# Part 1

5:02 PM

Proposition 2,6·2 ·

$[\![ A: \mathcal{H} \to 2^{\mathcal{H}} ]\!]$

(i) A : monorone $\Leftrightarrow$

(ii) A : accretive $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \ \forall_{\alpha\in[0,1]} \quad \|x-y+\alpha(u-v)\| \geq \|x-y\| \quad \Leftrightarrow$

(iii) $\forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \quad \|x-u\|^2+\|y-v\|^2 \leq \|x-v\|^2 + \|y-u\|^2$

Proof :

(ii) $\Leftrightarrow$ (i)

A: monotone $\Leftrightarrow$ $\forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \quad \langle x-y|u-v\rangle \geq 0$

$\Leftrightarrow \forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \quad \langle x-y|v-u\rangle \leq 0$

/* Lemma 2.12. ((charecterizes obtuse and perpendicular vectors)

(i) $\forall_{x,y\in\mathcal{H}} \quad \langle x|y\rangle \leq 0 \Leftrightarrow \forall_{\alpha\in\mathbb{R}_+} \ \|x\| \leq \|x-\alpha y\| \Leftrightarrow \forall_{\alpha\in[0,1]} \ \|x\| \leq \|x-\alpha y\|$

(ii) $\forall_{x,y\in\mathcal{H}} \quad x \perp y \Leftrightarrow \forall_{\alpha\in\mathbb{R}} \ \|x\| \leq \|x-\alpha y\| \Leftrightarrow \forall_{\alpha\in[-1,1]}. \ \|x\| \leq \|x-\alpha y\|$

$\Leftrightarrow \forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \ \forall_{\alpha\in[0,1]}$

$$\|x-y\| \leq \|(x-y)-\alpha(v-u)\|$$

$$= \|x-y+\alpha(u-v)\| \qquad \text{(ii)} \Leftrightarrow \text{(i)}$$

(iii) $\Leftrightarrow$ (ii) :

$\forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \quad \|x-u\|^2+\|y-v\|^2 \leq \|x-v\|^2+\|y-u\|^2$

$$\Leftrightarrow \|\cancel{x}\|^2+\|\cancel{u}\|^2-2\langle x|u\rangle+\|\cancel{y}\|^2+\|\cancel{v}\|^2-2\langle y|v\rangle \leq \|\cancel{x}\|^2+\|\cancel{v}\|^2-2\langle x|v\rangle+\|\cancel{y}\|^2+\|\cancel{u}\|^2 -2\langle y|u\rangle$$

$\Leftrightarrow \forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \quad -2\langle x|u\rangle-2\langle y|v\rangle+2\langle x|v\rangle+2\langle y|u\rangle \leq 0$

$\Leftrightarrow \forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \quad \langle x|v-u\rangle+\langle y|u-v\rangle \leq 0$

$\Leftrightarrow \forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \quad -\langle x|u-v\rangle+\langle y|u-v\rangle = \langle -(x-y)|(u-v)\rangle \leq 0$

$\Leftrightarrow \forall_{(x,u)\in \operatorname{gra} A} \ \forall_{(y,v)\in \operatorname{gra} A} \quad \langle x-y|u-v\rangle \leq 0$

$\Leftrightarrow$ A: monotone

■

Example 20.8.

$[ D: \neg\emptyset, \subseteq \mathcal{H}$

$T: D \to \mathcal{H}$

$A = \mathrm{Id} - T\ ]$

A: monotone $\Leftrightarrow$ T: pseudononexpansive

/* T: pseudononexpansive $\Leftrightarrow \forall_{x\in D} \ \forall_{y\in D} \ \|Tx-Ty\|^2 \leq \|x-y\|^2 + \|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2$ */

Proof: take T: pseudononexpansive $\Leftrightarrow$

$\forall_{x,y\in D}$

$$\|x-y\|^2+\underbrace{\|(2\mathrm{Id}-T)x-(2\mathrm{Id}-T)y\|^2}_{\|(x-y)-(Tx-Ty)\|^2=\|x-y\|^2-2\langle x-y|Tx-Ty\rangle+\|Tx-Ty\|^2} \geqslant \|Tx-Ty\|^2$$

$$\leftrightarrow\ 2\|x-y\|^2-2\langle x-y|Tx-Ty\rangle+\cancel{\|Tx-Ty\|^2} \geqslant \cancel{\|Tx-Ty\|^2}$$

$$\leftrightarrow\ \underbrace{\|x-y\|^2}_{\langle x-y|x-y\rangle}-\langle x-y|Tx-Ty\rangle \geqslant 0$$

$$\leftrightarrow\ \langle x-y|\underbrace{x-y-Tx+Ty}_{\underbrace{(2\mathrm{Id}-T)}_{A}x-\underbrace{(2\mathrm{Id}-T)}_{A}y}\rangle \geqslant 0 \ \leftrightarrow\ \langle x-y|Ax-Ay\rangle\geqslant 0$$

$$\therefore\ T:\text{nonexpansive} \leftrightarrow \forall_{x,y\in D}\ \langle x-y|Ax-Ay\rangle\geqslant 0 \overset{\text{def}}{\leftrightarrow} A:\text{monotone}.$$

▨

**Example 20.12. (Projection operator of any set is monotone)**

$[C:\text{nonempty},\subseteq \mathcal{H}]\ \Pi_C:\text{monotone}$

**Proof:**

$\forall_{(x,u),(y,v)\in \operatorname{gra}\Pi_C}$ //recall : $\Pi_C(x)=\{p\in C \mid \|x-p\|=d_C(x)=\inf_{y\in C}\|x-y\|\}$

