Part 1

11:05 AM

Theorem 21.1. (Minty's theorem)

$[A:\mathcal{H}\to 2^{\mathcal{H}}, \text{monotone}]$

A: maximally monotone $\leftrightarrow \operatorname{ran}(\mathrm{Id}+A)=\mathcal{H}$

Proof:

$(\Leftarrow)$ given $\operatorname{ran}(\mathrm{Id}+A)=\mathcal{H}$

goal A: maximally monotone $\leftrightarrow \forall (x,u)\ \big((x,u)\in \operatorname{gra}A \leftrightarrow \forall (\tilde{y},\tilde{v})\in\operatorname{gra}A\ \ \langle x-\tilde{y}\,|\,u-\tilde{v}\rangle \geq 0\big)$

A: monotone $\leftrightarrow \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\Big((x,u)\in\operatorname{gra}A \Rightarrow \forall_{(\tilde{y},\tilde{v})\in\operatorname{gra}A}\ \langle x-\tilde{y}\,|\,u-\tilde{v}\rangle\geq 0\Big)$

so, more specifically we want to show, $\underbrace{\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\ \forall_{(\tilde{y},\tilde{v})\in\operatorname{gra}A}\ \langle x-\tilde{y}\,|\,u-\tilde{v}\rangle\geq 0}_{\text{given (1)}} \Rightarrow (x,u)\in\operatorname{gra}A$ (goal (0))

first note that $\forall (x,u)\in\mathcal{H}\times\mathcal{H}\ \ \tilde{x}=x+u\in\mathcal{H}$

now given,

$\operatorname{ran}(\mathrm{Id}+A)=\mathcal{H}$ /* $\operatorname{ran}\tilde{A}=\tilde{A}(\mathcal{H})$ */

$\leftrightarrow \forall \tilde{x}\in\mathcal{H}\ \exists y\in\mathcal{H}\ \ (\mathrm{Id}+A)y\ni\tilde{x}$

now $\tilde{x}=x+u$ $\exists y\in\mathcal{H}\ \ (\mathrm{Id}+A)y\ni\tilde{x}=x+u$

$\leftrightarrow y+Ay\ni x+u$

$\leftrightarrow \exists_{v\in Ay}\ y+v=x+u\in(\mathrm{Id}+A)y$

$\leftrightarrow \exists \underset{(y,v)\in\operatorname{gra}A}{\underbrace{v\in Ay}},\ y+v=x+u\in(\mathrm{Id}+A)y$

so. for $x+u\in\mathcal{H}$ $\exists_{(y,v)\in\mathcal{H}\times\mathcal{H}}\big((y,v)\in\operatorname{gra}A \wedge y+v=x+u\big)$

in given (1): setting $(\tilde{y},\tilde{v}):=(y,v)\in\operatorname{gra}A \Rightarrow$

$0\leq\langle x-y\,|\,u-v\rangle=\langle x-y\,|\,y-x\rangle$ [from (2)]

$=-\|x-y\|^2\leq 0$

$\leftrightarrow \|x-y\|^2=0 \leftrightarrow x=y$ (3)

(2),(3) $\Rightarrow$ $u=v$ ...(4)

$\therefore$ $(x,u)=(y,v)\in\operatorname{gra}A$ /* from (2), (3), (4) */ // It does say something interesting $\forall_{(x,y)\in\mathcal{H}\times\mathcal{H}}\ \forall_{(\tilde{y},\tilde{v})\in\operatorname{gra}A}\ \langle x-\tilde{y}\,|\,u-\tilde{v}\rangle\geq 0 \Rightarrow (\tilde{y},\tilde{v})=(x,y)\in\operatorname{gra}A$:O

$\leftrightarrow$ goal (0) achieved.

($\Leftarrow$ direction showed) :)

$(\Rightarrow)$

given A: maximally monotone $\leftrightarrow \forall(x,u)\ \big((x,u)\in\operatorname{gra}A \leftrightarrow \forall(\tilde{y},\tilde{v})\in\operatorname{gra}A\ \ \langle x-\tilde{y}\,|\,u-\tilde{v}\rangle\geq 0\big)$ ... (1)

goal: $\operatorname{ran}(\mathrm{Id}+A)=\mathcal{H}$

/* recall maximally monotone operator's Fitzpatrick function representation:

$[A:\mathcal{H}\to 2^{\mathcal{H}}, \text{maximally monotone}]$ $\left.\begin{array}{l}\operatorname{gra}A=\{(x,u)\in\mathcal{H}\times\mathcal{H}\mid F_A(x,u)=\langle x|u\rangle\}\ \therefore (x,u)\in\operatorname{gra}A \leftrightarrow F_A(x,u)=\langle x|u\rangle\\ \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\ F_A(x,u)\geq\langle x|u\rangle\end{array}\right\}$ (5)

$F_A(x,y)=\langle x|y\rangle-\inf_{(y,v)\in\operatorname{gra}A}\langle x-y\,|\,u-v\rangle$; $[A: \text{monotone}]\ F_A\in\Gamma_0(\mathcal{H}\times\mathcal{H})$ ...(6)

