Part 1

9:53 AM

Definition. 23.1.

$[\![A:\mathcal{H}\to 2^{\mathcal{H}},\ \gamma\in\mathbb{R}_{++}]\!]$

J_A : resolvent of $A \overset{\text{def}}{\leftrightarrow} J_A = (\mathrm{Id}+A)^{-1}$

${}^{\gamma}A$: Yosida approximation of A with index $\gamma \leftrightarrow {}^{\gamma}A = \frac{1}{\gamma}(\mathrm{Id}-J_{\gamma A})$

Proposition 23.2.

$[A:\mathcal{H}\to 2^{\mathcal{H}},\ \gamma\in\mathbb{R}_{++},\ x\in\mathcal{H},\ p\in\mathcal{H}]$

(i) $\mathrm{dom}\, J_{\gamma A} = \mathrm{dom}\, {}^{\gamma}A = \mathrm{ran}(\mathrm{Id}+\gamma A)$;

$\mathrm{ran}\, J_{\gamma A} = \mathrm{dom}\, A$

(ii) $p\in J_{\gamma A}(x) \leftrightarrow x\in p+\gamma Ap$

$\leftrightarrow (x-p)\in\gamma Ap$

$\leftrightarrow \left(p, \frac{1}{\gamma}(x-p)\right)\in \mathrm{gra}\, A$

(iii) $p\in {}^{\gamma}A x \leftrightarrow p\in A(x-\gamma p) \leftrightarrow (x-\gamma p, p)\in \mathrm{gra}\, A$

Example 23.3.

$f\in\Gamma_0(\mathcal{H}),\ \gamma\in\mathbb{R}_{++} \Rightarrow \begin{cases} \mathrm{Prox}_{\gamma f} = J_{\gamma\partial f} \\ \nabla({}^{\gamma}f) = {}^{\gamma}(\partial f) \end{cases}$

Proof:

Proposition 16.34:

$[f\in\Gamma_0(\mathcal{H})\ ;\ x,p\in\mathcal{H}]$

$p = \mathrm{Prox}_f x \Leftrightarrow x-p\in\partial f(p)$, i.e.,

$\mathrm{Prox}_f = (\mathrm{Id}+\partial f)^{-1}$

$\Rightarrow \mathrm{Prox}_{\gamma f} = (\mathrm{Id}+\partial\gamma f)^{-1} = (\mathrm{Id}+\gamma\partial f)^{-1} = J_{\gamma\partial f}$ /* By def */

Proposition 12.29.

$[\ f\in\Gamma_0(\mathcal{H});\ \gamma\in\mathbb{R}_{++}]\!]$

${}^{\gamma}f:\mathcal{H}\to\mathbb{R}$, Frechet differentiable // ${}^{\gamma}f$: Moreau envelope of f ; ${}^{\gamma}f = f \mathbin{\square} \left(\frac{1}{2\gamma}\|\cdot\|^2\right)$

$\nabla({}^{\gamma}f) = \frac{1}{\gamma}(\mathrm{Id}-\mathrm{Prox}_{\gamma f})$: γ^{-1} Lipschitz continuous

$= \frac{1}{\gamma}(\mathrm{Id}-J_{\gamma\partial f}) = \underbrace{{}^{\gamma}(\partial f)}_{\text{Yosida approximation of } \partial f}$

∎

Example 23.4.

$[$ C: nonempty closed convex subset, $\subseteq\mathcal{H}$

$\gamma\in\mathbb{R}_{++}]\!]$

$J_{N_C} = P_C$ // N_C : normal cone operator

${}^{\gamma}N_C = \frac{1}{\gamma}(\mathrm{Id}-P_C)$

Proof:

/* Example 12.25: $[\![$ C: nonempty closed convex subset of $\mathcal{H}]\!]$ $\mathrm{Prox}_{\iota_C} = P_C$ */

in Example 23.4. set $f := \iota_C$ we have:

$J_{\gamma\partial f} = J_{\gamma N_C}$ // $\because N_C = \partial\iota_C$

$= \mathrm{Prox}_{\gamma f} = P_C$

$\therefore J_{N_C} = P_C$

and ${}^{\gamma}N_C = \frac{1}{\gamma}(\mathrm{Id}-J_{\gamma\partial f}) = \frac{1}{\gamma}(\mathrm{Id}-J_{\gamma N_C}) = \frac{1}{\gamma}(\mathrm{Id}-P_C)$

∎

Proposition 23.7.

$[$ D: nonempty, $\subseteq\mathcal{H}$

$T: D\to\mathcal{H}$

$A = T^{-1}-\mathrm{Id}]$

(i) $T = J_A$

(ii) T: firmly nonexpansive $\leftrightarrow$ A: monotone

(iii) $\begin{pmatrix} T: \text{firmly nonexpansive} \\ D = \mathcal{H} \end{pmatrix} \Rightarrow A$: maximally monotone

Proof:

(i) from Def $J_A = (Id + A)^{-1}$

$$\forall_{(x,u)} \quad (x,u) \in \operatorname{gra} J_A$$
$$\leftrightarrow J_A x \ni u$$
$$\leftrightarrow (Id+A)^{-1} x \ni u$$
$$\leftrightarrow x \in (Id+A)u$$
$$\leftrightarrow x \in u + Au$$
$$\leftrightarrow x \in u + (T^{-1} - Id)u$$
$$\leftrightarrow x - \cancel{u} \in T^{-1}u - \cancel{u}$$
$$\leftrightarrow x \in T^{-1}u$$
$$\leftrightarrow Tx \ni u$$
$$\leftrightarrow (x,u) \in \operatorname{gra} T$$

$\therefore \operatorname{gra} J_A = \operatorname{gra} T \leftrightarrow T = J_A$.

(ii)

First prove: T: firmly nonexpansive $\Rightarrow$ A: monotone

given T: firmly nonexpansive

$$\leftrightarrow \forall_{x \in D} \forall_{y \in D} \quad \|Tx - Ty\|^2 \leq \langle x - y \mid Tx - Ty \rangle$$

take $(x,u) \in \operatorname{gra} A \leftrightarrow Ax \ni u \leftrightarrow (T^{-1} - Id)x = T^{-1}x - x \ni u \leftrightarrow T^{-1}x \ni x + u \leftrightarrow x \in T(x+u) \leftrightarrow x = T(x+u)$

// as $T: D \to \mathcal{H}$ so its a single-valued
// operator

$(y,v) \in \operatorname{gra} A$, similarly $y = T(y+v)$, $y + v \in D$

as $T(x+u) = x \in \mathcal{H}$, we can say that $(x+u) \in D$

as

// recall that By def
// $T: D \to \mathcal{H}$ means that T maps every point $\tilde{x}$ in D
// to a point Tx in $\mathcal{H}$, as $T(x+u) = x \in \mathcal{H}$, $x + u \in D$

now, Proposition 4.2.(v) says: T: firmly nonexpansive

$$\leftrightarrow \forall_{x \in D} \forall_{y \in D} \quad \langle Tx - Ty \mid (Id - T)x - (Id - T)y \rangle \geq 0$$

$x + u \in D$, $y + v \in D$

$$\Rightarrow \underbrace{\langle T(x+u) - T(y+v) \mid (Id - T)(x+u) - (Id - T)(y+v) \rangle}_{\langle x - y \mid x + u - T(x+u) - (y+v) + T(y+v) \rangle} \geq 0$$

// ingenerally $T(x+y) \neq Tx + Ty$, if so then T: linear
// but $(T+R)x = Tx + Rx$

$$= \langle x - y \mid x + u - x - y - v + y \rangle = \langle x - y \mid u - v \rangle$$

$$\leftrightarrow \langle x - y \mid u - v \rangle \geq 0$$

$\therefore$ So we have proven that, $\forall_{(x,u) \in \operatorname{gra} A} \forall_{(y,v) \in \operatorname{gra} A} \langle x - y \mid u - v \rangle \geq 0 \overset{\text{def}}{\leftrightarrow} A$: monotone ($\Rightarrow$ proved)

Now let us show that,

A: monotone $\Rightarrow$ T: firmly nonexpansive

take $x, y \in D$

now, $T: D \to \mathcal{H}$

$$\Rightarrow \exists_{u \in \mathcal{H}} \; u = Tx = J_A x \quad \text{// By (i) } T = J_A = (Id + A)^{-1}$$
$$= (Id + A)^{-1} x$$

$$\leftrightarrow (Id + A)^{-1} x = u$$

// this says: $\forall_x \left(x \in D \leftrightarrow x \in \operatorname{ran}(Id + A) \right) \Leftrightarrow D = \operatorname{ran}(Id + A)$

$$\leftrightarrow x \in (Id + A)y = y + Ay$$

$$\leftrightarrow x - u \in Au \quad \text{// now } u = Tx$$

$$\leftrightarrow x - Tx \in A(Tx) \leftrightarrow (Tx, x - Tx) \in \operatorname{gra} A$$

similarly, $y - Ty \in A(Ty) \leftrightarrow (Ty, y - Ty) \in \operatorname{gra} A$

A: monotone $\overset{\text{def}}{\Leftrightarrow} \forall_{(x,u) \in \operatorname{gra} A} \forall_{(y,v) \in \operatorname{gra} A} \langle x - y \mid u - v \rangle \geq 0$

$$\langle Tx - Ty \mid (Id - T)x - (Id - T)y \rangle \geq 0$$

so we have shown that: Proposition 4.2(v) so, we have shown that:

$\forall_{x\in D} \forall_{y\in D} \ \langle Tx-Ty \mid (\mathrm{Id}-T)x-(\mathrm{Id}-T)y\rangle \geq 0 \xleftrightarrow{\text{Proposition 4.2(v)}} T:\text{firmly nonexpansive}$

($\Rightarrow$ proved)

(iii) $T: D\to\mathcal{H} \Rightarrow \operatorname{dom} T = D$ //see note page 2 of Bauschke

$\Leftrightarrow \operatorname{ran} T^{-1} = \operatorname{dom} T = D$

$\Leftrightarrow \operatorname{ran}(\mathrm{Id}+A) = D$ // $\because T^{-1} = \mathrm{Id}+A$

given $D=\mathcal{H}$

$\operatorname{ran}(\mathrm{Id}+A)=\mathcal{H} \xleftrightarrow{\text{Minty's theorem}} A:\text{maximally monotone}$ ∎

Proposition 23.9.

[$A:\mathcal{H}\to 2^{\mathcal{H}}$, $\operatorname{dom} A \neq \emptyset$,

$D = \operatorname{ran}(\mathrm{Id}+A)$

$T = J_{A|D}$]

(i) $A = T^{-1}-\mathrm{Id}$

(ii) A: monotone $\Leftrightarrow$ T: firmly nonexpansive

(iii) A: maximally monotone $\Leftrightarrow$ (T: firmly nonexpansive $\wedge$ $D=\mathcal{H}$)

Proof:

(i) $(x,u)\in\operatorname{gra} A \Leftrightarrow Ax \ni u \Leftrightarrow Ax+x \ni u+x \Leftrightarrow (A+\mathrm{Id})(x) \ni u+x \Rightarrow u+x \in \operatorname{ran}(\mathrm{Id}+A) = D$ . . . (1)

$(x,u)\in\operatorname{gra}(T^{-1}-\mathrm{Id}) \Leftrightarrow (T^{-1}-\mathrm{Id})(x)\ni u$

$\Leftrightarrow T^{-1}x - x \ni u$

$\Leftrightarrow T^{-1}x \ni x+u$

$\Leftrightarrow x \in T(x+u) = J_{A|_D}(x+u)$ // As $(x+u)\in D$, we have $J_{(A|D)}(x+u) = J_A(x+u)$

$= (\mathrm{Id}+A)^{-1}(x+u)$ {$(x+u)\in D$ // From (1)}

$\Leftrightarrow x = (\mathrm{Id}+A)^{-1}(x+u)$

$\Leftrightarrow (\mathrm{Id}+A)(x) \ni x+u$ {$x+u\in D$}

$\Leftrightarrow x+Ax \ni x+u$ {$x+u\in D$}

$\Leftrightarrow Ax \ni u \Leftrightarrow (x,u)\in\operatorname{gra} A$ {$x+u\in D$}

$\therefore \forall_{(x,u)} \left((x,u)\in\operatorname{gra} A \Leftrightarrow (x,u)\in(T^{-1}-\mathrm{Id})\right)$

$\Leftrightarrow A = T^{-1}-\mathrm{Id}$ ☺

(ii)

First we prove, A: monotone $\Rightarrow$ T: firmly nonexpansive

take, // At first we will prove that J_A is single valued, i.e., for $(x,u)\in\operatorname{gra}J_A, (y,v)\in\operatorname{gra}J_A$, if $x=y$, then $u=v$. To do this we find out the realtionship between x,u,y,v with A, and use its monotonicity.