$\updownarrow$ // $\therefore\ p\in\Pi_C(x)\leftrightarrow\|x-p\|=d_C(x)$

$$\left[\begin{array}{l}\Pi_C x\ni u \leftrightarrow \|x-u\|=d_C(x)=\inf_{\tilde u\in C}\|x-\tilde u\|\leqslant\|x-\tilde u\|\ \forall_{\tilde u\in C}\leqslant\|x-v\| \quad \text{//setting } \tilde u:=v\\ \\ \Pi_C y\ni v \leftrightarrow \|y-v\|=d_C(y)=\inf_{\tilde v\in C}\|y-\tilde v\|\leqslant\|y-\tilde v\|\ \forall_{\tilde v\in C}\leqslant\|y-u\| \quad \text{//setting } \tilde v:=u\end{array}\right.$$

$$\Rightarrow\ \underbrace{\|x-u\|^2}_{\leqslant\|x-v\|}+\underbrace{\|y-v\|^2}_{\leqslant\|y-u\|}\leqslant\|x-v\|^2+\|y-u\|^2$$

$$\leftrightarrow\ \cancel{\|x\|^2}+\cancel{\|u\|^2}-2\langle x|u\rangle+\cancel{\|y\|^2}+\cancel{\|v\|^2}-2\langle y|v\rangle\leqslant\cancel{\|x\|^2}+\cancel{\|v\|^2}-2\langle x|v\rangle+\cancel{\|y\|^2}+\cancel{\|u\|^2}-2\langle y|u\rangle$$

$$\leftrightarrow\ -\langle x|u\rangle-\langle y|v\rangle\leqslant-\langle x|v\rangle-\langle y|u\rangle$$

$$\leftrightarrow\ \langle x|u\rangle+\langle y|v\rangle\geqslant\langle x|v\rangle+\langle y|u\rangle$$

$$\leftrightarrow\ \langle x|u-v\rangle-\langle y|u-v\rangle\geqslant 0$$

$$\leftrightarrow\ \langle x-y|u-v\rangle\geqslant 0$$

so, $\forall_{(x,u),(y,v)\in\operatorname{gra}\Pi_C}\ \langle x-y|u-v\rangle\geqslant 0 \leftrightarrow \Pi_C:\text{monotone}.$

▨

# Part 2

9:07 AM

Theorem 20.21. // Any monotone operator has a //maximally monotone extension.

$[A: \mathcal{H}\to 2^{\mathcal{H}}$, monotone$]$

$\exists_{\tilde{A}:\mathcal{H}\to 2^{\mathcal{H}},\ \text{maximally monotone}}$ $\operatorname{gra} A \subseteq \operatorname{gra}\tilde{A}$

Proof:

(case 1)

$\operatorname{gra} A \neq \emptyset$

$\mathcal{M}=\{B:\mathcal{H}\to 2^{\mathcal{H}}$, monotone, $\operatorname{gra} A\subseteq \operatorname{gra} B\}$ : the set of all monotone operators containing $A$

$\mathcal{M}$: nonempty (contains $A$)

$\mathcal{M}$: partially ordered /* notation. $\forall_{B_1,B_2\subseteq \mathcal{M}}$ $\operatorname{gra} B_1\subseteq \operatorname{gra} B_2 \leftrightarrow B_1\preccurlyeq B_2$ */

verified the definition in 1,2,3

$\mathcal{C}$: $\underbrace{\text{chain}}$ in $\mathcal{M}$ // $\mathcal{C}\subseteq \mathcal{M}$

(totally ordered subset of a partially ordered set)

$\Rightarrow \forall_{B_1,B_2\in\mathcal{C}}$ $\operatorname{gra} B_1\subseteq \operatorname{gra} B_2$ or $\operatorname{gra} B_2\subseteq \operatorname{gra} B_1$

So, $\underbrace{\bigcup_{C\in\mathcal{C}} \operatorname{gra} C}_{\operatorname{gra} C_{\max,\mathcal{C}}} \supseteq \operatorname{gra} B \ \forall B\in\mathcal{C}$ // $\because \mathcal{C}\subseteq \mathcal{M}$, and each member of $\mathcal{M}$ contains $\operatorname{gra} B$. so $\bigcup_{C\in\mathcal{C}} C$ must contain $B$.

/* Caution. $C_{\max}$ is not necessarily in

$\Rightarrow \forall B\in\mathcal{C}$ $B\preccurlyeq C_{\max,\mathcal{C}}$ // i.e., $\forall_{B\in\mathcal{C}}$ $\operatorname{gra} B\subseteq \operatorname{gra}\bigcup_{C\in\mathcal{C}} C = C_{\max,\mathcal{C}}$

So, for $\forall\,\mathcal{C}$: chain in $\mathcal{M}$ $\forall B\in\mathcal{C}$ $B\preccurlyeq C_{\max,\mathcal{C}}$

/* Zorns lemma: [$A$: partially ordered set] ($\forall\,\mathcal{C}$: chain in $A$ $\exists$ upper bound for $\mathcal{C}$) //if

$\Rightarrow$ ($A$: contains a maximal element $a$) // then

$\Updownarrow$

$\forall c\in A$ $a\preccurlyeq c \Rightarrow c=a$ */

So, using Zorn's lemma we have: $\mathcal{M}$ will contain a maximal element

$\exists\tilde{A}\in\mathcal{M}$ $\forall B\in\mathcal{M}$ $B\succcurlyeq\tilde{A}$

$\updownarrow$

$\operatorname{gra} B\supseteq \operatorname{gra}\tilde{A}$

$\Rightarrow \operatorname{gra}\tilde{A}=\operatorname{gra} A$ $[\because A\in\mathcal{M}]$

now by definition of $\mathcal{M}$, there cannot be any other $B\in\mathcal{M}$ containing $\tilde{A}$

So, $\tilde{A}$: maximally monotone.

(case 2)

$\operatorname{gra} A=\emptyset$

$\tilde{A}$: $\operatorname{gra} A=\{(0,0)\}$ is a maximally monotone extension of $A$.

$\blacksquare$

Proposition 20.31.