*/

$\forall (x,u)\in\mathcal{H}\times\mathcal{H}$

$2F_A(x,u)+\|(x,u)\|^2$

$=2F_A(x,u)+\|x\|^2+\|u\|^2$ /* $\|(x,u)\|^2=\sum x_i^2+\sum u_i^2=\|u\|^2+\|x\|^2$ */

$\geq 2\langle x|u\rangle+\|x\|^2+\|u\|^2$

$=\|x+u\|^2\geq 0$

$\leftrightarrow \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\ \underbrace{F_A}_{\in\Gamma_0(\mathcal{H}\times\mathcal{H})\ [\text{from (6)}]}(x,u)+\frac{1}{2}\|(x,u)\|^2\geq 0$ // dividing both sides by 2

$\leftrightarrow$ $F_A+\frac{1}{2}\|\cdot\|^2\geq 0$, $F_A\in\Gamma_0$ ...(7)

/* Corollary 15.17. $[\, f \in \Gamma_0(\mathcal{H}) \,;\, q = \frac{1}{2}\|\cdot\|^2 \,;\, f+q \geq 0 \,]$ $\exists w \in \mathcal{H}$ $\forall x \in \mathcal{H}$ $f(x)+q(x) \geq q(x-w)$ */

From (7) and Corollary 15.17 we have: /* $w \in \mathcal{H} \Leftrightarrow \exists p \in \mathcal{H}$ $w=-p$ */

$$\exists_{(-v,-y)\in\mathcal{H}\times\mathcal{H}} \ \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \quad F_A(x,u) + \frac{1}{2}\|(x,u)\|^2 \geq \frac{1}{2}\|(x,u)+(v,y)\|^2$$

$$= \frac{1}{2}\|(x,u)\|^2 + \frac{1}{2}\|(v,y)\|^2 + \underbrace{\langle (x,u) \mid (v,y)\rangle}_{\langle x|v\rangle + \langle u|y\rangle} \quad \text{/* } \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} v \\ y \end{bmatrix} = [x^T \ u^T]\begin{bmatrix} v \\ y \end{bmatrix} = x^T v + u^T y \text{ */}$$

$$\Leftrightarrow F_A(x,u) \geq \underbrace{\frac{1}{2}\|(v,y)\|^2}_{\underbrace{\frac{1}{2}\|v\|^2 + \frac{1}{2}\|y\|^2}_{\geq -\langle v|y\rangle}} + \langle x|v\rangle + \langle u|y\rangle$$

/* $\|v+y\|^2 = \|v\|^2 + \|y\|^2 + 2\langle v|y\rangle \geq 0$
$\Rightarrow \frac{1}{2}\|v\|^2 + \frac{1}{2}\|y\|^2 \geq -\langle v|y\rangle$ */

$$\geq -\langle v|y\rangle + \langle x|v\rangle + \langle u|y\rangle \quad \ldots (8)$$

$$\therefore \exists_{(v,y)\in\mathcal{H}\times\mathcal{H}} \ \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \quad F_A(x,u) \geq -\langle v|y\rangle + \langle x|v\rangle + \langle u|y\rangle \ \ldots (9)$$

if $(x,u) \in \operatorname{gra} A \Leftrightarrow F_A(x,u) = \langle x|u\rangle$ [from (5)] ... (10)

(9), (10) $\Rightarrow$

$$\exists_{(v,y)\in\mathcal{H}\times\mathcal{H}} \ \forall_{(x,u)\in\operatorname{gra} A} \quad \langle x|u\rangle \geq -\langle v|y\rangle + \langle x|v\rangle + \langle u|y\rangle$$

$$\Leftrightarrow \langle x|u\rangle - \langle x|v\rangle + \langle v|y\rangle - \langle u|y\rangle \geq 0$$

$$\Leftrightarrow \langle x|u-v\rangle - \langle u-v|y\rangle = \langle x|u-v\rangle - \langle y|u-v\rangle = \langle x-y \mid u-v\rangle \geq 0$$

A: maximally monotone $\Rightarrow \underbrace{(y,v) \in \operatorname{gra} A}_{(12)} \Leftrightarrow Ay \ni v$ [From (11)]

$$\exists_{(v,y)\in\operatorname{gra} A} \ \forall_{(x,u)\in\operatorname{gra} A} \quad \langle x|u\rangle \geq \frac{1}{2}\|v\|^2 + \langle x|v\rangle + \frac{1}{2}\|y\|^2 + \langle y|u\rangle$$

/* set $(x,u) := (y,v) \in \operatorname{gra} A$ [from (11)] */

$$\longrightarrow \langle y|v\rangle \geq \frac{1}{2}\|v\|^2 + \langle y|v\rangle + \frac{1}{2}\|y\|^2 + \langle y|v\rangle$$

$$\Rightarrow 0 \geq \frac{1}{2}\|v\|^2 + \frac{1}{2}\|y\|^2 + \langle y|v\rangle$$

$$\Leftrightarrow 0 \geq \|v\|^2 + \|y\|^2 + 2\langle y|v\rangle = \|v+y\|^2$$

$$\Leftrightarrow v+y=0$$

$$\Leftrightarrow v=-y \ \ldots (13)$$

(12), (13) $\Rightarrow$

$(y,-y) \in \operatorname{gra} A$

$\Leftrightarrow Ay \ni -y \Leftrightarrow (\mathrm{Id}+A)y \ni 0$

$\therefore 0 \in (\mathrm{Id}+A)y$

$\Leftrightarrow \exists y \in \mathcal{H}$ $(\mathrm{Id}+A)(y) \ni 0$

$\Leftrightarrow 0 \in \operatorname{ran}(\mathrm{Id}+A)$ /* note that this holds for any maximally monotone operator */ ... (14)

take any $w \in \mathcal{H}$,

define $B: \mathcal{H} \to 2^{\mathcal{H}} : x \mapsto -w + Ax$: maximally monotone /* [$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