take $(x,u)\in\operatorname{gra}(J_A)$

$\Leftrightarrow J_A(x)\ni u$

$\Leftrightarrow (\mathrm{Id}+A)^{-1}x \ni u$ //note that as $\exists_u (\mathrm{Id}+A)u \ni x \Rightarrow x\in\operatorname{ran}(\mathrm{Id}+A)=D$

1 $\Leftrightarrow x\in(\mathrm{Id}+A)u = u+Au$

$\Leftrightarrow (x-u)\in Au \Leftrightarrow (u,x-u)\in\operatorname{gra} A$

similarly, $(y,v)\in\operatorname{gra} J_A \Leftrightarrow (y-v)\in Av \Leftrightarrow (v,y-v)\in\operatorname{gra} A$ // similarly, $y\in\operatorname{ran}(\mathrm{Id}+A)=D$

A: monotone $\overset{\text{def}}{\Leftrightarrow} \forall_{(a,b)\in\operatorname{gra}A} \forall_{(c,d)\in\operatorname{gra}A} \langle a-c \mid b-d\rangle \geq 0$

$\underbrace{\langle u-v \mid (x-u)-(y-v)\rangle}_{} \geq 0$

$\langle u-v \mid x-y-(u-v)\rangle = -\|u-v\|^2 + \langle u-v \mid x-y\rangle$

$\Leftrightarrow \langle u-v \mid x-y\rangle \geq \|u-v\|^2$ . . . (eq:3)

set $x:=y$ $\Rightarrow \|u-v\|^2 \leq 0 \Leftrightarrow \|u-v\|^2=0 \Leftrightarrow \underbrace{u}_{Tx}=\underbrace{v}_{Ty}$

so, $x=y \Rightarrow u=v \Rightarrow J_A$: single-valued $\Rightarrow T=J_{A|D}$: single valued

thus (eq:3) becomes: // by setting $u=Tx$, $v=Ty$:

$\langle Tx-Ty \mid x-y\rangle \geq \|Tx-Ty\|^2 \quad \forall_{x\in D} \forall_{y\in D}$

//Recall Proposition 4.2 (iv): T: firmly nonexpansive $\Leftrightarrow \forall \ \forall \ \langle x-y \mid Tx-Ty\rangle \geq \|Tx-Ty\|^2$

$\langle Tx - Ty \mid x - y \rangle \geq \|Tx - Ty\|^2 \quad \forall_{x \in D} \ \forall_{y \in D}$

// Recall Proposition 4.2.(iv): T: firmly nonexpansive $\Leftrightarrow \forall_{x \in D} \ \forall_{y \in D} \ \langle x - y \mid Tx - Ty \rangle \geq \|Tx - Ty\|^2$

T: firmly nonexpansive. ($\Rightarrow$ direction proved)

($\Leftarrow$)

T: firmly nonexpansive $\Rightarrow$ A: monotone

From 23.7 (ii): A: monotone

Proposition 23.7.
[D: nonempty subset of $\mathcal{H}$
$T: D \to \mathcal{H}$
$A = T^{-1} - \mathrm{Id}$]
(i) $T = J_A$
(ii) T: firmly nonexpansive $\Leftrightarrow$ A: monotone
(iii) (T: firmly nonexpansive, $D = \mathcal{H}$) $\Leftrightarrow$ A: maximally monotone

:)

(iii)
follows from (i), (ii) and

:)

■

Corollary 23.10.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, $\gamma \in \mathbb{R}_{++}$]

(i) $J_{\gamma A}: \mathcal{H} \to \mathcal{H}$: firmly nonexpansive, maximally monotone

$(\mathrm{Id} - J_{\gamma A}): \mathcal{H} \to \mathcal{H}$: firmly nonexpansive, maximally monotone

(ii) Reflected resolvent (Cayley operator)

$R_{\gamma A}: \mathcal{H} \to \mathcal{H}: x \mapsto 2J_{\gamma A}x - x$: nonexpansive

(iii) ${}^{\gamma}A: \mathcal{H} \to \mathcal{H}$: γ-cocoercive

(iv) ${}^{\gamma}A$: maximally monotone

(v) ${}^{\gamma}A: \mathcal{H} \to \mathcal{H}$: $\frac{1}{\gamma}$ Lipschitz continuous

Proof: /* Corollary 23.8.

T: firmly nonexpansive $\Leftrightarrow \exists_{A: \mathcal{H} \to 2^{\mathcal{H}}, \text{ maximally monotone}} \ T = J_A = (\mathrm{Id} + A)^{-1}$ */

using this

A: maximally monotone, $\gamma \in \mathbb{R}_{++}$
$\Downarrow$

$J_{\gamma A}$: firmly nonexpansive // $\because$ $J_{\gamma A} = (\mathrm{Id} + \gamma A)^{-1}$: (γA: maximally monotone)

$\Leftrightarrow (\mathrm{Id} - J_{\gamma A})$: firmly nonexpansive /* Proposition 4.2.

[D: nonempty, $\subseteq \mathcal{H}$; $T: D \to \mathcal{H}$] T: firmly expansive $\Leftrightarrow$

$\mathrm{Id} - T$: firmly nonexpansive */

/* Example 20.27.

$(T: \mathcal{H} \to \mathcal{H}, \ ]0, \frac{1}{2}]$ averaged$) \Rightarrow T$: maximally monotone

recall that $]0, \frac{1}{2}]$ averaged operators are firmly nonexpansive

*/

$J_{\gamma A}, \mathrm{Id} - J_{\gamma A}$: firmly nonexpansive, $\mathcal{H} \to \mathcal{H}$

$\Rightarrow J_{\gamma A}, \mathrm{Id} - J_{\gamma A}$: maximally monotone :)

(ii)

/* recall Proposition 4.2. (iii)

[$T: D \to \mathcal{H}$] T: firmly nonexpansive $\Leftrightarrow 2T - \mathrm{Id}$: nonexpansive

*/

$J_{\gamma A}$: firmly nonexpansive // in (i)

$\Leftrightarrow 2J_{\gamma A} - \mathrm{Id} = R_{\gamma A}$: nonexpansive

(iii) in (i) we have proven:

$(\mathrm{Id} - J_{\gamma A})$: firmly nonexpansive

(now by the definition of Yosida approximation: ${}^{\gamma}A = \frac{1}{\gamma}(\mathrm{Id} - J_{\gamma A}) \Leftrightarrow (\mathrm{Id} - J_{\gamma A}) = \gamma \, {}^{\gamma}A$

$(Id - J_{\gamma A})$: firmly nonexpansive

now By the definition of Yosida approximation: ${}^{\gamma}A = \frac{1}{\gamma}(Id - J_{\gamma A}) \Leftrightarrow (Id - J_{\gamma A}) = \gamma\, {}^{\gamma}A$

$\Leftrightarrow \gamma\, {}^{\gamma}A$: firmly nonexpansive

$\Leftrightarrow {}^{\gamma}A$: γ-cocoercive // T: β-cocoercive $\Leftrightarrow \beta T$: firmly nonexpansive

:)

(iv) // Example 20.28:

[[$T: \mathcal{H} \to \mathcal{H}$, β-cocoercive, $\beta \in \mathbb{R}_{++}$]] T: maximally monotone

in (iii): ${}^{\gamma}A : \underbrace{\gamma}_{\in \mathbb{R}_{++}}$-cocoercive $\Rightarrow {}^{\gamma}A$: maximally monotone

(v) in (iii), we have proven that: ${}^{\gamma}A$: γ-cocoercive

Cauchy-Schwarz

$$\overset{\text{def}}{\Leftrightarrow} \forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} \; \gamma \|{}^{\gamma}Ax - {}^{\gamma}Ay\|^2 \leq \langle x - y \mid {}^{\gamma}Ax - {}^{\gamma}Ay \rangle \leq \|x - y\| \, \|{}^{\gamma}Ax - {}^{\gamma}Ay\|$$

$$\Leftrightarrow \forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} \; \|{}^{\gamma}Ax - {}^{\gamma}Ay\| \leq \frac{1}{\gamma} \|x - y\|$$

$\therefore {}^{\gamma}A$: $\frac{1}{\gamma}$ Lipschitz continuous.

$\blacksquare$

Part 2

7:37 AM

Proposition 23.11.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone

$\beta \in \mathbb{R}_{++}$]

A: strongly monotone with constant $\beta \Leftrightarrow$

J_A: $(\beta+1)$ cocoercive $\Rightarrow J_A$: Lipschitz continuous with constant $\frac{1}{\beta+1} \in]0,1[$

Proof:

$x, y, u, v \in \mathcal{H}$

($\Rightarrow$ direction)

$A: \beta$-strongly monotone $\Rightarrow A$: monotone $\Rightarrow T = J_A$: firmly nonexpansive (single valued operator) on its domain

/* (Proposition 23.7. (ii): J_A: firmly nonexpansive $\leftrightarrow A$: monotone */

/* Proposition 23.2. // Basic relationships between A, ${}^{\gamma}A$, $J_{\gamma A}$

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, $\gamma \in \mathbb{R}_{++}$, $x \in \mathcal{H}$, $p \in \mathcal{H}$]

(i) $\operatorname{dom} J_{\gamma A} = \operatorname{dom} {}^{\gamma}A = \operatorname{ran}(\mathrm{Id} + \gamma A)$; $\operatorname{ran} J_{\gamma A} = \operatorname{dom} A$

(ii) $p \in J_{\gamma A}(x) \Leftrightarrow x \in p + \gamma A p \Leftrightarrow x - p \in \gamma A p \Leftrightarrow (p, \frac{1}{\gamma}(x-p)) \in \operatorname{gra} A$

(iii) $p \in {}^{\gamma}A x \Leftrightarrow p \in A(x - \gamma p) \Leftrightarrow (x - \gamma p, p) \in \operatorname{gra} A$ */

take (x,u): $J_A x = u \leftrightarrow x - u \in Au \leftrightarrow (u, x-u) \in \operatorname{gra} A$

(y,v): $J_A y = v \leftrightarrow y - v \in Av \leftrightarrow (v, y-v) \in \operatorname{gra} A$

$\therefore A: \beta$ strongly monotone $\overset{\text{def}}{\leftrightarrow} \forall_{(x,u) \in \operatorname{gra} A} \forall_{(y,v) \in \operatorname{gra} A} \langle x-y \mid u-v \rangle \geq \beta \|x-y\|^2$

$$\langle u-v \mid \underbrace{\underbrace{(x-u)-(y-v)}_{(x-y)-(u-v)}} \rangle \geq \beta \|u-v\|^2$$

$= \langle u-v \mid x-y \rangle - \|u-v\|^2$

$\leftrightarrow \langle u-v \mid x-y \rangle \geq (\beta+1)\|u-v\|^2 \quad \therefore \forall_{x \in \operatorname{dom} J_A} \forall_{y \in \operatorname{dom} J_A} \langle x-y \mid J_A x - J_A y \rangle \geq (\beta+1)\|J_A x - J_A y\|^2 \overset{\text{def}}{\leftrightarrow} J_A: (\beta+1)$ cocoercive on $\operatorname{dom} J_A = \operatorname{ran}(\mathrm{Id}+A) = (\mathrm{Id}+A)(\mathcal{H})$ // from 23.2 (i): $\operatorname{dom} J_{\gamma A}$

$\Rightarrow \|u-v\|\,\|x-y\| \geq \langle u-v \mid x-y \rangle \geq (\beta+1)\|u-v\|^2$

$\leftrightarrow \|x-y\| \geq (\beta+1)\|u-v\| \leftrightarrow \|J_A x - J_A y\| \leq \frac{1}{\beta+1}\|x-y\|$

$\therefore$ J_A: $(\beta+1)$ cocoercive $\Rightarrow$

J_A: $\underbrace{\frac{1}{\beta+1}}_{\in]0,1[}$ Lipschitz continuous.