$[A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone, $x\in\mathcal{H}]$

$Ax$: closed, convex

Proof:

case 1: $x\notin \operatorname{dom} A \Rightarrow$ $Ax=\emptyset$ : closed, convex and noninteresting

case 2: $x\in\operatorname{dom} A\Rightarrow$ $Ax\neq\emptyset=\{\dots,u,\dots\}$ $\Rightarrow\exists_u$ $u\in Ax$ $\leftrightarrow (x,u)\in\operatorname{gra} A \leftrightarrow \forall_{(y,v)\in\operatorname{gra} A}$ $\langle x-y\mid u-v\rangle\geq 0$ : true for $u\in Ax$

/* $[A:\mathcal{H}\to 2^{\mathcal{H}}]$ $A$: maximally monotone $\leftrightarrow \forall_{(x,u)}\left((x,u)\in\operatorname{gra} A \Leftrightarrow \forall_{(y,v)\in\operatorname{gra} A} \langle x-y\mid u-v\rangle\geq 0\right)$ */

So, $\forall_u\left(\overset{(x,u)\in\operatorname{gra} A}{u\in Ax} \Leftrightarrow \forall_{(y,v)\in\operatorname{gra} A} \langle x-y\mid u-v\rangle\geq 0\right)$

$\Rightarrow$ $Ax=\{u\in\mathcal{H}\mid \forall_{(y,v)\in\operatorname{gra} A}\langle x-y\mid u-v\rangle\geq 0\}$

$$=\underbrace{\bigcap_{(y,v)\in\operatorname{gra} A}\{u\in\mathcal{H}\mid \underbrace{\langle x-y\mid u-v\rangle\geq 0}_{\text{closed, convex set in } u}\}}_{\text{intersection of a bunch of closed, convex sets} \Rightarrow \text{convex, closed}}$$

$\therefore$ $Ax$: closed, convex $\blacksquare$

* Proposition 20.32.

$[A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone operator

$(x_b,u_b)_{b\in B}$: bounded net in $\operatorname{gra} A$

$(x,u)\in\mathcal{H}\times\mathcal{H}]$

(i) $(x_b\to x, u_b\rightharpoonup u)\Rightarrow (x,u)\in\operatorname{gra} A$

(ii) $(x_b\rightharpoonup x, u_b\to u)\Rightarrow (x,u)\in\operatorname{gra} A$

Proof:

$\forall_{(y,v)\in\operatorname{gra} A}$

$\leftrightarrow Ay\ni v$

/* $A$: maximally monotone $\leftrightarrow \forall_{(x,u)}\left((x,u)\in\operatorname{gra} A\leftrightarrow \forall_{(y,v)\in\operatorname{gra} A} \langle x-y\mid u-v\rangle\geq 0\right)$ ...(0)

$\because$ $(x_b,u_b)_{b\in B}\subseteq\operatorname{gra} A$

↳index set of the net B

$\forall_{b\in B}$ $\langle x_b-y\mid u_b-v\rangle\geq 0$ .. (i)

/* Lemma 2.36: $[(x_a)_{a\in A}$ : net in $\mathcal{H}$, bounded, $x_a\to\overset{\in\mathcal{H}}{x}$.

$(u_a)_{a\in A}$ : net in $\mathcal{H}$, $u_a\rightharpoonup \underset{\in\mathcal{H}}{u}]$ $\Rightarrow$ $\langle x_a\mid u_a\rangle\to\langle x\mid u\rangle$ */

using this in our case $\because$ $(x_b)_{b\in B},(u_b)_{b\in B}$ : bounded, $x_b\to x$, $u_b\rightharpoonup u$ $\Rightarrow$ $x_b-y\to x-y$, $u_b-v\rightharpoonup u-v$

$\overset{\text{Theorem 2.36}}{\Rightarrow}$ $\langle x_b-y\mid u_b-v\rangle\to\langle x-y\mid u-v\rangle$

$\overset{\text{from (i)}}{\Rightarrow}$ $\langle x-y\mid u-v\rangle\geq 0$

$\Rightarrow$ $(x,u)\in\operatorname{gra} A$

$\blacksquare$

(ii) Apply (i) to $A^{-1}$

$\blacksquare$

* Proposition 20.33.

$[A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone$]$

(i) $\operatorname{gra} A$: sequentially closed in $\mathcal{H}^{\text{strong}}\times\mathcal{H}^{\text{weak}}$

$\leftrightarrow$ $\forall_{(x_n,u_n)_n\subseteq\operatorname{gra} A,\ (x,u)\in\mathcal{H}\times\mathcal{H}\ :\ x_n\to x,\ u_n\rightharpoonup u}$ $(x,u)\in\operatorname{gra} A$

(ii) $\operatorname{gra} A$: sequentially closed in $\mathcal{H}^{\text{weak}}\times\mathcal{H}^{\text{strong}}$

$\leftrightarrow$ $\forall_{(x_n,u_n)_n\subseteq\operatorname{gra} A,\ (x,u)\in\mathcal{H}\times\mathcal{H}\ :\ x_n\rightharpoonup x,\ u_n\to u}$ $(x,u)\in\operatorname{gra} A$

(iii) $\operatorname{gra} A$: closed in $\mathcal{H}^{\text{strong}}\times\mathcal{H}^{\text{strong}}$

Proof: (i)

Lemma 2.38 : $[(x_n)_{n\in\mathbb{N}}\subseteq\mathcal{H}]$ $x_n\rightharpoonup\square$ $\leftrightarrow$ $(x_n)_{n\in\mathbb{N}}$ : bounded, has almost one weak sequential cluster point

/* (20.32 (i))

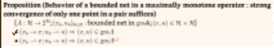
Proposition (Behavior of a bounded net in a maximally monotone operator : strong

(iii) gra A: closed in $\mathcal{H}$ × $\mathcal{H}$

Proof: (i)

Lemma 2.38 : $[(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}]$ $x_n \rightharpoonup \square \Leftrightarrow (x_n)_{n\in\mathbb{N}}$ : bounded, has almost one weak sequential cluster point

/*
(20.32 (i))



**Proposition (Behavior of a bounded net in a maximally monotone operator : strong convergence of only one point in a pair suffices)**

*/

given: $(x_n,u_n) \in \text{gra}\,A$, $(x,u) \in \mathcal{H}\times\mathcal{H}$ : $(x_n,u_n) \rightharpoonup (x,u)$

/* $\left.\begin{matrix} x_n \to x \Rightarrow x_n \rightharpoonup x \\ u_n \rightharpoonup u \end{matrix}\right\} \Rightarrow (x_n,u_n) \rightharpoonup (x,u)$ */

So Lemma 2.38 says $(x_n,u_n)_{n\in\mathbb{N}}$ : bounded net, has almost one weak sequential cluster point

now Proposition 20.32 (i) can be applied:

$(x,u) \in \text{gra}\,A$

(ii) $A^{-1}$ : maximally monotone

// $A$: maximally monotone ⇒ $A^{-1}$ : maximally monotone

now apply (i)'s logic to $A^{-1}$

(iii) follows trivially.