$u + \gamma A(\cdot + z)$: maximally monotone */ ($\gamma \in \mathbb{R}_{++}$, $u, z \in \mathcal{H}$)

Here comes the magic step:

now, (14) was derived for any maximally monotone operator, so B, too will satisfy (14)

$0 \in \operatorname{ran}(\mathrm{Id}+B)$

$\Leftrightarrow \exists z \in \mathcal{H}$ $(\mathrm{Id}+B)z = z + Bz = 0$

$\Leftrightarrow z - w + Az = 0$

$\Leftrightarrow (\mathrm{Id}+A)z = w$

$\Leftrightarrow w \in \operatorname{ran}(\mathrm{Id}+A)$

$\therefore \forall_{w\in\mathcal{H}}$ $w \in \operatorname{ran}(\mathrm{Id}+A)$

$\Leftrightarrow \mathcal{H} \subseteq \operatorname{ran}(\mathrm{Id}+A)$
also trivially $\operatorname{ran}(\mathrm{Id}+A) \subseteq \mathcal{H}$ $\Big] \Rightarrow \boxed{\operatorname{ran}(\mathrm{Id}+A) = \mathcal{H}}$

■

Theorem 21.7. (Debrunner-Flor)

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone, gra $A \neq \emptyset$]

$\forall_{w\in\mathcal{H}} \ \exists_{x\in \overline{\mathrm{conv}}\,\mathrm{dom}\,A} \ \inf_{(y,v)\in \mathrm{gra}A} \langle y-x \mid v-(w-x)\rangle \geq 0$

Proof:

$C = \overline{\mathrm{conv}}\,\mathrm{dom}\,A$

goal:

$\forall_{w\in\mathcal{H}} \ \exists_{x\in \overline{\mathrm{conv}}\,\mathrm{dom}\,A} \ \inf_{(y,v)\in \mathrm{gra}A} \langle y-x \mid v-(w-x)\rangle \geq 0 \quad \ldots (1)$

now $F_A(x,u) = \langle x \mid u\rangle - \inf_{(y,v)\in \mathrm{gra}A} \langle x-y \mid u-v\rangle$ /* By definition */

$\therefore F_A(x,w-x) = \langle x \mid w-x\rangle - \inf_{(y,v)\in \mathrm{gra}A} \underbrace{\langle x-y \mid w-x-v\rangle}$

$\frac{1}{(-1)^2}\langle -(x-y) \mid -(w-x-v)\rangle = \langle y-x \mid v-w+x\rangle$

$\leftrightarrow \ \langle x \mid w-x\rangle - F_A(x,w-x) = \inf_{(y,v)\in \mathrm{gra}A} \langle x-y \mid w-x+v\rangle \quad \ldots (2)$

from (1),(2) the goal is:

$\forall_{w\in\mathcal{H}} \ \exists_{x\in C} \ \underbrace{\langle x \mid w-x\rangle - F_A(x,w-x)} \geq 0$

$= -\underbrace{\langle x \mid x\rangle}_{\|x\|^2} + \langle x \mid w\rangle - F_A(x,w-x)$

$\leftrightarrow \forall_{w\in\mathcal{H}} \ \exists_{x\in C} \ F_A(x,w-x) + \|x\|^2 - \langle x \mid w\rangle \leq 0$

$\leftrightarrow \forall_{w\in\mathcal{H}} \ \min_{x\in\mathcal{H}} \left(F_A(x,w-x) + \|x\|^2 - \langle x \mid w\rangle + \iota_C(x)\right) \leq 0$ /* $\exists_{x\in C} \ P(x) \leq 0 \leftrightarrow \min_{x\in\mathcal{H}} P(x) + \iota_C(x) \leq 0$ */

... goal (3)

We divide the proof into two cases: $w=0$, $w\neq 0$

Case 1: $w=0$

then goal (3) becomes:

$\min_{x\in\mathcal{H}} \left(F_A(x,-x) + \|x\|^2 + \iota_C(x)\right) \leq 0 \quad \ldots$ (goal (4))

set. $q = \frac{1}{2}\|\cdot\|^2$

$f: \mathcal{H}\times\mathcal{H} \to]-\infty,+\infty]: f(y,x) = \frac{1}{2} \underbrace{F_A^*}(2y,2x) \ \in \Gamma_0(\mathcal{H}\times\mathcal{H})$