($\Leftarrow$)

J_A: $(\beta+1)$ cocoercive

$\overset{\text{def}}{\leftrightarrow} \forall_{x \in \operatorname{dom} J_A} \forall_{y \in \operatorname{dom} J_A} \langle x-y \mid J_A x - J_A y \rangle \geq (\beta+1)\|J_A x - J_A y\|^2$

now take $(x,u) \in \operatorname{gra} A \leftrightarrow Ax \ni u = (u+x) - x$ // now: Proposition 23.2. (ii) $\boxed{x} - p \in \gamma A p \leftrightarrow p \in J_{\gamma A}\boxed{x}$ */

$\leftrightarrow J_A(u+x) \ni x$ // now A: monotone $\Rightarrow J_A$: single valued, firmly nonexpansive operator

$\leftrightarrow J_A(u+x) = x$

similarly $(y,v) \in \operatorname{gra} A \leftrightarrow J_A(y+v) = y$

$(x,y) = (J_A(u+x), J_A(y+v))$ // now, clearly $(u+x) \in \operatorname{dom} J_A$, $(u+v) \in \operatorname{dom} J_A$

$\Rightarrow$

$\langle (u+x)-(y+v) \mid \underset{x}{J_A(u+x)} - \underset{y}{J_A(y+v)} \rangle \geq (\beta+1)\|\underbrace{J_A(u+x)}_{x} - \underbrace{J_A(y+v)}_{y}\|^2$

$\leftrightarrow \langle (u-v)+(x-y) \mid x-y \rangle \geq (\beta+1)\|x-y\|^2$

$\leftrightarrow \langle u-v \mid x-y \rangle + \|x-y\|^2 \geq (\beta+1)\|x-y\|^2$

$\leftrightarrow \langle u-v \mid x-y \rangle \geq \beta\|x-y\|^2$

$\therefore \forall_{(x,u) \in \operatorname{gra} A} \forall_{(y,v) \in \operatorname{gra} A} \langle x-y \mid u-v \rangle \geq \beta\|x-y\|^2 \overset{\text{def}}{\leftrightarrow} A: \beta$-strongly monotone

$A: \mathcal{H} \to 2^{\mathcal{H}}$: strongly monotone with constant $\beta \in \mathbb{R}_{++} \overset{\text{def}}{\leftrightarrow} (A - \beta\mathrm{Id})$: monotone $\leftrightarrow$ $(\forall (x,u) \in \operatorname{gra} A)(\forall (y,v) \in \operatorname{gra} A)$ $\langle x-y \mid u-v \rangle \geq \beta\|x-y\|^2$

/* Extending a firmly nonexpansive operator on $D \subseteq \mathcal{H}$ (corresponding to a monotone operator) to a

" " " " " $\mathcal{H}$ (" " " maximally monotone operator) */

Theorem 23.13.

[D: nonempty $\subseteq \mathcal{H}$; $T: D \to \mathcal{H}$, firmly nonexpansive] $\Rightarrow$

$\exists_{\tilde{T}: \mathcal{H} \to \mathcal{H}} (\tilde{T}$: firmly nonexpansive, $\tilde{T}|_D = T$, $\operatorname{ran} \tilde{T} \subseteq \overline{\operatorname{conv}}\operatorname{ran} T)$

Proof:

*An important assertion of Proposition 23.7. (ii)

[$T: \underbrace{D}_{\neq\emptyset \subseteq \mathcal{H}} \to \mathcal{H}$: firmly nonexpansive] $\exists_{A: \mathcal{H} \to 2^{\mathcal{H}},\ A = T^{-1} - \mathrm{Id}} (A$: monotone, $\operatorname{ran}(\mathrm{Id}+A) = D$, $T = J_A)$

using this we have : $\exists_{A: \mathcal{H} \to 2^{\mathcal{H}}: A = T^{-1} - \mathrm{Id}}$ (A: monotone, $\operatorname{ran}(\mathrm{Id}+A) = D$, $T = J_A$) ✓

Theorem 21.8.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone]

$\exists_{\tilde{A}: \text{maximally monotone extension of } A}$ $\operatorname{dom}\tilde{A} \subseteq \overline{\operatorname{conv}}\operatorname{dom} A$

$\exists$ $\tilde{A}$: maximally monotone extension of A

$\operatorname{dom}\tilde{A} \subseteq \overline{\operatorname{conv}}\operatorname{dom} A$

Theorem 21.8.
$[A : \mathcal{H} \to 2^{\mathcal{H}}$, monotone$]$
$\exists_{\tilde{A}:\text{ maximally monotone extension of } A}$ $\quad \operatorname{dom} \tilde{A} \subseteq \overline{\operatorname{conv}}\, \operatorname{dom} A$

$\rightsquigarrow \exists$ $\tilde{A}$: maximally monotone extension of A $\qquad \operatorname{dom} \tilde{A} \subseteq \overline{\operatorname{conv}}\, \operatorname{dom} A$

define $\tilde{T} = J_{\tilde{A}}$

$\Rightarrow J_{\tilde{A}} : \mathcal{H} \to \mathcal{H}$: firmly nonexpansive
$\Rightarrow$ single valued operator so, $\operatorname{dom} \tilde{T} = \mathcal{H}$

/* for a single valued operator $T : \mathcal{X} \to \mathcal{Y}$, $\operatorname{dom} T = \mathcal{X}$ by definition. */

/* Corollary 23.10. $[A : \mathcal{H} \to 2^{\mathcal{H}}$: maximally monotone; $\gamma \in \mathbb{R}_{++}]$
(i) $J_{\gamma A} : \mathcal{H} \to \mathcal{H}$: firmly nonexpansive, maximally monotone
$\operatorname{Id} - J_{\gamma A} : \mathcal{H} \to \mathcal{H}$: firmly nonexpansive, maximally monotone
(ii) $R_{\gamma A} : \mathcal{H} \to \mathcal{H} : x \mapsto 2J_{\gamma A}x - x$: nonexpansive
Reflected resolvent
(iii) ${}^{\gamma}A : \mathcal{H} \to \mathcal{H}$: γ-cocoercive
(iv) ${}^{\gamma}A$: maximally monotone
(v) ${}^{\gamma}A : \mathcal{H} \to \mathcal{H}$: $\frac{1}{\gamma}$ Lipschitz continuous. */

$$\begin{aligned} \operatorname{ran} \tilde{T} &= \operatorname{ran} J_{\tilde{A}} = \operatorname{dom} \tilde{A} \\ &\subseteq \overline{\operatorname{conv}}\, \operatorname{dom} A \\ &= \overline{\operatorname{conv}}\, \operatorname{ran} J_A \\ &= \overline{\operatorname{conv}}\, \operatorname{ran} T \end{aligned}$$

/* Proposition 23.2. // Basic relationships between $A, {}^{\gamma}A, J_{\gamma A}$
$[A : \mathcal{H} \to 2^{\mathcal{H}}, \gamma \in \mathbb{R}_{++}, x \in \mathcal{H}, p \in \mathcal{H}]$
(i) $\operatorname{dom} J_{\gamma A} = \operatorname{dom} {}^{\gamma}A = \operatorname{ran}(\operatorname{Id} + \gamma A)$; $\operatorname{ran} J_{\gamma A} = \operatorname{dom} A$
(ii) $p \in J_{\gamma A}(x) \Leftrightarrow x \in p + \gamma A p \Leftrightarrow x - p \in \gamma A p \Leftrightarrow (p, \frac{1}{\gamma}(x-p)) \in \operatorname{gra} A$
(iii) $p \in {}^{\gamma}A x \Leftrightarrow p \in A(x - \gamma p) \Leftrightarrow (x - \gamma p, p) \in \operatorname{gra} A$ */

$\therefore \operatorname{ran} \tilde{T} \subseteq \overline{\operatorname{conv}}\, \operatorname{ran} T$

Finally, let $x \in D \Rightarrow Tx = J_A x = u$ // say

// Using Proposition 23.2 (ii)

$\Leftrightarrow x - u \in A \underset{Tx}{\underset{\parallel}{u}} \Leftrightarrow x - Tx \in A(Tx) \subseteq \tilde{A}(Tx)$ // $\tilde{A}$: maximally monotone extension of A
// $\therefore A(Tx) \subseteq \tilde{A}(Tx)$

$\Rightarrow x - Tx \in \tilde{A}(Tx)$

$\Leftrightarrow Tx \in J_{\tilde{A}}(x)$ // now, $J_{\tilde{A}}$: firmly nonexpansive

$\Leftrightarrow Tx = J_{\tilde{A}}(x) = \tilde{T}x$

$\therefore \forall_{x \in D} \; \tilde{T}x = Tx$

$\Leftrightarrow \tilde{T}|_D = T$ ∎

Corollary 23.14. (Kirszbraun-Valentine theorem)
$[D : \neq \varnothing, \subseteq \mathcal{H}; T : D \to \mathcal{H}$, nonexpansive operator$] \Rightarrow \exists_{\tilde{T} : \mathcal{H} \to \mathcal{H}} (\tilde{T}$: nonexpansive, $\tilde{T}|_D = T$, $\operatorname{ran} \tilde{T} \subseteq \overline{\operatorname{conv}}\, \operatorname{ran} T)$

Proof:

set $R := \frac{1}{2}(\operatorname{Id} + T)$ // Proposition 4.2. (iii) T : firmly nonexpansive $\Leftrightarrow \tilde{T} = 2T - \operatorname{Id}$: expansive, i.e. $\tilde{T}$: expansive $\Leftrightarrow T = \frac{1}{2}(\tilde{T} + \operatorname{Id})$: firmly nonexpansive

$\Rightarrow R : D \to \mathcal{H}$: firmly nonexpansive /* now use Theorem 23.13.

Theorem 23.13. $[D : \text{nonempty}, \subseteq \mathcal{H}; T : D \to \mathcal{H}$, firmly nonexpansive$] \Rightarrow \exists_{\tilde{T} : \mathcal{H} \to \mathcal{H}} (\tilde{T}$: firmly nonexpansive, $\tilde{T}|_D = T$, $\operatorname{ran} \tilde{T} \subseteq \overline{\operatorname{conv}}\, \operatorname{ran} T)$

$\Rightarrow \exists_{\tilde{R} : \mathcal{H} \to \mathcal{H}} (\tilde{R}$: firmly nonexpansive, $\tilde{R}|_D = R$, $\operatorname{ran} \tilde{R} \subseteq \underbrace{\overline{\operatorname{conv}}\, \operatorname{ran} R}_{= C})$

$\tilde{R}$: firmly nonexpansive, $\tilde{R}|_D = R$, ran
$\Leftrightarrow \tilde{T} = (2\tilde{R} - \operatorname{Id})$: expansive
$\Updownarrow$
$\tilde{R} = \frac{1}{2}(\operatorname{Id} + \tilde{T})$

on D we have $\operatorname{Id}|_D = \operatorname{Id}$

$\tilde{R}|_D = R \Leftrightarrow \frac{1}{2}(\tilde{T} + \operatorname{Id})|_D = R = \frac{1}{2}(T + \operatorname{Id}) \Leftrightarrow \frac{1}{2}\tilde{T}|_D + \cancel{\frac{1}{2}\operatorname{Id}|_D} = \frac{1}{2}T + \cancel{\frac{1}{2}\operatorname{Id}} \Rightarrow \tilde{T}|_D = T$

$\operatorname{ran} \tilde{R} \subseteq \overline{\operatorname{conv}}\, \operatorname{ran} R$
$\hookrightarrow \operatorname{ran} \tilde{R} = \operatorname{ran} \frac{1}{2}(\operatorname{Id} + \tilde{T}) \subseteq \overline{\operatorname{conv}}\, \operatorname{ran} \frac{1}{2}(T + \operatorname{Id})$ // this part kind of gets complicated, that is why they have defined $\tilde{T} := P_C \circ (2\tilde{R} - \operatorname{Id})$

now define:

$\tilde{T} := P_C \circ (2\tilde{R} - \operatorname{Id})$ it would have the required properties $\quad \tilde{T}|_D = T$
$\operatorname{ran} \tilde{T} \subseteq \overline{\operatorname{conv}}\, \operatorname{ran} T = C$

∎

Proposition 23.18.
$[A : \mathcal{H} \to 2^{\mathcal{H}}$; maximally monotone; $\gamma \in \mathbb{R}_{++}] \Rightarrow$

$$\begin{cases} \operatorname{Id} = J_{\gamma A} + \gamma J_{\gamma^{-1} A^{-1}} \circ \gamma^{-1} \operatorname{Id} \\ J_{A^{-1}} = \operatorname{Id} - J_A \end{cases}$$

Proof: A : maximally monotone $\Rightarrow J_{\gamma A}$: firmly nonexpansive, $\operatorname{dom} J_{\gamma A} = \mathcal{H}$ // using Proposition 23.7.
$\Rightarrow J_{\gamma A}$:

[D : nonempty subset of $\mathcal{H}$
$T : D \to \mathcal{H}$ // for a single-valued operator $D = \operatorname{dom} T$
$A = T^{-1} - \operatorname{Id}$]
(i) $T = J_A \rightsquigarrow A = J_A^{-1} - \operatorname{Id}$
(ii) T : firmly nonexpansive $\Leftrightarrow A$: monotone // J_A : firmly nonexpansive $\Leftrightarrow A$: monotone
(iii) (T : firmly nonexpansive, $D = \mathcal{H}$) $\Leftrightarrow A$: maximally monotone

A^{-1} : maximally monotone $\Rightarrow J_{\gamma^{-1} A^{-1}}$: firmly nonexpansive, $\operatorname{dom} J_{\gamma^{-1} A^{-1}} = \mathcal{H}$
/* A : maximally monotone $\Rightarrow A^{-1}$: maximally monotone $\because$ Proposition 20.22 */

now ${}^{\gamma}A \overset{\text{def}}{:=} \frac{1}{\gamma}(\operatorname{Id} - J_{\gamma A})$