$\square$

Proposition 20.35.

[$A: \mathcal{H} \to 2^{\mathcal{H}}$, maximally monotone, almost single-valued

dom A : linear subspace /* $\forall_{\alpha,\beta\in\mathbb{R}} \forall_{x,y\in \text{dom}\,A \Leftrightarrow Ax,\, Ay \neq \emptyset}$ $\alpha x+\beta y \in \text{dom}\,A$ */

$A|_{\text{dom}\,A}$ : linear /* $\forall_{\alpha,\beta\in\mathbb{R}} \forall_{x,y\in\text{dom}\,A}$ $A(\alpha x+\beta y) = \alpha Ax + \beta Ay$ */

$\forall_{x\in\text{dom}A} \forall_{y\in\text{dom}A}$ $\langle x|Ay\rangle = \langle Ax|y\rangle$ /* on Eucledian space it means A: symmetric matrix */

$h:\mathcal{H}\to\,]-\infty,+\infty]: x \mapsto \begin{cases} \frac{1}{2}\langle x|Ax\rangle, & \text{if } x\in\text{dom}A \\ +\infty \end{cases}$

// Another way of saying this is: $h(x) = \frac{1}{2}\langle x \mid Ax\rangle + \iota_{\text{dom}A}$

$f:\mathcal{H}\to\,]-\infty,+\infty] . x \mapsto \sup_{y\in\text{dom}A} (\langle x|Ay\rangle - h(y))$ ]

⇒ (i) $f + \iota_{\text{dom}A} = h$

(ii) $f \in \Gamma_0(\mathcal{H})$

(iii) $\partial f = A$

(iv) $f = h^{**}$

Proof: dom A = dom h /* Caution: the first one is the domain of an operator $\text{dom}\,A = \{x\in\mathcal{H}: Ax \neq \emptyset\}$ }However definitionwise

Take $x \in \text{dom}A = \text{dom}\,h$ second " " " " " function $\text{dom}\,h = \{x\in\mathcal{H}: h(x) < +\infty\}$ }they are the same */

(i) $\forall_{y\in\text{dom}A}$ $0 \leq \langle x-y|Ax-Ay\rangle$ // A: maximally monotone we have just used the definition here

$= \langle x|Ax\rangle + \langle y|Ay\rangle - \underbrace{\langle y|Ax\rangle}_{\parallel} - \langle x|Ay\rangle$

// $\langle x|Ay\rangle$ as $x,y\in\text{dom}A$ by given

$= 2h(x) + 2h(y) - 2\langle x|Ay\rangle$ /* using definition of A */

$\Rightarrow \forall_{\substack{x\in\text{dom}A \\ =\text{dom}h}} \forall_{\substack{y\in\text{dom}A\\=\text{dom}\,h}}$ $0 \leq h(x) + h(y) - \langle x|Ay\rangle$

⇔ " " $\langle x|Ay\rangle - h(y) \leq h(x)$

$\Leftrightarrow \forall_{x\in\text{dom}A}$ $\underbrace{\sup_{y\in\text{dom}A} \langle x|Ay\rangle - h(y)}_{f(x)\ //\text{ by definition}} \leq h(x)$

$\Leftrightarrow \forall_{x\in\text{dom}A=\text{dom}\,h}$ $f(x) \leq h(x)$

$\Leftrightarrow \forall_{x\in\text{dom}A}$ $f(x) + \iota_{\text{dom}A}(x) \leq h(x)$ /* as $\forall x\in\text{dom}A$ $\iota_{\text{dom}A}(x) = 0$ */

note that even $\forall_{x\notin\text{dom}A}$ $f(x) + \underbrace{\iota_{\text{dom}A}(x)}_{=+\infty} = +\infty = h(x)$ // ∵ dom A = dom h

so, $\forall_{x\in\mathcal{H}}$ $f(x) + \iota_{\text{dom}A}(x) \leq h(x)$ . . . (i)

again by definition,

$\forall_{\substack{x\in\text{dom}A \\ \subseteq\mathcal{H}}}$ $f(x) = \sup_{y\in\text{dom}A} \left(\langle x|Ay\rangle - h(y)\right) \geq \langle x|Ay\rangle - h(y)$ $\forall_{y\in\text{dom}\,A}$

set $y := x \in \text{dom}A$

$\Rightarrow$ $f(x) \geq \underbrace{\langle x|Ax\rangle}_{2h(x)} - h(x) = h(x)$ // ∵ $x\in\text{dom}A$, using the definition

$\Rightarrow \forall_{x\in\text{dom}A=\text{dom}\,h}$ $f(x) - h(x) \geq 0$

$\Rightarrow \inf_{x\in\text{dom}A} (f(x) - h(x)) \geq 0$

$\Leftrightarrow \inf_{x\in\mathcal{H}} (f(x) - h(x) + \iota_{\text{dom}A}) \geq 0$ // just using the indicator function trick

$\Leftrightarrow \forall_{x\in\mathcal{H}}$ $f(x) - h(x) + \iota_{\text{dom}A} \geq 0$

$\Leftrightarrow \forall_{x\in\mathcal{H}}$ $f(x) + \iota_{\text{dom}A}(x) \geq h(x)$ . . . (ii)

From (i) and (ii) :

$\forall_{x\in\mathcal{H}}$ $f(x) + \iota_{\text{dom}A}(x) = h(x)$

$\Leftrightarrow \boxed{f + \iota_{\text{dom}A} = h}$ ■

(ii)

$\forall_{x\in\mathcal{H}}$ $f(x) = \sup_{y\in\text{dom}A} (\underbrace{\langle x|Ay\rangle - h(y)}$)

$\underbrace{\text{continuous affine}}$ function over $x$ (the variable) with parameter $y$