Conjugate of the Fitzpatrick function

$g = (q+\iota_C)^* = q - \frac{1}{2} d_C^2 \in \Gamma_0(\mathcal{H})$ /* Example 13.5. $(\frac{1}{2}\|\cdot\|^2 + \iota_C)^* = \frac{1}{2}\left(\|\cdot\|^2 - d_C^2\right)$ */

$L: \mathcal{H}\times\mathcal{H} \to \mathcal{H}: L(y,x) = x-y \ \in \mathcal{B}(\mathcal{H}\times\mathcal{H}, \mathcal{H})$

now we show that, $\inf_{(y,x)\in\mathcal{H}\times\mathcal{H}} \left(f(y,x) + g(L(y,x))\right) \geq 0 \quad \ldots (4)$

$\leftrightarrow \forall_{(y,x)\in\mathcal{H}\times\mathcal{H}} \ f(y,x) + g(L(y,x)) \geq 0$

lets expand the objective:

$(f+g\circ L)(y,x) = f(y,x) + g(\underbrace{L(y,x)}_{x-y}) = \frac{1}{2} F_A^*(2y,2x) + \underbrace{q(x-y)}_{\frac{1}{2}\|x-y\|^2} - \frac{1}{2}\underbrace{d_C^2(x-y)} \quad \ldots (5)$

/* $d_C(\square) = \|\square - P_C\square\| \leq \|\square - z\| \ \forall z\in C$ */

in (4)

we can confine $(y,x)\in \mathrm{dom}\, f \cap \underbrace{\mathrm{dom}\, g\circ L}_{\subseteq \mathrm{dom} f}$ /* outside it we will have $+\infty$ as the value */

now, $(y,x)\in \mathrm{dom}\, f \leftrightarrow (2y,2x)\in \mathrm{dom}\, F_A^*$ [from (5)] ... (6)

Now we show (4) as follows:

$\forall_{(y,x)\in\mathcal{H}\times\mathcal{H}}$

/* Proposition 20.51.(ii) : conjugate of Fitzpatrick function

$[A:\mathcal{H}\to 2^{\mathcal{H}}, \text{monotone}, \text{gra } A\neq\emptyset] \;\Big|\; \overline{\text{conv}}\text{ gra }A^{-1} \subseteq \text{dom } F_A^* \subseteq \overline{\text{conv}}\text{ gra }A^{-1} \subseteq \overline{\text{conv}}\text{ ran }A \times \overline{\text{conv}}\text{ dom }A$ */
$F_A^* \geq \langle\cdot|\cdot\rangle$

$\text{dom } F_A^* \subseteq \overline{\text{conv}}\text{ ran }A \times \underbrace{\overline{\text{conv}}\text{ dom }A}_{C}$. (7)

from (6),(7) $\Rightarrow$ $(2y,2x)\in\overline{\text{conv}}\text{ ran }A\times C \Rightarrow \boxed{2x\in C}$ ...(8)

now: $0 = \underbrace{4\langle y|x\rangle}_{\langle 2y|2x\rangle} + \|x-y\|^2 - \|x+y\|^2$ /* $= 4\langle y|x\rangle + \|x\|^2 - 2\langle x|y\rangle + \|y\|^2 - \|x\|^2 - 2\langle x|y\rangle - \|y\|^2$ */

$= \langle 2y|2x\rangle + \|x-y\|^2 - \|(x-y)-2x\|^2$

$\leq F_A^*(2y,2x) + \|x-y\|^2 - \|(x-y)-2x\|^2$

$\leq F_A^*(2y,2x) + \|x-y\|^2 - d_C^2(x-y)$

$= 2\left(f(y,x) + g(L(y,x))\right)$ [from (5)]

/* (8) $\Rightarrow$ $2x\in C$
now $d_C(x-y) = \|(x-y) - \underbrace{P_C(x-y)}_{\in C}\| \leq \|x-y-z\| \quad \forall z\in C$
$\Rightarrow \quad -d_C^2(x-y) \geq -\|x-y-z\|^2 \quad \forall z\in C$ */

so,
$\forall_{(x,y)\in\mathcal{H}\times\mathcal{H}} \quad f(y,x) + g(L(y,x)) \geq 0 \Leftrightarrow \inf (f+g\circ L)(\mathcal{H}\times\mathcal{H}) \geq 0$ ...(9)

now $g(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}d_C^2(x) < +\infty \quad \forall_{x\in\mathcal{H}}$

$\Rightarrow$ $\text{dom } g = \mathcal{H}$

as a result $\text{dom } g - L\text{dom } f = \mathcal{H} - L\text{dom } f = \mathcal{H}$ /* recall $C-D=\{x-y \mid x\in C, y\in D\}$, so $C=\mathcal{H}$ হলে some other set থেকে substract করলে কোন change হবে না */

so, $\text{int}(\text{dom } g - L\text{dom } f) = \text{int }\mathcal{H} = \mathcal{H}$

now $0\in\mathcal{H} = \text{int}(\text{dom } g - L\text{dom } f) \subseteq \text{sri}(\text{dom } g - L\text{dom } f)$ /* $\text{int } C \subseteq \text{sri } C$ for C: convex $\Leftarrow$ (6.11) */