/* Proposition 23.6 (ii):
$[A : \mathcal{H} \to 2^{\mathcal{H}}; \gamma \in \mathbb{R}_{++}; \mu \in \mathbb{R}_{++}]$ ${}^{\gamma}A = (\gamma \operatorname{Id} + A^{-1})^{-1} = (J_{\gamma^{-1} A^{-1}}) \circ \gamma^{-1} \operatorname{Id}$ */

$\frac{1}{\gamma}(\operatorname{Id} - J_{\gamma A}) = (J_{\gamma^{-1} A^{-1}}) \circ \gamma^{-1} \operatorname{Id}$

$\Leftrightarrow \operatorname{Id} = J_{\gamma A} + \gamma (J_{\gamma^{-1} A^{-1}}) \circ \gamma^{-1} \operatorname{Id}$

... single valued $\Rightarrow {}^{\gamma}A$ ... $(\operatorname{Id} - ...)$: single valued ... $\forall$ ... $\exists$ ... ${}^{\gamma}A x = u$

$\frac{1}{\gamma}(\mathrm{Id}-J_{\gamma A}) = (J_{\gamma^{-1}A^{-1}}) \circ \gamma^{-1}\mathrm{Id}$

$\leftrightarrow \quad \mathrm{Id} = J_{\gamma A} + \gamma (J_{\gamma^{-1}A^{-1}}) \circ \gamma^{-1}\mathrm{Id}$

$\because J_{\gamma A}$: single-valued $\Rightarrow {}^{\gamma}A = \frac{1}{\gamma}(\mathrm{Id}-J_{\gamma A})$: single valued $\therefore \forall_{x \in \operatorname{dom} {}^{\gamma}A} \exists_{u \in \mathcal{H}} \ {}^{\gamma}Ax = u$

$\frac{1}{\gamma}(\mathrm{Id}-J_{\gamma A}) = (\gamma \mathrm{Id} + A^{-1})^{-1}$

$\overset{\gamma:=1}{\Rightarrow} \quad (\mathrm{Id}-J_A) = (\mathrm{Id}+A^{-1})^{-1}$ // By definition, $J_{A^{-1}} = (\mathrm{Id}+A^{-1})^{-1}$

$= J_{A^{-1}}$

$\leftrightarrow \quad \boxed{\mathrm{Id} = J_A + J_{A^{-1}}}$

■

Proposition 23.20.

[[$T:\mathcal{H}\to\mathcal{H}$; $\gamma \in \mathbb{R}_{++}$]]

T: γ-cocoercive $\Leftrightarrow \exists_{A:\mathcal{H}\to 2^{\mathcal{H}},\text{ maximally monotone}} \quad T = {}^{\gamma}A = \frac{1}{\gamma}(\mathrm{Id}-J_{\gamma A})$

Proof: ($\Leftarrow$ direction)

/* Corollary 23.10 (iii)

[[$A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone ; $\gamma \in \mathbb{R}_{++}$]] ${}^{\gamma}A:\mathcal{H}\to\mathcal{H}$: γ-cocoercive */

using this $T = {}^{\gamma}A$: γ-cocoercive

($\Rightarrow$ direction)

given: T : γ-cocoercive

goal: $A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone, $T = {}^{\gamma}A = \frac{1}{\gamma}(\mathrm{Id}-J_{\gamma A})$

/* Corollary 23.8. [[$\tilde{T}:\mathcal{H}\to\mathcal{H}$]] $\tilde{T}$: firmly nonexpansive $\Leftrightarrow \exists_{A:\mathcal{H}\to 2^{\mathcal{H}}:\text{ maximally monotone}} \quad \tilde{T} = J_A$ */

γT : firmly nonexpansive // By definition

// Coming from

Proposition 23.18.

[$A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone ; $\gamma \in \mathbb{R}_{++}$] $\Rightarrow$

$\mathrm{Id} = J_{\gamma A} + \gamma J_{\gamma^{-1}A^{-1}} \circ \gamma^{-1}\mathrm{Id}$

$J_{A^{-1}} = \mathrm{Id} - J_A$

$\overset{\text{cor 23.8}}{\Leftrightarrow} \exists_{B:\mathcal{H}\to 2^{\mathcal{H}}:\text{ maximally monotone}} \quad \gamma T = J_B = \mathrm{Id} - J_{B^{-1}}$

$\leftrightarrow \quad T = \frac{1}{\gamma}(\mathrm{Id}-J_{B^{-1}}) = {}^{\gamma}(B^{-1})$

as B: maximally monotone $\Leftrightarrow B^{-1}$: maximally monotone

$\therefore A := B^{-1} : \mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone, $T = {}^{\gamma}A$

■

Proposition 23.27.

[$A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone ; $\gamma \in \mathbb{R}_{++}$; $B = \gamma^{-1}\mathrm{Id} - {}^{\gamma}A = \gamma^{-1}J_{\gamma A}$]

$B:\mathcal{H}\to\mathcal{H}$: maximally monotone

$J_B = \mathrm{Id} - \frac{1}{\gamma} J_{\frac{\gamma^2}{\gamma+1}A} \circ \left(\frac{\gamma}{\gamma+1}\mathrm{Id}\right)$

Proof. Corollary 23.10 [$A:\mathcal{H}\to 2^{\mathcal{H}}$; maximally monotone ; $\gamma \in \mathbb{R}_{++}$]
(i) $J_{\gamma A}:\mathcal{H}\to\mathcal{H}$: firmly nonexpansive, maximally monotone
$\mathrm{Id} - J_{\gamma A}:\mathcal{H}\to\mathcal{H}$: firmly nonexpansive, maximally monotone

So, $J_{\gamma A}$: firmly nonexpansive $\Rightarrow$ single valued
maximally monotone, $\operatorname{dom} J_{\gamma A} = \mathcal{H}$

$\Rightarrow B = \gamma^{-1}J_{\gamma A}$: single valued.
maximally monotone
$\operatorname{dom} \gamma^{-1}J_{\gamma A} = \mathcal{H}$

/* $\because$ [$A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone ; $z, u \in \mathcal{H}$; $\gamma \in \mathbb{R}_{++}$] • A^{-1} : maximally monotone • $u + \gamma A(\cdot + z)$: maximally monotone */

$\forall_{x\in\mathcal{H}} \forall_{p\in\mathcal{H}} \quad p = J_B x$

\# Proposition 23.2. // Basic relationships between A, ${}^{\gamma}A$, $J_{\gamma A}$
[$A:\mathcal{H}\to 2^{\mathcal{H}}$, $\gamma \in \mathbb{R}_{++}$, $x \in \mathcal{H}$, $p \in \mathcal{H}$]
(i) $\operatorname{dom} J_{\gamma A} = \operatorname{dom} {}^{\gamma}A = \operatorname{ran}(\mathrm{Id}+\gamma A)$; $\operatorname{ran} J_{\gamma A} = \operatorname{dom} A$
(ii) $p \in J_{\gamma A}(x) \Leftrightarrow x \in p + \gamma Ap \Leftrightarrow x - p \in \gamma Ap \Leftrightarrow (p, \frac{1}{\gamma}(x-p)) \in \operatorname{gra} A$
(iii) $p \in {}^{\gamma}Ax \Leftrightarrow p \in A(x-\gamma p) \Leftrightarrow (x-\gamma p, p) \in \operatorname{gra} A$

$\Leftrightarrow \quad x = p + \overset{\text{single valued}}{Bp}$

$\Leftrightarrow \quad x - p = Bp = \gamma^{-1}J_{\gamma A}p$

$\Leftrightarrow \quad \gamma(x-p) = J_{\gamma A}p \quad \Leftrightarrow \quad p \in \gamma(x-p) + \gamma A(\gamma(x-p)) \quad \Leftrightarrow \quad p - \gamma(x-p) \in \gamma A(\gamma(x-p))$ // now multiply both sides γ our end goal is putting it in $\tilde{x} - \tilde{p} \in \tilde{\gamma}\tilde{A}\tilde{p}$ form to use the equivalence $\tilde{p} \in J_{\tilde{\gamma}\tilde{A}}\tilde{x}$

$\Leftrightarrow \quad \gamma^2 A(\gamma(x-p)) \ni \gamma p - \gamma^2(x-p) = \gamma p - \gamma^2 x + \gamma^2 p + \gamma x - \gamma x = \gamma x - \gamma^2(x-p) - \gamma x + \gamma p = \gamma x - \underbrace{\gamma^2(x-p) - \gamma(x-p)}_{-(x-p)\gamma(\gamma+1)} = \gamma x - \gamma(\gamma+1)(x-p)$ // now divide both sides by $(\gamma+1)$

$\Leftrightarrow \quad \underbrace{\frac{\gamma^2}{\gamma+1}}_{\tilde{\gamma}} \underbrace{A}_{\tilde{A}} \underbrace{(\gamma(x-p))}_{\tilde{p}} \ni \underbrace{\frac{\gamma}{\gamma+1}x}_{\tilde{x}} - \underbrace{\gamma(x-p)}_{\tilde{p}}$ // now $\tilde{x} - \tilde{p} \in \tilde{\gamma}\tilde{A}\tilde{p} \Leftrightarrow \tilde{p} \in J_{\tilde{\gamma}\tilde{A}}\tilde{x}$

$\Leftrightarrow \quad \gamma(x-p) \in J_{\frac{\gamma^2}{\gamma+1}A}\left(\frac{\gamma}{\gamma+1}x\right)$ // now $J_{\frac{\gamma^2}{\gamma+1}A}(\cdot)$: single valued, firmly nonexpansive, monotone, so "$\in$" can be // replaced with "$=$"

$\Leftrightarrow \quad \gamma(x-p) = J_{\frac{\gamma^2}{\gamma+1}A}\left(\frac{\gamma}{\gamma+1}x\right)$

$\Leftrightarrow \quad x - p = \frac{1}{\gamma} \cdot J_{\frac{\gamma^2}{\gamma+1}A}\left(\frac{\gamma}{\gamma+1}x\right)$

$\Leftrightarrow \quad p = x - \frac{1}{\gamma} J_{\frac{\gamma^2}{\gamma+1}A}\left(\frac{\gamma}{\gamma+1}x\right)$

So, $\forall_{x\in\mathcal{H}} \forall_{p\in\mathcal{H}} \quad p = J_B x \Leftrightarrow p = x - \frac{1}{\gamma} J_{\frac{\gamma^2}{\gamma+1}A}\left(\frac{\gamma}{\gamma+1}x\right) = J_B x$

$$\therefore \quad J_B = \mathrm{Id} - \frac{1}{\gamma} J_{\frac{\gamma^2}{\gamma+1}A}\left(\frac{\gamma}{\gamma+1}\mathrm{Id}\right) \quad \blacksquare$$

Proposition 23.32.

[$\mathcal{K}$: real Hilbert space

$L \in \mathcal{B}(\mathcal{H},\mathcal{K}) : LL^* = \mu\,\mathrm{Id}$; $f \in \Gamma_0(\mathcal{K})$; $x \in \mathcal{H}$]

$f \circ L \in \Gamma_0(\mathcal{H})$

$\mathrm{Prox}_{f\circ L}\, x = x + \mu^{-1} L^*\left(\mathrm{Prox}_{\mu f}(Lx) - Lx\right)$

Proof :

Fact 2.18. (Important info on linear continuous operator)

[$T \in \mathcal{B}(\mathcal{H},\mathcal{K})$] $\Rightarrow$ ($\mathcal{K}$: real Hilbert space)

(i) $T^{**} = T$

(ii) $\|T^*\| = \|T\| = \sqrt{\|T^*T\|}$

(iii) $(\ker T)^\perp = \overline{\mathrm{ran}}\, T$

(iv) $(\mathrm{ran}\, T)^\perp = \ker T^*$

(v) $\ker T^*T = \ker T$, $\overline{\mathrm{ran}}\, TT^* = \overline{\mathrm{ran}}\, T$.