$\underbrace{\Rightarrow \text{lower semicontinuous convex} \in \Gamma(\mathcal{H})}$

sup over the parameter $y \in \text{dom}A \Rightarrow$ so $f(\cdot)$ : convex // $\sup(\cdot) = -\inf(-\,\cdot)$

by definition, $f:\mathcal{H}\to\,]-\infty,+\infty] \Rightarrow -\infty \notin f(\mathcal{H})$, also $\text{dom}\,f \neq \emptyset$ as $f(0) = \sup_{y\in\text{dom}A} (-h(y)) = -\inf_{\substack{y\in\text{dom}A \\ =\text{dom}\,h}} -(-h(y)) = -\inf_{y\in\text{dom}h} \overset{-\frac{1}{2}\langle y|Ay\rangle}{\overset{\parallel}{h(y)}} = -\frac{1}{2} \underbrace{\inf_{y\in\text{dom}A} \langle y|Ay\rangle}$ =0 /* Because A: maximally monotone, $A|_{\text{dom}A}$ : linear ⇒ A: positive semidefinite on domA

$\Downarrow$ =0 comes from: Example 20.15. A: monotone, linear on $S \Leftrightarrow \forall_{x\in S} \langle Ax|x\rangle \geq 0$

$f$ : proper so, $y = 0$ will yield the minimum value of 0 */

∴ $f$ : proper, $\in \Gamma(\mathcal{H}) \Leftrightarrow f \in \Gamma_0(\mathcal{H})$ ■

(iii) $\forall_{y\in\mathcal{H}}$ $f(x) + \langle y-x|Ax\rangle$ // recall that we have taken $x\in\text{dom}A$

// using (i)

$= h(x) - \underbrace{\iota_{\text{dom}A}(x)}_{0} + \langle y-x|Ax\rangle$ // $x\in\text{dom}A \Rightarrow \iota_{\text{dom}A}(x) = 0$

$= \underbrace{h(x)}_{\frac{1}{2}\langle x|Ax\rangle} + \underbrace{\langle y-x|Ax\rangle}_{\langle y|Ax\rangle - \langle x|Ax\rangle} = \langle y|Ax\rangle - \overbrace{\frac{1}{2}\langle x|Ax\rangle}^{h(x)} = \langle y|Ax\rangle - h(x) \leq \underbrace{\sup_{\tilde{x}\in\text{dom}A} (\langle y|A\tilde{x}\rangle - h(\tilde{x}))}_{\overset{\parallel}{f(y)}\ /*\text{ by definition }*/} = f(y)$

/* $\forall x\in\text{dom}\,A$ $h(x) = \frac{1}{2}\langle x|Ax\rangle$ */

so, we have:

$\forall_{y\in\mathcal{H}}$ $f(y) \geq f(x) + \langle Ax|y-x\rangle \Leftrightarrow Ax \in \partial f(x) \Leftrightarrow (x, Ax) \in \text{gra}\,\partial f$

/* recall that $u \in \partial f(x) \Leftrightarrow (\forall y\in\mathcal{H})$ $f(y) \geq f(x) + \langle u|y-x\rangle$ */

/* $\forall x\in \text{dom} A \quad h(x)=\frac{1}{2}\langle x|Ax\rangle$ */ $\qquad\qquad$ $\|$ $f(y)$ /* by definition */

So, we have:

$\forall_{y\in\mathcal{H}} \quad f(y)\geq f(x)+\langle Ax|y-x\rangle \Leftrightarrow Ax\in\partial f(x) \Leftrightarrow (x,Ax)\in \text{gra}\,\partial f$

/* recall that $g\in\partial f(x) \Leftrightarrow (\forall y\in\mathcal{H}) \quad f(y)\geq f(x)+\langle g|y-x\rangle$ */

Now, $\text{gra } A=\{(x,Ax) \mid x\in \text{dom} A\}$ /* for $x\notin \text{dom} A \quad (x,Ax)=(x,\emptyset)$
which is a trivial member of all set.
so we do not need to consider that */

So, $\forall_{(x,Ax)\in \text{gra} A} \quad (x,Ax)\in \text{gra}\,\partial f$
/* $\partial f$: monotone for any proper function */

$\Leftrightarrow \quad \text{gra} A\subseteq \text{gra}\,\partial f \quad \ldots (i)$

now A: maximally monotone

$\Leftrightarrow \neg\left(\exists_{B:\mathcal{H}\to 2^{\mathcal{H}},\text{monotone}} \text{gra} A\subsetneq \text{gra} B\right) \quad \ldots (ii)$

From (i) and (ii) the only possibility is:

$\text{gra} A=\text{gra}\,\partial f$

$\Leftrightarrow \quad A=\partial f$

(iv) from $\underbrace{\text{(ii)}}_{\text{proof}}$ we have proven : $f\in\Gamma_0(\mathcal{H}) \Rightarrow f=\left(\overbrace{f+\iota_{\text{dom}\,\partial f}}^{\underset{\vee}{h \text{ (from (i))}}}\right)^{**}=h^{*}$

/* from corollary 16.31. $f\in\Gamma_0(\mathcal{H}) \Rightarrow f=(f+\iota_{\text{dom}\,\partial f})^{**}$ */ ▨

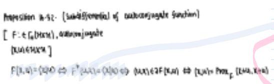
*Corollary 20.39.

$[F:\in\Gamma_0(\mathcal{H}\times\mathcal{H})$, autoconjugate

$A: \text{gra} A=\{(x,u)\in\mathcal{H}\times\mathcal{H}: F(x,u)=\langle x|u\rangle\}$ /* $(x,u)\in A \Leftrightarrow Ax\ni u \Leftrightarrow F(x,u)=\langle x|u\rangle$ */

A: maximally monotone

/* Proposition 13.31. $[F:\in\Gamma(\mathcal{H}\times\mathcal{H})$, autoconjugate$]$ $\forall (x,u)\in\mathcal{H}\times\mathcal{H} \quad F(x,u)\geq\langle x|u\rangle,\ F^{*}(u,x)\geq\langle x|u\rangle$

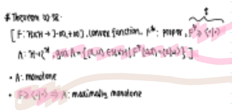


Proposition 16.52. [subdifferential of autoconjugate function]

[ $F:\in\Gamma_0(\mathcal{H}\times\mathcal{H})$, autoconjugate

$(x,u)\in\mathcal{H}\times\mathcal{H}$ ]

$F(x,u)=\langle x|u\rangle \Leftrightarrow F^{*}(u,x)=\langle x|u\rangle \Leftrightarrow (u,x)\in\partial F(x,u) \Leftrightarrow (x,u)=\text{Prox}_F(x+u,x+u)$