Theorem 15.23. $g\in\Gamma_0(\mathcal{H})$
$[f\in\Gamma_0(\cancel{\mathcal{H}})$ $\in\Gamma_0(\mathcal{H}\times\mathcal{H})$, $g\in\Gamma_0(\mathcal{K})$, $L\in\mathcal{B}(\cancel{\mathcal{H},\mathcal{K}})$ $\mathcal{B}(\mathcal{H}\times\mathcal{H},\mathcal{H})$
$0\in\text{sri}(\text{dom } g - L(\text{dom } f))]$
$\inf(f+g\circ L)(\cancel{\mathcal{H}})^{(\mathcal{H}\times\mathcal{H})} = -\min(f^*\circ L^* + g^{*\vee})(\cancel{\mathcal{K}})^{\mathcal{H}}$ ✓ /* $\varphi^\vee$: reversal of $\varphi \overset{\text{def}}{\Leftrightarrow} \varphi^\vee : x\mapsto\varphi(-x)$ */

$\inf(f+g\circ L)(\mathcal{H}) = -\min(f^*\circ L^* + g^{*\vee})(\mathcal{H})$

$f:\mathcal{H}\times\mathcal{H}\to]-\infty,+\infty]: f(y,x) = \frac{1}{2}\underbrace{F_A^*}_{}(2y,2x) \in\Gamma_0(\mathcal{H}\times\mathcal{H}) \Rightarrow f^* = \frac{1}{2}F_A$

conjugate of the Fitzpatrick function

$g = (q+\iota_C)^* = q - \frac{1}{2}d_C^2 \in\Gamma_0(\mathcal{H}) \Rightarrow g^* = q + \iota_C$

$L:\mathcal{H}\times\mathcal{H}\to\mathcal{H}: L(y,x) = x-y \in\mathcal{B}(\mathcal{H}\times\mathcal{H},\mathcal{H}) \Rightarrow L^*:\mathcal{H}\to\mathcal{H}\times\mathcal{H}: x\mapsto(x,-x)$

using 15.23 we have

$$\min_{x\in\mathcal{H}} \left(f^*(-L^*x) + g^*(x)\right) \leq 0$$ [could not figure this step]

$$\Rightarrow \min_{x\in\mathcal{H}} \left(F_A(x,-x) + (\|x\|^2 + \iota_C(x))\right) \leq 0$$

$\therefore$ goal (4) achieved. ☺

(b)
$W\neq 0$
..

$B: \mathcal{H} \to 2^{\mathcal{H}}$: gra $B = -(0,w) +$ gra A

as in (a) $\exists_{(x,-x) \in \mathcal{H} \times \mathcal{H}}$ $\{(x,-x)\} \cup$ gra B : monotone

$\Leftrightarrow$ $\{(x, w-x)\} \cup$ gra A : monotone

■

Theorem 21.15. (Rockafellar–Veselý)

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone

$x \in \mathcal{H}$]

A: locally bounded at $x \leftrightarrow x \notin$ bdry dom A

Proof:

First we prove $x \notin$ bdry dom $A \leftrightarrow x \in \overline{\text{dom}}\, A \Rightarrow A$: locally bounded at x . . . (goal 1)

S: set of all points at which A is locally bounded $\Leftrightarrow x \in S \leftrightarrow \exists_{M_x > 0} \exists_{\delta > 0} \forall_{y : \|y-x\|}$ $\|Ay\| < M_x$

$\mathcal{H} \setminus \overline{\text{dom}}\, A \subseteq S$ . . . (1)

/* $\forall x$ $x \in \mathcal{H} \setminus \overline{\text{dom}}\, A \leftrightarrow x \in \mathcal{H}, x \notin \overline{\text{dom}}\, A =$ dom $A \cup$ limpoints(dom$A$)

$\updownarrow$

$x \notin$ dom A, $x \notin$ limpoints (dom A) /*recall: $x \in$ limpoints $(E) \leftrightarrow \exists_{(x_a)_{a \in A} \subseteq E : x_a \neq x}$ $x_a \to x$ */

$\updownarrow$ $\qquad$ $\updownarrow$

$Ax = \{\}$ $\qquad$ $\forall (x_a)_{a \in A} \subseteq$ dom$A$: $x_a \neq x$ $\;x_a \not\to x$

$\updownarrow$ $\qquad$ $\exists \varepsilon > 0 \;\forall N \;\exists b \geq N \;\|x_b - x\| \geq \varepsilon$

$(x_a)_{a \in A}$: $Ax_a \neq \{\}$: $x_a \neq x$

So, $x \in \mathcal{H} \setminus \overline{\text{dom}}\, A \leftrightarrow x \in \mathcal{H}$, $Ax = \{\}$, $\forall_{(x_a)_{a \in A} : Ax_a \neq \{\}, x_a \neq x} \exists_{\varepsilon > 0} \forall_N \exists_{b \geq N} \|x_b - x\| \geq \varepsilon$

clearly at such x, Ax will be bounded (as $Ax = \{\}$ does not even have a point, so it cannot be unbounded */

$x \in S$ */

Proposition 21.10. (Sufficient condition for local boundedness of an operator)

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, monotone;

$\theta_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H} : (x,u) \mapsto x$

$z \in$ int θ_1(dom F_A)]