Fact 2.19. [$\mathcal{K}$: real Hilbert space ; $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$] $\mathrm{ran}\, T$: closed $\Leftrightarrow$ $\mathrm{ran}\, T^*$: closed $\Leftrightarrow$ $\mathrm{ran}\, TT^*$: closed $\Leftrightarrow$ $\mathrm{ran}\, T^*T$: closed $\Leftrightarrow \exists_{\alpha\in\mathbb{R}_{++}} \forall_{x\in(\ker T)^\perp} \|Tx\| \geq \alpha\|x\|$

$\mathrm{ran}\, L$: closed $\Leftrightarrow$ $\mathrm{ran}\, L = \overline{\mathrm{ran}}\, L$

$\mathrm{ran}\, LL^*$: closed $\Leftrightarrow$ $\mathrm{ran}\, LL^* = \overline{\mathrm{ran}}\, LL^*$

$\overline{\mathrm{ran}}\, LL^* = \overline{\mathrm{ran}}\, L$

$\Rightarrow \underbrace{\mathrm{ran}\, \underbrace{LL^*}_{\mu\mathrm{Id}}}_{=\mathcal{K}} = \mathrm{ran}\, L$

$\therefore \mathrm{ran}\, L = \mathcal{K}$

Proposition 16.42. [$\cancel{g}^{f} \in \Gamma_0(\mathcal{K})$; $L \in \mathcal{B}(\mathcal{H},\mathcal{K})$; one of the following holds : (i) $0 \in \mathrm{sri}(\mathrm{dom}\, \cancel{g}^{f} - \cancel{\mathrm{ran} L}^{\mathcal{K}})$ (ii) $\mathcal{K}$: finite-dimensional, $\cancel{g}^{f}$: polyhedral, $\mathrm{dom}\, \cancel{g}^{f} \cap \mathrm{ran}\, L \neq \emptyset$] $\partial(\cancel{g}^{f}\circ L) = L^* \circ (\partial \cancel{g}^{f}) \circ L$

now $f \in \Gamma_0(\mathcal{K})$; $L \in \mathcal{B}(\mathcal{H},\mathcal{K})$; lets check : $0 \overset{?}{\in} \mathrm{sri}(\mathrm{dom}\, f - \mathrm{ran}\, L) = \mathrm{sri}(\mathrm{dom}\, f - \mathcal{K})$ // $\because \mathrm{ran}\, L = \mathcal{K}$

Proposition 15.24. [$\cancel{f}^{:=1\in\Gamma_0(\mathcal{K})} \in \Gamma_0(\cancel{\mathcal{H}}^{\mathcal{K}})$, $g^{:=f\in\Gamma_0(\mathcal{K})} \in \Gamma_0(\mathcal{K})$, $\underset{\mathrm{Id}}{L} \in \mathcal{B}(\cancel{\mathcal{H}}^{\mathcal{K}},\mathcal{K})$] ($\mathrm{dom}\, \cancel{g}^{f} \cap \cancel{L}^{\mathrm{Id}}(\cancel{\mathrm{dom} f}^{\mathcal{K}})) \neq \emptyset$ and one of the following holds:

Clearly. $(\mathrm{dom}\, f \cap \mathrm{Id}(\mathcal{K})) = (\mathrm{dom}\, f \cap \mathcal{K}) \neq \emptyset$

(i) $\mathrm{cone}(\mathrm{dom}\, g - L(\mathrm{dom}\, f)) = \overline{\mathrm{span}}\,(\mathrm{dom}\, g - L(\mathrm{dom}\, f))$

(ii) $\mathrm{dom}\, g - L(\mathrm{dom}\, f)$: closed linear subspace

(iii) $(\mathrm{dom}\, f, \mathrm{dom}\, g)$: linear subspaces,

$\mathrm{dom}\, g + L(\mathrm{dom}\, f)$: closed

(iv) $\mathrm{dom}\, g$: cone, $\mathrm{dom}\, g - \mathrm{cone}\, L(\mathrm{dom}\, f)$: closed linear subspaces

(v) $0 \in \mathrm{core}(\mathrm{dom}\, g - L(\mathrm{dom}\, f))$

(vi) $0 \in \mathrm{int}(\mathrm{dom}\, g - L(\mathrm{dom}\, f))$

(vii) $\mathrm{cont}\, \cancel{g}^{f} \cap \cancel{L}^{\mathrm{Id}}(\cancel{\mathrm{dom} f}^{\mathcal{K}}) \neq \emptyset$ $\quad \mathrm{cont}\, f \cap \mathrm{Id}(\mathcal{K}) = \mathrm{cont}\, f \cap \mathcal{K} \neq \emptyset$

(viii) $\mathcal{K}$: finite dimensional $\wedge$ $(\mathrm{ri}\,\mathrm{dom}\, g) \cap (\mathrm{ri}\, L(\mathrm{dom}\, f)) \neq \emptyset$

(ix) $\mathcal{K}$: finite dimensional $\wedge$ $(\mathrm{ri}\,\mathrm{dom}\, g) \cap L(\mathrm{qri}\,\mathrm{dom}\, f) \neq \emptyset$)

$\Rightarrow$ $0 \in \mathrm{sri}(\mathrm{dom}\, \cancel{g}^{f} - \cancel{L}^{\mathrm{Id}(\mathcal{K})}(\mathrm{dom}\, f))$ // which will imply $\inf(f + g\circ L)(\mathcal{H}) = -\min(f^* \circ L^* + g^{*\vee})(\mathcal{K})$

$\Leftrightarrow 0 \in \mathrm{sri}(\mathrm{dom}\, f - \mathcal{K})$

So, all the antecedents of Proposition 16.42. holds, so: $\partial(f\circ L) = L^* \circ \partial f \circ L = L^* \partial f L$

. . . (eq: 1)

now :

Proposition 23.25.

[$\mathcal{K}$: real Hilbert space,

$L \in \mathcal{B}(\mathcal{H},\mathcal{K})$, LL^*: invertible as $LL^* = \mu\,\mathrm{Id}$

$\cancel{A}^{\partial f} : \mathcal{K} \to 2^{\mathcal{K}}$: maximally monotone as $f \in \Gamma_0(\mathcal{K})$

$B = L^* \cancel{A}^{\partial f} L$] $\Rightarrow$

(i) $B : \mathcal{H} \to 2^{\mathcal{H}}$: maximally monotone

(ii) $J_B = \mathrm{Id} - L^* \circ (LL^* + A^{-1})^{-1} \circ L$

(iii) $\exists_{\mu\in\mathbb{R}_{++}} LL^* = \mu\,\mathrm{Id} \Rightarrow J_B = \mathrm{Id} - L^* \circ {}^{\mu}A \circ L$

$\therefore \underbrace{J_{(L^*\partial f L)}}_{J_{\partial(f\circ L)}} = \mathrm{Id} - L^* \circ \underbrace{{}^{\mu}\partial f}_{=\frac{1}{\mu}(\mathrm{Id} - J_{\mu A})} \circ L = \mathrm{Id} - L^* \circ \frac{1}{\mu}(\mathrm{Id} - J_{\mu A}) \circ L$

// from (eq: 1)

$= \mathrm{Id} - \frac{1}{\mu} L^* \circ (\mathrm{Id} - J_{\cdot\cdot\cdot}) \circ L$

// from (eq: 1) $= Id - \frac{1}{\mu} L^* \circ (Id - J_{\mu\partial f}) \circ L$

$\Leftrightarrow J_{\partial(f \circ L)} = Id - \frac{1}{\mu} L^* \circ (Id - J_{\underbrace{\mu\partial f}_{\partial(\mu f)}}) \circ L$ // now $J_{\partial f} = \mathrm{Prox}_f$ for Γ_0 type f

$\Leftrightarrow \mathrm{Prox}_{f \circ L} = Id - \frac{1}{\mu} L^* \circ (Id - \mathrm{Prox}_{\mu f}) \circ L$

$\therefore \forall_{x \in \mathcal{H}} \quad \mathrm{Prox}_{f \circ L} x = x - \frac{1}{\mu} L^* (Lx - \mathrm{Prox}_{\mu f}(Lx))$

$= x + \frac{1}{\mu} L^* (Lx + \mathrm{Prox}_{\mu f}(Lx))$ ✓

and by composition law: $f \circ L \in \Gamma_0(\mathcal{H})$. ✓

$\boxtimes$

Part 3

1:00 PM

Zeros of monotone operator :

Proposition 23.35.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, stricly monotone] $\operatorname{zer} A$: almost a singleton

Proof:

Per absurdum lets assume $\operatorname{zer} A$ has atleast two elements $x, y : x \neq y$

$$\begin{cases} x \in \operatorname{zer} A \Leftrightarrow Ax \ni 0 \Leftrightarrow (x,0) \in \operatorname{gra} A \\ y \in \operatorname{zer} A \Leftrightarrow Ay \ni 0 \Leftrightarrow (y,0) \in \operatorname{gra} A \end{cases}, x \neq y$$

A: strictly monotone $\overset{\text{def}}{\Leftrightarrow} \forall_{(x,u) \in \operatorname{gra} A} \ \forall_{(y,v) \in \operatorname{gra} B} \ (x \neq y \Rightarrow \langle x-y \mid u-v \rangle > 0)$

so, $\langle x-y \mid 0-0 \rangle = 0 > 0 \to$ contradiction

$\therefore \operatorname{zer} A$: atmost a singleton.

■

Proposition 23.36.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone,

one of the following holds:

(i) A^{-1}: locally bounded everywhere

(ii) $\lim_{\|x\| \to +\infty} \inf \|Ax\| = +\infty$

(iii) $\operatorname{dom} A$: bounded] $\Rightarrow$

$\operatorname{zer} A \neq \emptyset$

Proof: $\operatorname{zer} A \neq \emptyset \Leftrightarrow \exists_x \ Ax \ni 0$, if we show that A: surjective (onto) $\Leftrightarrow \forall_{y \in \mathcal{H}} \exists_{x \in \mathcal{H}} \ Ax \ni y$, this would imply $\operatorname{zer} A \neq \emptyset$

Now we have the following results regarding surjectivity of A: maximally monotone operators

Corollary 21.19. /* surjective :: onto */

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

A: surjective $\Leftrightarrow A^{-1}$: locally bounded everywhere on $\mathcal{H}$

Corollary 21.20.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, $\lim_{\|x\| \to +\infty} \inf \|Ax\| = +\infty$] A: surjective

Corollary 21.21.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, bounded domain]

A: surjective

thus $\big((i) \vee (ii) \vee (iii) \big) \Rightarrow A$: surjective $\Rightarrow \operatorname{zer} A \neq \emptyset$

■

Corollary 23.37.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone

one of the following holds :

(i) A: uniformly monotone with a supercoercive modulus

(ii) A: strongly monotone]

A: singleton

Proof:

Proposition 22.8.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, one of the following holds $\left|\begin{array}{l} A: \text{uniformly monotone with a supercoerciv[e]} \\ \quad \text{modulus} \\ A: \text{strongly monotone} \end{array}\right.$]

$\lim_{\|x\| \to +\infty} \inf \|Ax\| = +\infty$, A: surjective (onto) /* A: onto $\Leftrightarrow \forall_{x \in \mathcal{H}} \exists_{y \in \mathcal{H}} \ Ay \ni x$ */

using this:

A : onto

$\Leftrightarrow \forall_{x \in \mathcal{H}} \exists_{y \in \mathcal{H}} \ Ay \ni x$

$\Rightarrow \operatorname{zer} A \neq \emptyset$ . . . (eq: 1)

strong monotonicity $\Rightarrow$ uniform monotonicity $\Rightarrow$ strict monotonicity $\Rightarrow$ paramonotonicity $\Rightarrow$ monotonicity

A: strictly monotone

$\Downarrow$

$\operatorname{zero} A$: almost a singleton (eq: 2)

so, from (eq: 1) and (eq: 2) we have:

$\operatorname{zer} A$: singleton.

