},v) \in \operatorname{dom} F_B \Rightarrow$ Simon's antecedent
$\Downarrow$
goal :) ■

Corollary 24.23.
[A, B : maximally monotone, $\mathcal{H} \to 2^{\mathcal{H}}$, $A+B$: maximally monotone
B : uniformly monotone with a supercoercive modulus ϕ,
One of the following holds:
(i) A : 3* monotone
(ii) $\operatorname{dom} A \subseteq \operatorname{dom} B$] $\Rightarrow$

(i) $\operatorname{ran}(A+B) = \mathcal{H}$
(ii) $\operatorname{zer}(A+B)$: singleton.

Proof:
/*

Proposition 22.8.
[$\tilde{A}: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, one of the following holds | $\tilde{A}$: uniformly monotone with a supercoercive modulus | $\tilde{A}$: strongly monotone]
$\lim_{\|x\| \to +\infty} \inf \|\tilde{A}x\| = +\infty$, $\tilde{A}$: surjective (onto) /* $\tilde{A}$: onto $\leftrightarrow \forall_{x \in \mathcal{H}} \exists_{y \in \mathcal{H}} \; \tilde{A}y \ni x$ */

*/

/* $f: \mathcal{H} \to [-\infty, +\infty]$: supercoercive $\overset{\text{def}}{\leftrightarrow} \lim_{\|x\| \to +\infty} (f(x)/\|x\|) = +\infty$
$\tilde{A}$: uniformly monotone with modulus $\phi \overset{\text{def}}{\leftrightarrow} \phi$: increasing, vanishes only at 0, $\forall_{(x,u),(y,v) \in \operatorname{gra} A} \langle x-y | u-v \rangle \geq \phi(\|x-y\|)$
★ Example 24.11. [$\tilde{A}: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, uniformly monotone with supercoercive modulus ϕ] $\Rightarrow$ $\tilde{A}$: 3* monotone.
*/

setting $\tilde{A} := B$

B : 3* monotone, surjective (onto)

★ Corollary 24.22. [A, B : monotone, $\mathcal{H} \to 2^{\mathcal{H}}$, $A+B$: maximally monotone, A or B : $\overbrace{\text{surjective}}^{\text{onto}}$
One of the following holds: (i) A, B : 3* monotone, (ii) $\operatorname{dom} A \subseteq \operatorname{dom} B$, B : 3* monotone
] $\Rightarrow$ $A+B$: surjective

+ given (i) $\vee$ (ii)

$\Rightarrow$ $A+B$: surjective $\leftrightarrow \operatorname{ran}(A+B) = \mathcal{H}$.

(ii) B : uniformly monotone $\Rightarrow$ B : strictly monotone
A : monotone
$\overline{}$
$A+B$: strictly monotone
set $\tilde{A} := A+B$

Proposition 23.35. [$\tilde{A}: \mathcal{H} \to 2^{\mathcal{H}}$, strictly monotone] $\operatorname{zer} \tilde{A}$: almost a singleton

zer(A+B) : singleton (as $\operatorname{zer}(A+B) \neq \emptyset$ by (i))

■

// Parallel sum of resolvents:

Proposition 24.28.
[A, B : monotone operators, $\mathcal{H} \to 2^{\mathcal{H}}$, $A+B$: maximally monotone]
$J_A \square J_B = J_{\frac{1}{2}(A+B)} \circ \frac{1}{2} \operatorname{Id}$

Proof:

$(x,u)\in \operatorname{gra}(J_A \square J_B) \leftrightarrow (J_A\square J_B)x \ni u$ /* now,

Elementary properties of parallel sum

$\tilde{A} := J_A,\ \tilde{B} = J_B$

Proposition 24.?: [$\tilde{A},\tilde{B}$: monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$] ⇒

(i) $(\tilde{A}\square\tilde{B})x = \bigcup_{y\in\mathcal{H}} (\tilde{A}y \cap \tilde{B}(x-y))$

(ii) $u\in(\tilde{A}\square\tilde{B})x \Leftrightarrow \exists_{y\in\mathcal{H}} (y\in\tilde{A}^{-1}u \wedge x-y\in\tilde{B}^{-1}u)$ */

$\leftrightarrow \exists_{y\in\mathcal{H}} (y\in J_A^{-1}u) \wedge (x-y\in J_B^{-1}u)$

$\leftrightarrow \exists_{y\in\mathcal{H}}\ J_Ay\ni u \wedge J_B(x-y)\ni u$ ... (eq:0)

/* now, A,B: monotone ⇒ J_A, J_B: firmly nonexpansive ⇒ single-valued operator */

$\leftrightarrow \exists_{y\in\mathcal{H}}\ J_Ay = u \wedge J_B(x-y) = u$

$\leftrightarrow \exists_{y\in\mathcal{H}}\ u = J_Ay = J_B(x-y)$ ... (eq:1)

/* now

Proposition 23.2: Basic relationships between A, γA, $J_{\gamma A}$

[$A:\mathcal{H}\to 2^{\mathcal{H}}$, $\gamma\in\mathbb{R}_{++}$, ...]