A: locally bounded at $z \leftrightarrow z \in S$ $\quad\therefore$ int θ_1(dom F_A) $\subseteq S$

Proposition 21.11. (Representing domain of a maximally monotone operator via Fitzpatrick function)

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone

$\theta_1 : \mathcal{H} \times \mathcal{H} \to \mathcal{H} : (x,u) \mapsto x$]

int dom $A \subseteq$ int θ_1(dom F_A) $\subseteq$ dom $A \subseteq \theta_1$(dom F_A) $\subseteq \overline{\text{dom}}\, A$

int dom $A =$ int θ_1(dom F_A)

$\overline{\text{dom}}\, A = \overline{\theta_1(\text{dom}\, F_A)}$

int dom $A \subseteq S$

Now we claim

$S \cap \overline{\text{dom}}\, A = S \cap$ dom A . . . (goal 0)

* Showing $S \cap \overline{\text{dom}}\, A \subseteq S \cap$ dom A

let, $x \in S \cap \overline{\text{dom}}\, A \leftrightarrow x \in \overline{\text{dom}}\, A$, $x \in S$

$\updownarrow$

A locally bounded at x

now $\overline{\text{dom}}\, A$: convex, closed

$\Rightarrow \overline{\text{dom}}\, A$: sequentially closed $\leftrightarrow \forall_{(x_n)_{n \in \mathbb{N}} \subseteq \overline{\text{dom}}\, A : x_n \to \tilde{x}}$ $\tilde{x} \in \overline{\text{dom}}\, A$

$\Rightarrow \exists_{(x_n, u_n)_{n \in \mathbb{N}} \subseteq \text{gra} A : x_n \to x}$ $x \in \overline{\text{dom}}\, A \wedge (u_n)_{n \in \mathbb{N}}$: bounded

A bounded sequence has a weakly convergent sequence (Lemma 2.37)

So, we can construct a subsequence (still expressed by (x_n, u_n) for convenience)

such that:

$\exists_{(x_n, u_n)_{n \in \mathbb{N}} \subseteq \text{gra} A : x_n \to x, u_n \rightharpoonup u}$ $(x,u) \in$ gra$A$, also u: bounded $\Rightarrow$ $x \in$ dom$A$, locally bounded

$\Rightarrow x \in S \cap$ dom A

Proposition 20.33. (Used heavily by Davis)

$\therefore S \cap \overline{\text{dom}}\, A \subseteq S \cap$ dom A . . . (2)

> **Proposition 20.33.** (Used heavily by Davis)
> [$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]
> (i) $\operatorname{gra} A$: sequentially closed in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{weak}}$
> $\overset{\text{def}}{\leftrightarrow} \forall_{(x_n,u_n)_{n\in\mathbb{N}} \subseteq \operatorname{gra} A} \ \forall_{(x,u)\in\mathcal{H}\times\mathcal{H} : x_n \to x,\, u_n \rightharpoonup u} \ (x,u) \in \operatorname{gra} A$
> (ii) $\operatorname{gra} A$: sequentially closed in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$
> $\overset{\text{def}}{\leftrightarrow} \forall_{(x_n,u_n)_{n\in\mathbb{N}} \subseteq \operatorname{gra} A} \ \forall_{(x,u)\in\mathcal{H}\times\mathcal{H} : x_n \rightharpoonup x,\, u_n \to u} \ (x,u) \in \operatorname{gra} A$
> (iii) $\operatorname{gra} A$: closed in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{strong}}$

$\Rightarrow x \in S \cap \operatorname{dom} A$

$\therefore S \cap \overline{\operatorname{dom} A} \subseteq S \cap \operatorname{dom} A \ldots (2)$

now as $\operatorname{dom} A \subseteq \overline{\operatorname{dom}} A$

$\Rightarrow S \cap \operatorname{dom} A \subseteq S \cap \overline{\operatorname{dom} A} \ldots (3)$

So, from (2), (3):

$S \cap \overline{\operatorname{dom}} A = S \cap \operatorname{dom} A \ldots$ (goal (0) achieved)

$\Downarrow$

(goal (1) achieved)

($\Rightarrow$)

Now, we show

$S \cap \operatorname{bdry} \operatorname{dom} A = \emptyset$

$\leftrightarrow \neg (\exists_x \ x \in S \wedge x \in \operatorname{bdry} \operatorname{dom} A)$

$\leftrightarrow \forall_x \ \underbrace{x \notin S}_{\neg(x\in S)} \vee x \notin \operatorname{bdry} \operatorname{dom} A$

$\leftrightarrow \forall_x \ x \in S \Rightarrow x \notin \operatorname{bdry} \operatorname{dom} A$

$\leftrightarrow \forall_{x : x \in S} \ x \notin \operatorname{bdry} \operatorname{dom} A$

Per absurdum $\exists_{x : x\in S} \ x \in \operatorname{bdry} \operatorname{dom} A$

$\updownarrow$

$\exists \delta \in \mathbb{R}_{++} \ A(B(x; 2\delta))$: bounded

$C = \overline{\operatorname{dom}} A$: convex, closed, nonempty [Corollary 21.12].