# Theorem 20.38.

[ $F:\mathcal{H}\times\mathcal{H}\to\,]-\infty,+\infty]$, convex function, $F^{*}$: proper, $\overbrace{F^{*}\geq\langle\cdot|\cdot\rangle}^{\ddagger}$

$A:\mathcal{H}\to 2^{\mathcal{H}}$, $\text{gra} A=\{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F^{*}(u,x)=\langle x|u\rangle\}$ ]

- A: monotone
- $F\geq\langle\cdot|\cdot\rangle \Rightarrow$ A: maximally monotone

Proof: // Proof blueprint: Direct application of Theorem 16.52

$(x,u)\in \text{gra} A \Leftrightarrow F(x,u)=\langle x|u\rangle$

$\Leftrightarrow F^{*}(u,x)=\langle x|u\rangle$

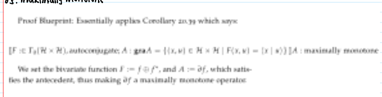
$\left.\begin{array}{l}\text{gra} A=\{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F^{*}(u,x)=\langle x|u\rangle\} \\ F:\text{autoconjugate} \Rightarrow F^{*}\geq\langle\cdot|\cdot\rangle,\ F\geq\langle\cdot|\cdot\rangle\end{array}\right\} \xRightarrow{\text{Th } 20.38}$ A: maximally monotone.

▨

*Theorem 20.40.

$[f\in\Gamma_0(\mathcal{H})]$

$\partial f$: maximally monotone



Proof Blueprint: Essentially applies Corollary 20.39 which says:

$[F :\in \Gamma_0(\mathcal{H}\times\mathcal{H})$, autoconjugate; $A : \text{gra}A = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F(x,u)=\langle x \mid u\rangle\}]\,A$ : maximally monotone

We set the bivariate function $F := f\oplus f^{*}$, and $A := \partial f$, which satisfies the antecedent, thus making $\partial f$ a maximally monotone operator.

Proof: $f\in\Gamma_0(\mathcal{H}) \Rightarrow f^{*}\in\Gamma_0(\mathcal{H})$ /* using Fenchel-Moreau corollary */

$\underbrace{(f\oplus f^{*})}: (x,y)\mapsto f(x)+f^{*}(y)$ /* $\oplus$: Hilbert direct sum */

$\in\Gamma_0(\mathcal{H}\times\mathcal{H})$ /* Proposition 8.25. convexity is preserved under Hilbert direct sum */ $\ldots(0)$

$(f\oplus f^{*})$ : autoconjugate ..(1) /* F: autoconjugate $\Leftrightarrow F^{*}=F^{T}$ where $F^{T}(u,x)=F(x,u)$ */

/* $\overset{\text{proof}}{(f\oplus f^{*})}(x,y)=f(x)+f^{*}(y)$ /* recall: Hilbert direct sum: $(f_1\oplus f_2)(x,y)=f_1(x)+f_2(y)$ */

$(f\oplus f^{*})^{T}(x,y)=(f\oplus f^{*})(y,x)=f(y)+f^{*}(x)$

$(f\oplus f^{*})^{*}(x,y)=(f^{*}\oplus f^{**})(x,y)$ /* proposition 13.27. $\Big(\bigoplus_{\underset{\hookrightarrow \text{finite}}{i\in I}} \overset{f_i:\mathcal{H}\to]-\infty,+\infty]}{f_i}\Big)^{*}=\bigoplus_{i\in I} f_i^{*}$ */

$=f^{*}(x)+\underbrace{f^{**}(y)}_{f(y)}$ /* Fenchel-Moreau corollary: $f\in\Gamma_0(\mathcal{H})\Rightarrow f^{**}=f$ */

$=f^{*}(x)+f(y)$

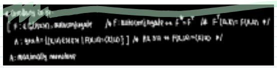
$\therefore (f\oplus f^{*})^{T}(x,y)=(f\oplus f^{*})^{*}(x,y) \Leftrightarrow (f\oplus f^{*})$ : autoconjugate */

/* Proposition 16.9. $\qquad\qquad$ this is a beautifully symmetric result.

$[f:\mathcal{H}\to]-\infty,+\infty]$, proper$]$ $\overset{(x,u)\in\text{gra}\,\partial f}{\overset{\updownarrow}{u\in\partial f(x)}} \Leftrightarrow x\in\partial f^{*}(u) \Leftrightarrow \underbrace{f(x)+f^{*}(u)}_{(f\oplus f^{*})(x,u)}=\langle x|u\rangle$ */

So, $\text{gra}\,\partial f=\{(x,u)\in\mathcal{H}\times\mathcal{H} \mid (f\oplus f^{*})(x,u)=\langle x|u\rangle\} \ldots(2)$

(0), (1), $\Rightarrow$



*Corollary 20.39.

[ $F:\in\Gamma_0(\mathcal{H}\times\mathcal{H})$, autoconjugate /* F: autoconjugate $\Leftrightarrow F^{*}=F^{T}$ /* $F^{T}(u,x)=F(x,u)$ */

$A: \text{gra}A=\{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F(x,u)=\langle x|u\rangle\}$ ] /* $Ax\ni u \Leftrightarrow F(x,u)=\langle x|u\rangle$ */

A: maximally monotone

$\left.\begin{array}{l}\bullet (f\oplus f^{*}): \text{autoconjugate}, \in\Gamma_0(\mathcal{H}\times\mathcal{H}) \\ (2)\end{array}\right\} \Rightarrow \partial f$ : maximally monotone /

▨

Example: 20.41

[C: nonempty closed convex subset of $\mathcal{H}$]

$N_C$ : maximally monotone

Proof:

$f\in\Gamma_0(\mathcal{H}) \Rightarrow \partial f$ : maximally monotone (Theorem 20.40)

$\left.\begin{array}{l}\text{set } f=\iota_C\in\Gamma_0(\mathcal{H}) \\ \therefore \partial f=N_C \text{ /* C: nonempty}\end{array}\right\}$

$N_C$: maximally monotone.

▨

* Example 20.45.