(i) $\operatorname{dom} J_{\gamma A} = \operatorname{ran}(\operatorname{Id}+\gamma A)$; $\operatorname{ran} J_{\gamma A} = \operatorname{dom} A$

(ii) ... $\Leftrightarrow (p, \frac{1}{\gamma}(x-p))\in\operatorname{gra}A$

(iii) ... */

$\leftrightarrow (u, y-u)\in\operatorname{gra}A \wedge (u, x-y-u)\in\operatorname{gra}B$ ... (eq:2)

now: $A+B$: maximally monotone ⇒ $\operatorname{gra}(A+B)\neq\emptyset$

let. $(z,w)\in\operatorname{gra}(A+B)$

$\leftrightarrow (A+B)z = Az+Bz \ni w$

$\Leftrightarrow \exists_{w_A\in Az} \exists_{w_B\in Bz}\ w = w_A + w_B \in Az+Bz$

$\leftrightarrow \exists_{(z,w_A)\in\operatorname{gra}A} \exists_{(z,w_B)\in\operatorname{gra}B}\ w = w_A+w_B$ ... (eq:3)

A,B: monotone /* A: monotone $\overset{\text{def}}{\leftrightarrow} \forall_{(\tilde{x},\tilde{u}),(\tilde{y},\tilde{v})\in\operatorname{gra}A} \langle\tilde{x}-\tilde{y}\mid\tilde{u}-\tilde{v}\rangle\geq 0$ */

$$\left.\begin{array}{l}\langle u-z\mid y-u-w_A\rangle\geq 0\\ \langle u-z\mid x-y-u-w_B\rangle \geq 0\end{array}\right\}\overset{\oplus}{\Rightarrow} \langle u-z\mid \underbrace{\cancel{y}-u-w_A+x-\cancel{y}-u-w_B}_{\substack{\| \\ x-2u-\underbrace{(w_A+w_B)}_{w}\\ =\ x-2u-w}}\rangle\geq 0 \leftrightarrow \langle u-z\mid (x-2u)-w\rangle\geq 0\ \forall_{(z,w)\in\operatorname{gra}(A+B)}$$

/* now $(z,w)\in\operatorname{gra}(A+B)$, and

recall that $\tilde{A}$: maximally monotone $\overset{\text{def}}{\leftrightarrow} \forall_{(\tilde{x},\tilde{u})} \left((\tilde{x},\tilde{u})\in\operatorname{gra}\tilde{A} \leftrightarrow \forall_{(\tilde{y},\tilde{v})\in\operatorname{gra}\tilde{A}} \langle\tilde{x}-\tilde{y}\mid\tilde{u}-\tilde{v}\rangle\geq 0\right)$ */

set $\tilde{A} := A+B$

$\Leftrightarrow (\underbrace{u, x-2u}_{(u,\,2(\frac{x}{2}-u))})\in\operatorname{gra}(A+B)$

/* recall that according to Proposition 23.2 (ii): $(p, \frac{1}{\gamma}(x-p))\in\operatorname{gra}A \leftrightarrow p\in J_{\gamma A}(x)$ */

$\leftrightarrow u = J_{\frac{1}{2}(A+B)}(\tfrac{1}{2}x) = (J_{\frac{1}{2}(A+B)}\circ\tfrac{1}{2}\operatorname{Id})(x) \leftrightarrow (x,u)\in\operatorname{gra}(J_{\frac{1}{2}(A+B)}\circ\tfrac{1}{2}\operatorname{Id})$

so, we have proven that:

$\forall_{(x,u)} \left((x,u)\in\operatorname{gra}(J_A\square J_B) \leftrightarrow (x,u)\in\operatorname{gra}(J_{\frac{1}{2}(A+B)}\circ\tfrac{1}{2}\operatorname{Id})\right)$ //as along the way we have maintained ⇔

$\leftrightarrow J_A\square J_B = J_{\frac{1}{2}(A+B)}\circ\frac{1}{2}\operatorname{Id}$

∎

Part 2

12:38 PM

Proposition 24.26.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, atmost single valued,

$B: \mathcal{H} \to \mathcal{H}$, linear]

$A \square B = A(A+B)^{-1}B$

Proof:

$\forall_{(x,u) \in \mathcal{H} \times \mathcal{H}}$

$(x,u) \in \text{gra}(A \square B)$ /* $A \square B = (A^{-1}+B^{-1})^{-1}$

$\leftrightarrow$ $(A \square B)x \ni u \leftrightarrow (A^{-1}+B^{-1})^{-1}x \ni u$

$\leftrightarrow$ $x \in (A^{-1}+B^{-1})u = A^{-1}u + B^{-1}u$

$\leftrightarrow \exists_{y \in \mathcal{H}}$ $y \in A^{-1}u \wedge x \in y + B^{-1}u$

$\leftrightarrow \exists_{y \in \mathcal{H}}$ $y \in A^{-1}u \wedge x - y \in B^{-1}u$

$\leftrightarrow$ $Ay \ni u \wedge B(x-y) \ni u$ /* A : atmost single valued $\therefore Ay = u$

B: $\mathcal{H} \to \mathcal{H}$, linear $\therefore B(x-y) = u$ */

$\leftrightarrow$ $u = Ay = Bx - By$

$\leftrightarrow$ $u = Ay$, $Ay + By = (A+B)y = Bx$

$\leftrightarrow$ $u = Ay$, $y \in (A+B)^{-1}Bx$

$\leftrightarrow$ $u = Ay \in A(A+B)^{-1}Bx$

$\leftrightarrow$ $(x,u) \in \text{gra}\left(A(A+B)^{-1}B\right)$

$\therefore (A \square B) = A(A+B)^{-1}B$ ∎