$\Rightarrow \operatorname{spts} C = P_C(\underbrace{\mathcal{H} \setminus \overline{\operatorname{dom}} A}_{\operatorname{bdry} C}) = P_C(\operatorname{bdry} C)$

$\overline{\operatorname{spts} C} = \operatorname{bdry} C$

ব্যাপার কী?

$\Rightarrow \exists_{z \in \operatorname{spts} C} \ z \in \operatorname{bdry} C \cap B(x; \delta)$

$w \in N_C z \setminus \{0\}$

$B(z; \delta) \subseteq B(x; 2\delta) \Rightarrow z \in \operatorname{bdry} C$

$\vdots$

(incomplete)

■

⊛ **Corollary 21.17.**

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, at most single-valued]

A : strong-to-weak continuous everywhere on $\operatorname{int} \operatorname{dom} A$

Proof: We want to show:

$\forall_{x \in \operatorname{int}\operatorname{dom} A} \ \forall_{(x_n)_{n\in\mathbb{N}} : x_n \to x} \ A x_n \rightharpoonup A x$

$\leftrightarrow$ A: strong-to-weak continuous everywhere on $\operatorname{int} \operatorname{dom} A$.

Fix

$x \in \operatorname{int} \operatorname{dom} A \Rightarrow x \notin \operatorname{bdry} \operatorname{dom} A \leftrightarrow A$: locally bounded at $x \leftrightarrow \exists_{\delta \in \mathbb{R}_{++}} \ A(B(x;\delta))$: bounded

> Theorem 21.15. (Rockafellar–Veselý)
> [$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone
> $x \in \mathcal{H}$]
> A : locally bounded at $x \leftrightarrow x \notin \operatorname{bdry} \operatorname{dom} A$

Construct, $(x_n)_{n\in\mathbb{N}} \subseteq B(x;\delta) : x_n \to x$

$\therefore (A x_n)_{n\in\mathbb{N}}$: bounded

$\Rightarrow \exists_{\text{subsequence that weakly converges}}$ [Lemma 2.37]

$\leftrightarrow \exists_{(A x_{k_n})_{n\in\mathbb{N}}} \ \exists_y \ A x_{k_n} \rightharpoonup y$

$x_{k_n} \to x$ /* any subsequence of a convergent sequence goes to the same limit point */

$\Leftrightarrow \exists_{(Ax_{k_n})_{n\in\mathbb{N}}} \exists_y \; Ax_{k_n} \rightharpoonup y$

$x_{k_n} \to x$ /* any subsequence of a convergent sequence goes to the same limit point */

$(x_{k_n}, Ax_{k_n}) \in \operatorname{gra} A : x_{k_n} \to x,\ Ax_{k_n} \rightharpoonup y$

$\Rightarrow \boxed{(x, y) \in \operatorname{gra} A}$ /* using

Proposition 20.33. (Used heavily by Davis)

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

(i) $\operatorname{gra} A$: sequentially closed in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{weak}}$

$\overset{\text{def}}{\Leftrightarrow} \forall_{(x_n, u_n)_{n\in\mathbb{N}} \subseteq \operatorname{gra} A} \forall_{(x,u) \in \mathcal{H}\times\mathcal{H} : x_n \to x,\ u_n \rightharpoonup u} (x,u) \in \operatorname{gra} A$

(ii) $\operatorname{gra} A$: sequentially closed in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$

$\overset{\text{def}}{\Leftrightarrow} \forall_{(x_n, u_n)_{n\in\mathbb{N}} \subseteq \operatorname{gra} A} \forall_{(x,u) \in \mathcal{H}\times\mathcal{H} : x_n \rightharpoonup x,\ u_n \to u} (x,u) \in \operatorname{gra} A$

(iii) $\operatorname{gra} A$: closed in $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{strong}}$

*/

$\Rightarrow y = Ax$

[$\because$ A: almost single-valued]

$Ax_{k_n} \rightharpoonup Ax$

now note that $(Ax_{k_n})_{n\in\mathbb{N}}$ is an arbitrary subsequence of $(Ax_n)_{n\in\mathbb{N}}$

so: any subsequence of $(x_n)_{n\in\mathbb{N}}$ will weakly converge to $Ax = y$.

$\therefore$ Ax : unique cluster point of $(Ax_n)_{n\in\mathbb{N}}$

$Ax_n \rightharpoonup Ax$

using lemma 2.38.

* lemma 2.38. ★★★

[$(x_n)_{n\in\mathbb{N}}$: sequence in $\mathcal{H}$]

$(x_n)_{n\in\mathbb{N}}$: converges weakly $\Leftrightarrow$ $(x_n)_{n\in\mathbb{N}}$: bounded, posseses at most one weak sequential cluster point.

So, we have proven that,

$\forall_{x \in \operatorname{int}\operatorname{dom} A} \forall_{(x_n)_{n\in\mathbb{N}} : x_n \to x} \; Ax_n \rightharpoonup Ax$

$\Leftrightarrow$ A: strong-to-weak continuous everywhere on $\operatorname{int}\operatorname{dom} A$.