■

Proposition 23.38.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone ;

$\gamma \in \mathbb{R}_{++}$]

$\operatorname{Fix} J_{\gamma A} = \operatorname{zer} A = \operatorname{zer}^{\gamma} A$

Proof: $x \in \mathcal{H}$

$x \in \operatorname{zer} A \Leftrightarrow Ax \ni 0 \Leftrightarrow \gamma Ax \ni 0 = x - x \Leftrightarrow x \in x + \gamma A \Leftrightarrow x \in J_{\gamma A}(x)$ /* now Proposition 23.37 J_A : firmly nonexpansive $\Leftrightarrow A$: monotone */

$\Leftrightarrow x = J_{\gamma A}(x) \Leftrightarrow x \in \operatorname{Fix} J$ . . . (eq: 1)

Proof: $x \in \mathcal{H}$

$x \in \operatorname{zer} A \Leftrightarrow Ax \ni 0 \Leftrightarrow \gamma Ax \ni 0 = x - x \Leftrightarrow x \in x + \gamma A \Leftrightarrow x \in J_{\gamma A}(x)$ /* now Proposition 23.37 J_A: firmly nonexpansive $\leftrightarrow$ A: monotone */

$\Leftrightarrow x = J_{\gamma A}(x) \Leftrightarrow x \in \operatorname{Fix} J_{\gamma A}$ ... (eq: 1)

***
* Proposition 23.2. //Basic relationships between A, ${}^{\gamma}A$, $J_{\gamma A}$
[$A: \mathcal{H} \to 2^{\mathcal{H}}$, $\gamma \in \mathbb{R}_{++}$, $x \in \mathcal{H}$, $p \in \mathcal{H}$]
(i) $\operatorname{dom} J_{\gamma A} = \operatorname{dom} {}^{\gamma}A = \operatorname{ran}(\mathrm{Id} + \gamma A)$; $\operatorname{ran} J_{\gamma A} = \operatorname{dom} A$
(ii) $p \in J_{\gamma A}(x) \Leftrightarrow x \in p + \gamma A p \Leftrightarrow x - p \in \gamma A p \Leftrightarrow (p, \frac{1}{\gamma}(x-p)) \in \operatorname{gra} A$
(iii) $p \in {}^{\gamma}Ax \Leftrightarrow p \in A(x - \gamma p) \Leftrightarrow (x - \gamma p, p) \in \operatorname{gra} A$

again $0 \in Ax \Leftrightarrow 0 \in A(x - \gamma 0) \Leftrightarrow 0 \in {}^{\gamma}Ax \Leftrightarrow x \in \operatorname{zer} {}^{\gamma}A$ ... (eq: 2)

from (i), (ii): $\forall_x$ $(x \in \operatorname{zer} A \Leftrightarrow x \in \operatorname{Fix} J_{\gamma A} \Leftrightarrow x \in \operatorname{zer} {}^{\gamma}A)$

$\Leftrightarrow$ $\operatorname{zer} A = \operatorname{Fix} J_{\gamma A} = \operatorname{zer} {}^{\gamma}A$

■

Proposition 23.39.
[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone] $\operatorname{zer} A$: closed, convex

Proof: A: maximally monotone

$\Rightarrow$ A^{-1}: maximally monotone ... (eq: 1)

$x \in \operatorname{zer} A \Leftrightarrow Ax \ni 0 \Leftrightarrow x \in A^{-1}0$ $\therefore \operatorname{zer} A = A^{-1}0$ ... (eq: 2)

* Proposition 20.31. /*Just beautiful! Output set of a maximally monotone operator on a point is closed, convex */
[$\tilde{A}: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone ; $\tilde{x} \in \mathcal{H}$]
$\tilde{A}\tilde{x}$: closed, convex

$\tilde{A} := A^{-1}$
$\tilde{x} = 0$

using this, (eq: 1), (eq: 2)

$A^{-1}0$: closed convex

■

Example 23.40. (Classical Proximal Point Algorithm)
[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone ; $\gamma \in \mathbb{R}_{++}$; $x_0 \in \mathcal{H}$; $\forall_{n \in \mathbb{N}}$ $x_{n+1} = J_{\gamma A} x_n$] $\Rightarrow$
(i) $\operatorname{zer} A \neq \emptyset \Rightarrow x_n \rightharpoonup x : x \in \operatorname{zer} A$
(ii) A: strongly monotone with $\beta \in \mathbb{R}_{++} \Rightarrow x_n \xrightarrow{\text{linearly}} \underbrace{x}_{\text{unique point in zer}A}$

Proof:

(i) Proposition 23.38. [$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone ; $\gamma \in \mathbb{R}_{++}$] $\operatorname{Fix} J_{\gamma A} = \operatorname{zer} A = \operatorname{zer} {}^{\gamma}A$

so, $\operatorname{Fix} J_{\gamma A} = \operatorname{zer} A = \operatorname{zer} {}^{\gamma}A$

Also
**
Corollary 23.10. [$A: \mathcal{H} \to 2^{\mathcal{H}}$: maximally monotone ; $\gamma \in \mathbb{R}_{++}$]
(i) $J_{\gamma A}: \mathcal{H} \to \mathcal{H}$: firmly nonexpansive, maximally monotone
$\mathrm{Id} - J_{\gamma A}: \mathcal{H} \to \mathcal{H}$: firmly nonexpansive, maximally monotone

$\therefore J_{\gamma A}: \mathcal{H} \to \mathcal{H}$: firmly nonexpansive

recall Example 5.17. [$\underset{\substack{\parallel \\ J_{\gamma A}}}{T}: \mathcal{H} \to \mathcal{H}$, firmly nonexpansive ✓, $\underbrace{\operatorname{Fix} \overset{J_{\gamma A}}{T} \neq \emptyset}_{\text{by given}}$; $x_0 \in \mathcal{H}$; $\forall_{n \in \mathbb{N}}$ $x_{n+1} = Tx_n$] $x_n \rightharpoonup \underbrace{x}_{\text{point in Fix}T}$

$T := J_{\gamma A}$

we arrive at the claim that $x_n \rightharpoonup \underbrace{x}_{\text{points}}$:)

(ii)

Proposition 23.11.
[$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone ; $\beta \in \mathbb{R}_{++}$]
A: strongly monotone with constant $\beta \Leftrightarrow J_A : (\beta+1)$-cocoercive
$\Rightarrow J_A$: Lipschitz continuous with constant $\frac{1}{\beta+1} \in]0,1[$

A: strongly monotone with constant $\beta \in \mathbb{R}_{++}$

$\Leftrightarrow$

$\forall_{(x,u) \in \operatorname{gra} A}$ $\forall_{(y,v) \in \operatorname{gra} A}$ $\langle x-y \mid u-v \rangle \geq \beta \|x-y\|^2$

$(x,u) \in \operatorname{gra} \overset{\in \mathbb{R}_{++}}{\gamma} A \Leftrightarrow \gamma A x \ni u \Leftrightarrow A(\gamma x) \ni u \Leftrightarrow (\gamma x, u) \in \operatorname{gra} A$

$(y,v) \in \operatorname{gra} \gamma A \Leftrightarrow (\gamma y, v) \in \operatorname{gra} A$

$\langle \gamma x - \gamma y \mid u - v \rangle \geq \beta \|\gamma x - \gamma y\|^2 = \beta \gamma^2 \|x-y\|^2$

$\Leftrightarrow \gamma \langle x-y \mid u-v \rangle \geq \beta \gamma^2 \|x-y\|^2$

$\Leftrightarrow \langle x-y \mid u-v \rangle \geq \beta \gamma \|x-y\|^2$

$\therefore \forall_{(x,u) \in \operatorname{gra} \gamma A}$ $\forall_{(y,v) \in \operatorname{gra} \gamma A}$ $\langle x-y \mid u-v \rangle \geq \beta\gamma \|x-y\|^2$

$\Leftrightarrow \ \langle x-y \mid u-v\rangle \geq \beta\gamma \|x-y\|^2$

$\therefore \forall_{(x,y)\in \operatorname{gra}\gamma A} \forall_{(y,v)\in \operatorname{gra}\gamma A} \ \langle x-y \mid u-v\rangle \geq \beta\gamma\|x-y\|^2$

$\Leftrightarrow$ γA: strongly monotone with constant $\beta\gamma$

$\Rightarrow$ $J_{\gamma A}$: Lipschitz continuous with constant $\frac{1}{\beta\gamma+1} \in \,]0,1[$

γA: strongly monotone $\Rightarrow$ $J_{\gamma A}$: firmly nonexpansive

$\therefore$ $J_{\gamma A}$: contractive with contraction parameter $\frac{1}{\beta\gamma+1} \in \,]0,1[$

$\therefore \forall_{n\in\mathbb{N}} \ x_{n+1} = J_{\gamma A} x_n$: will be equivalent to classical Banach-Picard iteration.

(Banach-Picard iteration for contractive operator)

Theorem 1.48. [$(\mathcal{X}, d)$: complete metric space; $T:\mathcal{X}\to\mathcal{X}$: Lipschitz continuous with constant $\beta\in[0,1[$

$x_0\in\mathcal{X}$; $\forall_{n\in\mathbb{N}} \ x_{n+1} = Tx_n$]

(i) $\exists_x$ x: unique fixed point of T

(ii) $\forall_{n\in\mathbb{N}}$ $d(x_{n+1},x) \leq \beta d(x_n,x)$

(iii) $\forall_{n\in\mathbb{N}}$ $d(x_n,x) \leq \beta^n (x_0,x)$ // $x_n\to x$ linearly

(iv) Priori error estimate: $\forall_{n\in\mathbb{N}}$ $d(x_n,x) \leq \beta^n d(x_0,x_1)/(1-\beta)$

(v) Posteriori error estimate: $\forall_{n\in\mathbb{N}}$ $d(x_n,x) \leq d(x_n,x_{n+1})/(1-\beta)$

(vi) $d(x_0,x_1)/(1+\beta) \leq d(x_n,x) \leq d(x_0,x_1)/(1-\beta)$

Proof: see onenote 11/19/2016 7:16PM

we arrive at the claim.

■

Theorem 23.41: (Proximal-point algorithm)

[$A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone, $\operatorname{zer} A \neq \emptyset$;

$(\gamma_n)_{n\in\mathbb{N}} :\subseteq \mathbb{R}_{++}$, $\sum_{n\in\mathbb{N}} \gamma_n^2 = +\infty$;

$x_0\in\mathcal{H}$;

$\forall_{n\in\mathbb{N}} \quad x_{n+1} = J_{\gamma_n A} x_n$]

$\therefore$ (eq:1)

(i) $x_n \rightharpoonup \underbrace{x}_{\in \operatorname{zer} A}$

(ii) A: uniformly monotone on every bounded subset of $\mathcal{H}$ $\Rightarrow$ $x_n \to \underbrace{x}_{\text{unique point in } \operatorname{zer} A}$

Proof:

Proof Sketch for Theorem 23.41

First we work on the gap sequence $u_n = \frac{x_n - x_{n+1}}{\gamma_n}$, and show $(x_{n+1}, u_n) \in$ **gra**A. Then using the monotonicity of A applied to (x_{n+1}, u_n) and (x_{n+2}, u_{n+1}) in **gra**A, we show that $\|u_{n+1}\| \leq \|u_n\|$, which implies $\|u_n\|$ converges.

Next, we show that $(x_n)_{n\in\mathbf{N}}$ is Fejer monotone w.r.t **zer**A = **Fix**$J_{\gamma_n A}$ using the firmly nonexpansive nature of $J_{\gamma_n A}$. As $(x_n)_{n\in\mathbf{N}}$ is bounded due to Fejer monotonicity, it will have a weak sequential cluster point.

Third, going back to the initial inequality to prove the Fejer monotone nature of the $(x_n)_{n\in\mathbf{N}}$, we sum over all the indices and show $\sum_{n\in\mathbf{N}} \gamma_n^2 \|u_n\|^2 < +\infty$, which along with given $\sum_{n\in\mathbf{N}} \gamma_n^2 = +\infty$ implies $\underline{\lim}\|u_n\| = 0$, which along with $\lim\|u_n\|$ exists implies $u_n \to 0$.

Now we are in business. We use Theorem 5.5 as our key theorem with $C :=$ **zer**A. Take any weak sequential cluster point x of $(x_n)_{n\in\mathbf{N}}$. Using Proposition 20.3, regarding closedness of the **gra**A, we show that $(x, 0) \in$ **zer**A, which suffices for claim (i). Claim (ii) follows from claim (i), Lemma 2.41 and definition of uniform monotonicity of A.

Theorem 5.5.
$[(x_n)_{n\in\mathbf{N}} :\subseteq \mathcal{H}$, Fejer monotone w.r.t. C, every weak sequential cluster point of the sequence is in $C]$
$\Rightarrow (x_n)_{n\in\mathbf{N}}$: converges weakly to a point in C.

Proposition 20.33.
$[A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone$]$
gra$A$: sequentially closed in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$, i.e.,
$\forall_{(x_n,u_n)_{n\in\mathbf{N}} \subseteq \mathbf{gra}A} \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}: x_n \rightharpoonup x, u_n \to u} (x,u) \in$ **gra**A.

Lemma 2.41. $[(x_n)_{n\in\mathbf{N}}, (u_n)_{n\in\mathbf{N}} :\subseteq \mathcal{H}; x, u \in \mathcal{H}]$
$(x_n \rightharpoonup x, u_n \to x) \Rightarrow \langle x_n \mid u_n\rangle \to \langle x \mid u\rangle$.

(stage 0) before we prove (i),(ii)

Set: $\forall_{n\in\mathbb{N}} \ u_n = \frac{x_n - x_{n+1}}{\gamma_n}$ ... (eq:0) : our first goal is showing that $u_n\to 0$, before we start proving (i) and (ii).

now from (eq:1) $x_{n+1} = J_{\gamma_n A} x_n$

$\Leftrightarrow \ x_n \in x_{n+1} + \gamma_n A x_{n+1}$ /* Proposition 23.2.