$[A: \text{monotone}, \in\mathcal{B}(\mathcal{H});\ q_A:\mathcal{H}\to\mathbb{R}: x\mapsto\frac{1}{2}\langle x|Ax\rangle]$

$\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \quad F_A(x,u)=2q_A^{*}\left(\frac{1}{2}u+\frac{1}{2}A^{*}x\right)$

**Proof Blueprint:** Uses definition of conjugate function, adjoint operator. The rest is the direct manipulation of the definition of the Fitzpatrick function.

Proof: $\forall (x,u)\in\mathcal{H}\times\mathcal{H}$

$F_A(x,u)=\sup_{\overset{(y,v)\in\text{gra}A}{\overset{\updownarrow}{Ay=v}}} \left(\langle y|u\rangle+\langle x|v\rangle-\langle y|v\rangle\right)$ /* $A\in\mathcal{B}(\mathcal{H}) \Leftrightarrow Ax$ : single valued from $\mathcal{H}$ to $\mathcal{H}$. */

$=\sup_{(y,v):Ay=v} \left(\langle y|u\rangle+\langle x|Ay\rangle-\langle y|Ay\rangle\right)$

$=\sup_{y\in\mathcal{H}} \left(\langle y|u\rangle+\langle x|Ay\rangle-\langle y|Ay\rangle\right)$

/* As y is the only independent variable */

$=2\sup_{y\in\mathcal{H}} \Big(\underbrace{\frac{1}{2}\langle y|u\rangle+\frac{1}{2}\underbrace{\langle x|Ay\rangle}_{\substack{\langle A^{*}x|y\rangle \\ =\langle y|A^{*}x\rangle}}}_{\frac{1}{2}\langle y|(u+A^{*}x)\rangle}-\frac{1}{2}\langle y|Ay\rangle\Big)$ /* Adjoint of a linear bounded operator $\langle x|Ay\rangle=\langle A^{*}x|y\rangle$ */

$=2\sup_{y\in\mathcal{H}} \left(\underbrace{\frac{1}{2}\langle y|u+A^{*}x\rangle}-\underbrace{\frac{1}{2}\langle y|Ay\rangle}\right)$

/* recall the definition of conjugate function:

$f^{*}(u)=\sup\left(-f(x)+\langle x|u\rangle\right)$

$\langle A'x|y\rangle$ /* Adjoint of a linear bounded operator
$= \langle y|A^*x\rangle$ $\quad \langle x|Ay\rangle = \langle A^*x|y\rangle$ */

$\underbrace{\tfrac{1}{2}\langle y|(u+A^*x)\rangle}$

$$= 2 \sup_{y\in\mathcal{H}} \Big( \underbrace{\tfrac{1}{2}\langle y|u+A^*x\rangle}_{\langle y|\frac{1}{2}(u+A^*x)\rangle} - \underbrace{\tfrac{1}{2}\langle y|Ay\rangle}_{q_A(y)} \Big)$$

/* recall the definition of conjugate function:

$$f^*(u) = \sup_{x\in\mathcal{H}} \big(-f(x) + \langle x|u\rangle\big)$$

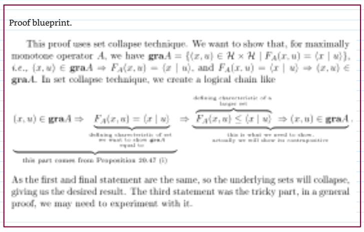
now $q_A(x) = \tfrac{1}{2}\langle x|Ax\rangle$

$$q_A^*(u) = \sup_{x\in\mathcal{H}} \big(-\tfrac{1}{2}\langle x|Ax\rangle + \langle x|u\rangle\big)$$

$$= 2 \sup_{y\in\mathcal{H}} \Big(-q_A(y) + \langle y|\tfrac{1}{2}(u+A^*x)\rangle\Big)$$

$$= 2\, q_A^*\big(\tfrac{1}{2}u + \tfrac{1}{2}A^*x\big)$$

■



Proof blueprint.

This proof uses set collapse technique. We want to show that, for maximally monotone operator $A$, we have $\mathbf{gra}A = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_A(x,u) = \langle x \mid u\rangle\}$, i.e., $(x,u)\in\mathbf{gra}A \Rightarrow F_A(x,u) = \langle x \mid u\rangle$, and $F_A(x,u) = \langle x \mid u\rangle \Rightarrow (x,u)\in\mathbf{gra}A$. In set collapse technique, we create a logical chain like

$$\underbrace{(x,u)\in\mathbf{gra}A \Rightarrow \underbrace{F_A(x,u) = \langle x \mid u\rangle}_{\substack{\text{defining characteristic of set}\\ \text{we want to show } \mathbf{gra}A\\ \text{equal to}}}}_{\text{this part comes from Proposition 20.47 (i)}} \Rightarrow \underbrace{\overbrace{F_A(x,u) \leq \langle x \mid u\rangle}^{\substack{\text{defining characteristic of a}\\ \text{larger set}}} \Rightarrow (x,u)\in\mathbf{gra}A}_{\substack{\text{this is what we need to show,}\\ \text{actually we will show its contrapositive}}}.$$

As the first and final statement are the same, so the underlying sets will collapse, giving us the desired result. The third statement was the tricky part, in a general proof, we may need to experiment with it.

* Proposition 20.48. (Fitzpatrick function of maximally monotone operators)

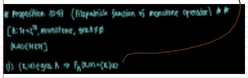
$[A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone operator$]$

- $F_A \geq \langle\cdot|\cdot\rangle$
- $\text{gra}\,A = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_A(x,u) = \langle x|u\rangle\}$

Proof:

$(x,u)\in\mathcal{H}\times\mathcal{H}$

$(x,u)\in\text{gra}\,A \Rightarrow F_A(x,u) = \langle x|u\rangle$ (1)



* Proposition 20.47 (Fitzpatrick function of monotone operator) * *
$[A:\mathcal{H}\to 2^{\mathcal{H}}$, monotone, $\text{gra}A\neq\emptyset]$
$[(x,u)\in\mathcal{H}\times\mathcal{H}]$
(i) $(x,u)\in\text{gra}\,A \Rightarrow F_A(x,u) = \langle x|u\rangle$

now if $(x,u)\notin\text{gra}\,A \Rightarrow \{(x,u)\}\cup\text{gra}\,A$ : not monotone (2)