■

Corollary 21.20.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$; maximally monotone; $\lim_{\|x\| \to +\infty} \inf \|Ax\| = +\infty$]

A: surjective $\Leftrightarrow$ A: onto

Proof:

/* Corollary 21.19.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone]

A: surjective $\Leftrightarrow A^{-1}$: locally bounded everywhere on $\mathcal{H}$ */

goal: A^{-1}: locally bounded on $\mathcal{H}$

Per absurdum A^{-1}: not locally bounded at $u \in \mathcal{H}$

$\Rightarrow \exists_{(x_n, u_n)_{n\in\mathbb{N}} \in \operatorname{gra} A : u_n \to u} \; \|x_n\| \to +\infty$

finite

Hence,

$+\infty = \lim_{\|x\| \to +\infty} \inf \|Ax\|$

$= \lim \inf \|Ax_n\| \leq \lim \|Ax_n\| = \lim \|u_n\| = \|u\|$

contradiction.

■

Part 2

4:23 PM

Theorem 21.8.

$[\![A: \mathcal{H} \to 2^{\mathcal{H}},\ \text{monotone}]\!]$

$\exists_{\tilde{A}: \text{maximally monotone extension of } A}$ $\quad \operatorname{dom} \tilde{A} \subseteq \overline{\operatorname{conv}} \operatorname{dom} A$

Proof:

set $C = \overline{\operatorname{conv}} \operatorname{dom} A$

$\mathcal{M} = \{ B \mid B: \text{monotone extension of } A,\ \operatorname{dom} B \subseteq C = \overline{\operatorname{conv}} \operatorname{dom} A \}$ //this set is not empty as //atleast A is there

order $\mathcal{M}$ partially: $\forall_{B_1 \in \mathcal{M}} \forall_{B_2 \in \mathcal{M}} \ B_1 \preccurlyeq B_2 \leftrightarrow (B_1 \ \operatorname{gra} B_1 \subseteq \operatorname{gra} B_2)$

Every chain $\mathcal{C}$ in $\mathcal{M}$ has its union as an upper bound //see a detailed explanation of this logic at of $\mathcal{C}$

/* Recall Zorn's lemma: $[\![A: \text{partially ordered set};\ \forall_{C: \text{chain in } A} \exists_{\text{upper bound for } C}]\!]$ A: contains a maximal element m

$\leftrightarrow \forall_{x \in A.\ x \succcurlyeq m} \ x = m$ */

Using Zorn's lemma we will have a maximal element $\tilde{A} \in \mathcal{M}$. now we show that $\tilde{A}$: maximally monotone. $\tilde{A}$: monotone by construction

take $w \in \mathcal{H}$, assume $w \in \mathcal{H} \setminus \operatorname{ran}(\operatorname{Id} + \tilde{A})$

Theorem 21.7. (Debrunner-Flor)

$[A: \mathcal{H} \to 2^{\mathcal{H}},\ \text{monotone operator},\ \operatorname{gra} A \neq \varnothing]$

$\forall_{w \in \mathcal{H}} \exists_{x \in \overline{\operatorname{conv}} \operatorname{dom} A} \inf_{(y,v) \in \operatorname{gra} A} \langle y - x \mid v - (w - x) \rangle \geq 0$

Now use, Debrunner-Flor theorem:

$\Rightarrow$

$$\exists_{x \in \overline{\operatorname{conv}} \operatorname{dom} \tilde{A}} \inf_{(y,v) \in \operatorname{gra} \tilde{A}} \langle y - x \mid v - (w - x) \rangle \geq 0 \quad \ldots \ (1)$$

Now by construction, $\operatorname{dom} \tilde{A} \subseteq \overline{\operatorname{conv}} \operatorname{dom} A$

$\Rightarrow \overline{\operatorname{conv}} \operatorname{dom} \tilde{A} \subseteq \overline{\operatorname{conv}} (\overline{\operatorname{conv}} \operatorname{dom} A) = \overline{\operatorname{conv}} \operatorname{dom} A$ $\quad // \because A \subseteq B \Rightarrow \overline{\operatorname{conv}} A \subseteq \overline{\operatorname{conv}} B$

$= C$, so (1) becomes:

$$\exists_{x \in C} \inf_{(y,v) \in \operatorname{gra} \tilde{A}} \underbrace{\langle y - x \mid v - (w - x) \rangle \geq 0}_{\langle x - y \mid (w - x) - v \rangle \geq 0}$$

now, $w \notin \operatorname{ran}(\operatorname{Id} + \tilde{A})$

$\leftrightarrow \forall_{\tilde{x} \in \mathcal{H}} \underbrace{(\operatorname{Id} + \tilde{A}) \tilde{x} \not\ni w}$

$\overset{\tilde{x} := x}{\Rightarrow} \ x + \tilde{A} x \not\ni w \leftrightarrow \tilde{A} x \not\ni w - x \leftrightarrow (x, w - x) \notin \operatorname{gra} \tilde{A}$

so, $\{(x, w - x)\} \cup \operatorname{gra} \tilde{A}$: graph of an operator in $\mathcal{M}$ that properly extends $\tilde{A}$

this contradicts the maximality of $\tilde{A}$

$\therefore \operatorname{ran}(\operatorname{Id} + \tilde{A}) = \mathcal{H}$

$\Uparrow$ Minty's theorem

$\tilde{A}$: maximally monotone.

■