$\Leftrightarrow \ \underbrace{\frac{x_n - x_{n+1}}{\gamma_n}}_{u_n} \in A x_{n+1}$

★★★

\# Proposition 23.2. //Basic relationships between A, ${}^{\gamma}A$, $J_{\gamma A}$

$[A:\mathcal{H}\to 2^{\mathcal{H}}, \gamma\in\mathbb{R}_{++}, x\in\mathcal{H}, p\in\mathcal{H}]$

(i) $\operatorname{dom} J_{\gamma A} = \operatorname{dom} {}^{\gamma}A = \operatorname{ran}(\mathrm{Id}+\gamma A)$; $\operatorname{ran} J_{\gamma A} = \operatorname{dom} A$

(ii) $p \in J_{\gamma A}(x) \Leftrightarrow x \in p + \gamma A p \Leftrightarrow x - p \in \gamma A p \Leftrightarrow (p, \frac{1}{\gamma}(x-p)) \in \operatorname{gra} A$

(iii) $p \in {}^{\gamma}A x \Leftrightarrow p \in A(x - \gamma p) \Leftrightarrow (x - \gamma p, p) \in \operatorname{gra} A$

$\ldots u_n \in Ax_{n+1} \quad \ldots (\text{eq}:2)$

$\Updownarrow$

from (eq:0): $(x_{n+1}, u_n) \in \text{gra}A \quad \forall_{n\in\mathbb{N}}$ [eq: 2.5]

$$u_{n+1} = \frac{x_{n+1}-x_{n+2}}{\gamma_{n+1}}$$

$$\Leftrightarrow \quad x_{n+1}-x_{n+2} = \gamma_{n+1} u_{n+1} \quad \ldots (\text{eq}:3)$$

A: maximally monotone $\Leftrightarrow \forall_{(x,u)} \Big((x,u)\in\text{gra}A \Leftrightarrow \forall_{(y,v)\in\text{gra}A} \langle x-y \mid u-v\rangle \geq 0 \Big)$

$\because (x_{n+1}, u_n), (x_{n+2}, u_{n+1}) \in \text{gra}A \ \forall_{n\in\mathbb{N}}$

$$\Rightarrow \forall_{n\in\mathbb{N}} \ \underbrace{\langle \underbrace{x_{n+1}-x_{n+2}}_{\gamma_{n+1}u_{n+1} \text{ // from eq:3}} \mid u_n - u_{n+1}\rangle}_{\shortparallel} \geq 0$$

$$\gamma_{n+1}\langle u_{n+1} \mid u_n - u_{n+1}\rangle = \gamma_{n+1}\Big(\underbrace{\langle u_{n+1}\mid u_n\rangle}_{\substack{\text{cauchy-schwarz} \\ \leq \|u_{n+1}\|\,\|u_n\|}} - \|u_{n+1}\|^2\Big) \leq \gamma_{n+1}\Big(\|u_{n+1}\|\,\|u_n\| - \|u_{n+1}\|^2\Big) = \gamma_{n+1}\|u_{n+1}\|\Big(\|u_n\| - \|u_{n+1}\|\Big)$$

$$\therefore \forall_{n\in\mathbb{N}} \ \underbrace{\gamma_{n+1}}_{\in\mathbb{R}_{++}}\|u_{n+1}\|\big(\|u_n\|-\|u_{n+1}\|\big) \geq \langle x_{n+1}-x_{n+2} \mid u_n - u_{n+1}\rangle \geq 0$$

$$\Leftrightarrow \forall_{n\in\mathbb{N}} \ \underbrace{\|u_{n+1}\|}_{\geq 0}\big(\|u_n\|-\|u_{n+1}\|\big) \geq 0$$

$$\Rightarrow \forall_{n\in\mathbb{N}} \quad \|u_n\| - \|u_{n+1}\| \geq 0 \ \Leftrightarrow \ \|u_{n+1}\| \leq \|u_n\| \quad \ldots (\text{eq}:4)$$

so, the sequence $(\|u_n\|)_{n\in\mathbb{N}}$: monotonic decreasing, and clearly $\|u_n\| \geq 0$. so, $\|u_n\|$ converges as (Rudin)

and $\|u_n\| \leq \|u_0\|$

$\therefore (\|u_n\|)_{n\in\mathbb{N}}$: bounded in $[0, \|u_0\|]$

* Theorem 3.14
[$(s_n)_{n\in\mathbb{N}}$: monotonic] $(s_n)_{n\in\mathbb{N}}$: converges $\Leftrightarrow$ $(s_n)_{n\in\mathbb{N}}$: bounded

$$\therefore \exists_{C\geq 0} \ \|u_n\| = \frac{1}{\gamma_n}\|x_n - x_{n+1}\| \to C \quad \ldots (\text{eq}: 4.5)$$

$$\Leftrightarrow \quad \lim \|u_n\| = C \quad \ldots (\text{eq}: 4.6)$$

given $\text{zer}A \neq \emptyset$

$$\Leftrightarrow \exists_z \ z\in \text{zer}A \quad \ldots (\text{eq}:5)$$

now: Proposition 23.38. [$A: \mathcal{H}\to 2^{\mathcal{H}}$, monotone; $\gamma\in\mathbb{R}_{++}$] $\text{Fix}\, J_{\gamma A} = \text{zer}A = \text{zer}\,{}^{\gamma}A$

$$\therefore \forall_{n\in\mathbb{N}} \quad \text{Fix}\, J_{\gamma_n A} = \text{zer}A \quad \ldots (\text{eq}:6)$$

From (eq:5) and (eq:6):

$$z\in \text{Fix}\, J_{\gamma_n A} \quad \forall_{n\in\mathbb{N}}$$

$$\Leftrightarrow \quad z \in \bigcap_{n\in\mathbb{N}} \text{Fix}\, J_{\gamma_n A} \quad \ldots (\text{eq}:7)$$

Now, A: maximally monotone $\Leftrightarrow \underbrace{J_{\gamma_n A} \overset{\mathcal{H}\to\mathcal{H}}{} \text{: firmly nonexpansive, maximally monotone}}$ $\forall_{n\in\mathbb{N}}$ /* Corollary 23.10 */

$\Updownarrow$

$$\forall_{n\in\mathbb{N}} \ \forall_{x\in\mathcal{H}} \ \forall_{y\in\mathcal{H}} \ \|J_{\gamma_n A}x - J_{\gamma_n A}y\|^2 + \|(\text{Id}-J_{\gamma_n A})x - (\text{Id}-J_{\gamma_n A})y\|^2 \leq \|x-y\|^2$$

set $x := x_n$, $y := z \in \text{Fix}\, J_{\gamma_n A}$

$$\|J_{\gamma_n A}x_n - J_{\gamma_n A}z\|^2 \leq \|x_n - z\|^2 - \|(\text{Id}-J_{\gamma_n A})x_n - \underbrace{(\text{Id}-J_{\gamma_n A})z}_{(z-J_{\gamma_n A}z)=0 \ /* \because z = J_{\gamma_n A}z */}\|^2 = \|x_n - z\|^2 - \|x_n - J_{\gamma_n A}x_n\|^2$$

now $\underset{\shortparallel}{J_{\gamma_n A}x_n}$ // by iteration scheme $\in \text{Fix}\, J_{\gamma_n A} \Leftrightarrow J_{\gamma_n}Az = z$

$$\forall_{n\in\mathbb{N}} \ \|x_{n+1} - z\|^2$$

$$= \|J_{\gamma_n A}x_n - J_{\gamma_n A}z\|^2 \leq \|x_n - z\|^2 - \underbrace{\|x_n - \underbrace{J_{\gamma_n A}x_n}_{x_{n+1} \text{ // by the iteration scheme}}\|^2}_{\|x_n - x_{n+1}\|^2 = \gamma_n^2\|u_n\|^2 \ /* \because u_n = \frac{x_n - x_{n+1}}{\gamma_n} */} = \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 = \|x_n - z\|^2 - \underbrace{\gamma_n^2\|u_n\|^2}_{\text{negative number. so removing it will give a larger number}} \leq \|x_n - z\|^2$$

x_{n+1} // by the iteration scheme

negative number. so removing it will give a larger number

$$\|x_n - x_{n+1}\|^2 = \gamma_n^2 \|u_n\|^2 \quad /* \because u_n = \frac{x_n - x_{n+1}}{\gamma_n} */$$

$\therefore \forall_{n\in\mathbb{N}} \ \forall_{z\in \mathrm{Fix}\, J_{\gamma_n A}} \ \|x_{n+1}-z\|^2 \leq \|x_n - z\|^2$

Definition. (Fejer monotone sequence)
Let
- C : nonempty subset of $\mathcal{H}$.
- $(x_n)_{n\in\mathbb{N}}$: sequence in $\mathcal{H}$.

Then $(x_n)_{n\in\mathbb{N}}$ is Fejer monotone with respect to C if

$(\forall x \in C)\ (\forall n \in \mathbf{N})\quad \|x_{n+1} - x\| \leq \|x_n - x\|$.

$\leftrightarrow$ $(x_n)_{n\in\mathbb{N}}$: Fejer monotone sequence in $\mathcal{H}$ w.r.t. $\underbrace{\mathrm{Fix} J_{\gamma_n A}}_{=\ \mathrm{zer}\, A}$ $\forall_{n\in\mathbb{N}}$ /* from eq:6 */

$\mathrm{zer} A \neq \emptyset$

• Proposition 5.4. (some key properties of Fejer monotone sequences)
[C : nonempty subset of $\mathcal{H}$
$(x_n)_{n\in\mathbb{N}}$: Fejer monotone sequence w.r.t. C, $\mathcal{H}$] $\Rightarrow$
(i) $(x_n)_{n\in\mathbb{N}}$: bounded
(ii) $\forall_{z\in C} (\|x_n - z\|)_{n\in\mathbb{N}}$: converges /* note that it does not necessarily mean convergence in C */
(iii) $(d_C(x_n))_{n\in\mathbb{N}}$: decreasing and converges

so, $(x_n)_{n\in\mathbb{N}}$: bounded ... (eq:8), so,

*lemma 2.37. **
[$(x_n)_{n\in\mathbb{N}}$: bounded sequence in $\mathcal{H}$] $\Rightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}} : \text{subsequence of } (x_n)_{n\in\mathbb{N}}} (x_{k_n})_{n\in\mathbb{N}}$: weakly convergent

$\leftrightarrow$ $(x_n)_{n\in\mathbb{N}}$: posseses a weak sequential cluster point ... (eq:85)

also, $\forall_{n\in\mathbb{N}} \ \|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \gamma_n^2 \|u_n\|^2$

$\leftrightarrow \forall_{n\in\mathbb{N}} \ \gamma_n^2 \|u_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2$

$$\leftrightarrow \sum_{n=0}^{m} \gamma_n^2 \|u_n\|^2 \leq \sum_{n=0}^{m} \|x_n - z\|^2 - \sum_{0\leq n\leq m} \|x_{n+1} - z\|^2 = \sum_{n=0}^{m} \|x_n - z\|^2 - \sum_{\tilde{n}=1}^{m+1} \|x_{\tilde{n}} - z\|^2$$

$\leftrightarrow 1 \leq n+1 \leq m+1$, $\underbrace{n+1}_{\tilde{n}}$
$\leftrightarrow 1 \leq \tilde{n} \leq m+1$

$$= \|x_0 - z\|^2 + \cancel{\sum_{n=1}^{m} \|x_n - z\|^2} - \cancel{\sum_{\tilde{n}=1}^{m} \|x_{\tilde{n}} - z\|^2} - \|x_{m+1} - z\|^2$$

$$\leftrightarrow \sum_{n=0}^{m} \gamma_n^2 \|u_n\|^2 \leq \underbrace{\|x_0 - z\|^2}_{\text{finite}} - \underbrace{\|x_{m+1} - z\|^2}_{\text{finite as } (x_n)_{n\in\mathbb{N}} \text{ :bounded per (eq:8) } \forall m}$$

$$\Rightarrow \lim_{m\to\infty} \sum_{n=0}^{m} \gamma_n^2 \|u_n\|^2 \leq \|x_0 - z\|^2 - \mathrm{Lt}_{m\to\infty} \|x_{m+1} - z\|^2 = \text{finite}$$

$\therefore \sum_{n\in\mathbb{N}} \gamma_n^2 \|u_n\|^2 < +\infty$

But given $\sum_{n\in\mathbb{N}} \gamma_n^2 = +\infty$

$\Rightarrow \underline{\lim} \|u_n\| = 0$, but $\lim \|u_n\| = c$ (via eq:4.6), now when lim exists $\underline{\lim} \|u_n\| = \lim \|u_n\| = \overline{\lim} \|u_n\| = 0$

$\therefore \lim \|u_n\| = 0$
$\Leftrightarrow \|u_n\| \to 0$
$\Leftrightarrow u_n \to 0$

*An important result used in the proof of KM iteration
[$(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$; $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$; $\sum_{n\in\mathbb{N}} a_n b_n < +\infty$; $\sum_{n\in\mathbb{N}} a_n = +\infty$] $\underline{\lim}\, b_n = 0$.

(I)
assume x : any weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ /* from (eq:5)

$\leftrightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}}} \ x_{k_n} \rightharpoonup x \Leftrightarrow x_{k_n+1} \rightharpoonup x$

now in (eq:2) $\forall_{n\in\mathbb{N}} \ Ax_{n+1} \ni u_n$, and $u_n \to 0$

$\Rightarrow Ax_{k_n+1} \ni u_{k_n}$, and $u_{k_n} \to 0$

$\therefore (x_{k_n+1}, u_{k_n}) \in \mathrm{gra} A$, $u_{k_n} \to 0$, $x_{k_n+1} \rightharpoonup x$

denote $x_{k_n+1} \equiv y_{k_n}$

$(y_{k_n}, u_{k_n}) \in \mathrm{gra} A$, $u_{k_n} \to 0$, $y_{k_n} \rightharpoonup x$ /* now recall :

Proposition 20.33. (Used heavily by Davis) **
[$A : \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]
(i) gra A : sequentially closed in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{weak}}$
$\overset{\text{def}}{\leftrightarrow} \forall_{(x_n, u_n)_{n\in\mathbb{N}} \subseteq \mathrm{gra} A} \ \forall_{(x,u)\in\mathcal{H}\times\mathcal{H} : x_n \to x, u_n \rightharpoonup u} \ (x,u) \in \mathrm{gra} A$
(ii) gra A : sequentially closed in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$
$\overset{\text{def}}{\leftrightarrow} \forall_{(x_n, u_n)_{n\in\mathbb{N}} \subseteq \mathrm{gra} A} \ \forall_{(x,u)\in\mathcal{H}\times\mathcal{H} : x_n \rightharpoonup x, u_n \to u} \ (x,u) \in \mathrm{gra} A$
(iii) gra A : closed in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{strong}}$

*/

$\therefore (x, 0) \in \mathrm{gra} A \leftrightarrow Ax \ni 0 \leftrightarrow x \in \mathrm{zer} A$

now use Theorem 5.5.

$\overset{\text{def}}{\iff} \forall_{(x_n,u_n)_{n\in\mathbb{N}} \subseteq \operatorname{gra} A} \ \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}: x_n \rightharpoonup x,\, u_n \to u} \ (x,u) \in \operatorname{gra} A$

(iii) $\operatorname{gra} A$: closed in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{strong}}$

Theorem 5.5. ★★ ✓ $C := \operatorname{zer} A$ $\mathcal{H} \phi \neq C \subseteq \mathcal{H}$

[$(x_n)_{n\in\mathbb{N}}$: sequence in $\mathcal{H}$, Fejer monotone w.r.t. C, every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in $\underset{\operatorname{zer} A}{\overset{\|}{C}}$] $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$: converges weakly to a point in $\cancel{C}$ $\operatorname{zer} A$

$\therefore$ x_n converges weakly to a point in $\operatorname{zer} A$ ☺

(ii) A : uniformly monotone on every bounded subset of $\mathcal{H}$

strong monotonicity $\Rightarrow$ uniform monotonicity $\Rightarrow$ strict monotonicity $\Rightarrow$ paramonotonicity $\Rightarrow$ monotonicity

Proposition 23.35. [$A: \mathcal{H} \to 2^{\mathcal{H}}$, strictly monotone] $\operatorname{zer} A$: almost a singleton

$\operatorname{zer} A$: nonempty by given

$\operatorname{zer} A$: Singleton

let, x : weak limit of $(x_n)_{n\in\mathbb{N}}$ in (i)

as $x \in \operatorname{zer} A \iff Ax \ni 0$

Also, [eq: 2.5] $\Rightarrow$ $(x_{n+1}, u_n) \in \operatorname{gra} A \iff Ax_{n+1} \ni u_n$

also $(x_n)_{n\in\mathbb{N}}$: bounded from (eq:8) $\Rightarrow \exists_{\tilde{C}}$ $(x_n)_{n\in\mathbb{N}}$: bounded on $\tilde{C}$

$\therefore$ $(x_n)_{n\in\mathbb{N}} \cup \{x\}$: bounded on $C := \tilde{C} \cup \{x\}$

A : uniformly monotone on $C \iff$ $\left| \begin{array}{l} \exists \phi: \mathbb{R}_+ \to [0,+\infty], \text{ increasing, vanishing only at zero} \\ \forall_{x\in C} \forall_{y\in C} \forall_{u\in Ax} \forall_{v\in Ay} \ \langle x-y \mid u-v\rangle \geq \phi(\|x-y\|) \end{array} \right.$

$x := x;\ y := x_{n+1};$
$u := 0;\ v := u_n$

$\langle x - x_{n+1} \mid 0 - u_n \rangle \geq \phi(\|x - x_{n+1}\|)$

$\therefore \forall_{n\in\mathbb{N}}$ $\langle x_{n+1} - x \mid u_n \rangle \geq \phi(\|x_{n+1} - x\|)$ // now $\phi \geq 0$. ϕ : vanishes only at zero

now $u_n \to 0$, $x_{n+1} \rightharpoonup x$, so

$\Downarrow$

$x_{n+1} - x \rightharpoonup 0$

Lemma 2.41. ★★

[$(x_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$; $x, u \in \mathcal{H}$]

(i) $(x_n \rightharpoonup x,\ \overline{\lim} \|x_n\| \leq \|x\|) \iff x_n \to x$ } When weak convergence is equivalent to strong convergence

(ii) [$\mathcal{H}$: finitedimensional] $x_n \rightharpoonup x \iff x_n \to x$

(iii) $(x_n \rightharpoonup x,\ u_n \to u) \Rightarrow \langle x_n \mid u_n \rangle \to \langle x \mid u \rangle$ } one weak convergence and one strong convergence implies strong convergence of the inner product.

says, $\lim \langle x_{n+1} - x \mid u_n \rangle = 0$

$$\Rightarrow \underbrace{\lim_{n\to\infty} \langle x_{n+1} - x \mid u_n \rangle}_{\overset{\|}{0}} \geq \lim_{n\to\infty} \phi(\|x_{n+1} - x\|) \geq 0$$

$$\Rightarrow \lim_{n\to\infty} \phi(\|x_{n+1} - x\|) = 0$$

$$\Leftrightarrow \|x_{n+1} - x\| \to 0$$

$$\Leftrightarrow x_{n+1} \to x$$

$\therefore$ $x_n \to \underset{\text{unique point in } \operatorname{zer} A}{x}$ ☺

■

Corollary 23.46.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone ; $x \in \mathcal{H}$]

Exactly one of the following holds:

(i) $x \in \operatorname{dom} A$, ${}^{\gamma}Ax \to {}^{0}Ax$ and $\|{}^{\gamma}Ax\| \uparrow \|{}^{0}Ax\|$ as $\gamma \downarrow 0$

(ii) $x \notin \operatorname{dom} A$, $\|{}^{\gamma}Ax\| \uparrow +\infty$ as $\gamma \downarrow 0$

Proof:

define $B: y \mapsto By = A^{-1}y - x$

Proposition 20.22. (Maximally monotonicity preserving operation)

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone ; $z, u \in \mathcal{H}$; $\gamma \in \mathbb{R}_{++}$] $\left| \begin{array}{l} \bullet\ A^{-1} : \text{maximally monotone} \\ \bullet\ u + \gamma A(\cdot + z) : \text{maximally monotone} \end{array} \right.$

so. B : maximally monotone

$\forall_z$ $(z \in \operatorname{zer} B \iff Bz \ni 0 \iff A^{-1}z - x \ni 0 \iff A^{-1}z \ni x$

$\iff z \in Ax)$

$\forall_z$ $(z \in \operatorname{zer} B \iff z \in Ax) \iff \operatorname{zer} B = Ax$ . . . (eq:1)

$\Leftrightarrow z \in Ax)$

$\forall_z \, (z \in \operatorname{zer} B \Leftrightarrow z \in Ax) \Leftrightarrow \operatorname{zer} B = Ax$ . . . (eq:1)

now using Theorem 23.44 :

> perturbation result: set $\tilde{A} := B$, $\tilde{x} = 0$
>
> Theorem 23.44 [$\tilde{A}: H \to 2^H$, maximally monotone ; $\tilde{x} \in H$] $\Rightarrow$
>
> • $\forall_{\gamma \in]0,1[}$ $\tilde{A}x_\gamma + \gamma(x_\gamma - \tilde{x}) \ni 0$ define a unique curve $(x_\gamma)_{\gamma \in]0,1[}$
>
> • Exactly one of the following holds:
>
> (i) $\operatorname{zer} \tilde{A} \neq \emptyset$, $x_\gamma \to P_{\operatorname{zer}\tilde{A}} \tilde{x}$ as $\gamma \downarrow 0$
>
> (ii) $\operatorname{zer} \tilde{A} = \emptyset$, $\|x_\gamma\| \uparrow +\infty$ as $\gamma \downarrow 0$

[eq:0]

$\forall_{\gamma \in]0,1[}$ $Bx_\gamma + \gamma x_\gamma \ni 0$ will define a unique curve $(x_\gamma)_{\gamma \in]0,1[}$

> Proposition 23.6 .
>
> [$A: H \to 2^H$; $\gamma \in \mathbb{R}_{++}$; $\mu \in \mathbb{R}_{++}$]
>
> (i) $\operatorname{gra} {}^\gamma A \subseteq \operatorname{gra}(A \circ J_{\gamma A})$
>
> (ii) ${}^\gamma A = (\gamma \operatorname{Id} + A^{-1})^{-1} = (J_{\gamma^{-1}A^{-1}}) \circ \frac{1}{\gamma} \operatorname{Id}$
>
> (iii) ${}^{\gamma+\mu}A = {}^\gamma({}^\mu A)$
>
> (iv) $J_{\gamma({}^\mu A)} = \operatorname{Id} + \frac{\gamma}{\gamma+\mu}(J_{(\gamma+\mu)A} - \operatorname{Id})$

. now ${}^\gamma A = (J_{\underbrace{\gamma^{-1}A^{-1}}_{\text{maximally monotone}}}) \circ \overbrace{\frac{1}{\gamma} \operatorname{Id}}^{\text{firmly nonexpansive}}$: $w \mapsto {}^\gamma A w = J_{\gamma^{-1}A^{-1}}\left(\frac{1}{\gamma} w\right)$: singlevalued

firmly nonexpansive $\Rightarrow$ singlevalued

$0 \in Bx_\gamma + \gamma x_\gamma = A^{-1}x_\gamma - x + \gamma x_\gamma = (A^{-1} + \gamma \operatorname{Id})x_\gamma - x \Leftrightarrow x \in (A^{-1} + \gamma \operatorname{Id})x_\gamma \Leftrightarrow \underbrace{(A^{-1} + \gamma \operatorname{Id})^{-1}}_{{}^\gamma A} x \ni x_\gamma \Leftrightarrow {}^\gamma Ax \ni x_\gamma \Leftrightarrow {}^\gamma Ax = x_\gamma$ . . . (eq:2)

(i) suppose $x \in \operatorname{dom} A$

$\Leftrightarrow Ax = \operatorname{zer} B \neq \emptyset$ (from eq:1) $\Rightarrow \underset{{}^\gamma Ax \text{ (from eq:2)}}{x_\gamma} \to P_{\operatorname{zer} B} 0$ as $\gamma \downarrow 0$

as this is projection on $\operatorname{zer} B$. $P_{\operatorname{zer} B} 0 \in \operatorname{zer} B$

now, ${}^0Ax = \min_{y \in Ax = \operatorname{zer} B} \|y\| = \min_{y \in \operatorname{zer} B} \|y\|$

and by definition: $P_{\operatorname{zer} B} 0 = \operatorname{argmin}_{y \in \operatorname{zer} B} \frac{1}{2}\|y - 0\|^2 = \operatorname{argmin}_{y \in \operatorname{zer} B} \|y\| = {}^0Ax$

$\therefore {}^\gamma Ax \to {}^0Ax$ as $\gamma \downarrow 0$ ✓

> Proposition 23.43. [$A: H \to 2^H$, maximally monotone ; $\gamma \in \mathbb{R}_{++}$; $\mu \in \mathbb{R}_{++}$; $x \in \operatorname{dom} A$]
>
> (i) $\|{}^\gamma Ax\| \leq \inf \|Ax\|$
>
> (ii) $\|{}^{\gamma+\mu}Ax\| \leq \|{}^\gamma Ax\| \leq \|{}^0Ax\|$

$\Rightarrow \|{}^\gamma Ax\| \uparrow \|{}^0Ax\|$ as $\gamma \downarrow 0$ ✓

☺

(ii) suppose $x \notin \operatorname{dom} A$

$\Leftrightarrow Ax = \operatorname{zer} B = \emptyset$ (from eq:1)

From [eq:0] : $\|x_\gamma\| \uparrow +\infty$ as $\gamma \downarrow 0$

$\|{}^\gamma Ax\|$ by (eq:2)

$\therefore \|{}^\gamma Ax\| \uparrow +\infty$ as $\gamma \downarrow 0$ ☺

∎