/* $A$ : maximal monotone $\overset{\text{def}}{\Leftrightarrow} \forall(\tilde{x},\tilde{u})\ \big((\tilde{x},\tilde{u})\in\text{gra}\,A \Leftrightarrow \forall(y,v)\in\text{gra}\,A\ \langle\tilde{x}-y|\tilde{u}-v\rangle \geq 0\big)$

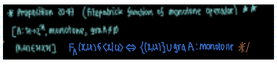
so, $(x,u)\notin\text{gra}\,A \Leftrightarrow \exists(y,v)\in\text{gra}\,A\ \langle x-y|u-v\rangle < 0$

so if $\text{gra}\,B = \{(x,u)\}\cup\text{gra}\,A$ then

$(x,u)\in\text{gra}B$, $(y,v)\in\text{gra}A\subseteq\text{gra}B$ but $\langle x-y|u-v\rangle<0$

so, $\text{gra}\,B = \{(x,u)\}\cup\text{gra}\,A$ : not monotone */

From, (2) and proposition 20.47 (iii) /*



* Proposition 20.47 (Fitzpatrick function of monotone operator) * *
$[A:\mathcal{H}\to 2^{\mathcal{H}}$, monotone, $\text{gra}A\neq\emptyset]$
$[(x,u)\in\mathcal{H}\times\mathcal{H}]$ $F_A(x,u)\leq\langle x|u\rangle \Leftrightarrow \{(x,u)\}\cup\text{gra}\,A$ : monotone */

$F_A(x,u) > \langle x|u\rangle$

So, we have:

$(x,u)\in\text{gra}\,A \Rightarrow F_A(x,u) = \langle x|u\rangle$, . . . (2')

$(x,u)\notin\text{gra}\,A \Rightarrow F_A(x,u) > \langle x|u\rangle$ // this is the interesting part . . (3)

$S = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_A(x,u) = \langle x|u\rangle\} \subseteq \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_A(x,u)\leq\langle x|u\rangle\} = S'$ .. (4)

(2') says $\text{gra}\,A\subseteq S$ . . . (5)

contrapositive (3) : $F_A(x,u)\leq\langle x|u\rangle \Rightarrow (x,u)\in\text{gra}\,A$

$\Leftrightarrow S'\subseteq\text{gra}\,A$

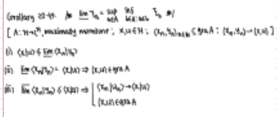
$\Rightarrow S\subseteq S'\subseteq\text{gra}\,A$ (from (4))

. . (6)

from (5) and (6):

$$\text{gra}\,A = S = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_A(x,u) = \langle x|u\rangle\}$$

■



Corollary 20.49. /* $\underline{\lim}\,\tau_n = \sup_{n\in\mathbb{N}} \inf_{k\in\mathbb{N}:k\geq n} \tau_k$ */
$[A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone; $x,u\in\mathcal{H}$; $(x_n,u_n)_{n\in\mathbb{N}}\subseteq\text{gra}A$; $(x_n,u_n)\to(x,u)]$
(i) $\langle x|u\rangle \leq \underline{\lim}\langle x_n|u_n\rangle$
(ii) $\underline{\lim}\langle x_n|u_n\rangle = \langle x|u\rangle \Rightarrow (x,u)\in\text{gra}A$
(iii) $\overline{\lim}\langle x_n|u_n\rangle \leq \langle x|u\rangle \Rightarrow \begin{cases}\langle x_n|u_n\rangle\to\langle x|u\rangle\\ (x,u)\in\text{gra}A\end{cases}$

Proof:

(i)

Proposition 20.47 (ii) : $[A:\overset{\text{maximally monotone}}{\mathcal{H}\to 2^{\mathcal{H}}}, \text{gra}\,A\neq\emptyset]\ F_A\in\underbrace{\Gamma_0(\mathcal{H}\times\mathcal{H})}^{\text{set of proper lower semicontinuous convex functions}} \Rightarrow F_A$: lower semicontinuous

Theorem 9.1: For a convex function all types of lower semicontinuity are equivalent. $] \Rightarrow F_A$: weakly sequentially lower semicontinuous

/* $f$: weakly sequentially lower semicontinuous $\leftrightarrow \forall x\in\mathcal{H}\ \forall(x_n)_{n\in\mathbb{N}}\subseteq\mathcal{H}: x_n\rightharpoonup x\ \ f(x)\leq\underline{\lim} f(x_n)$ */ . . (eq: 1)

Proposition 20.48.

$[A:\mathcal{H}\to 2^{\mathcal{H}}$, maximally monotone$]$ $\Big|$ $\forall(x,u)\ F_A(x,u)\geq\langle x|u\rangle$; $\text{gra}\,A = \{(x,u) \mid F_A(x,u) = \langle x|u\rangle\}$ . . . (4)

now,

$$\langle x|u\rangle \leq \overbrace{F_A(x,u) \leq \underline{\lim}\,F(x_n,y_n)}^{\text{// From (eq: 1)}} \quad \text{/* } (x_n,y_n)\to(x,u) \text{ */}$$

$$= \underline{\lim}\,\langle x_n|y_n\rangle \quad \text{/* } (x_n,y_n)\in\text{gra}\,A \Leftrightarrow F_A(x_n,y_n) = \langle x_n|y_n\rangle \text{ */}$$

. . . (3) ☺

(ii) given: $\langle x|u\rangle = \underline{\lim}\,\langle x_n|y_n\rangle$, so the inequality in (3) collapses. and we have:

$F_A(x,u) = \langle x|u\rangle \Leftrightarrow (x,u)\in\text{gra}\,A$ /* (4) */

(iii) given: $\overline{\lim}\,\langle x_n|y_n\rangle \leq \langle x|u\rangle$ . . . (5)

(3), (5) $\Rightarrow$

$$\overline{\lim}\,\langle x_n|y_n\rangle \leq \langle x|u\rangle \leq \underbrace{\underline{\lim}\,\langle x_n,y_n\rangle \leq \overline{\lim}\,\langle x_n,y_n\rangle}_{\text{by defn}} \Rightarrow \text{collapse}$$

$$\Rightarrow \overline{\lim}\,\langle x_n|y_n\rangle = \underline{\lim}\,\langle x_n|y_n\rangle = \underbrace{\lim\langle x_n|y_n\rangle = \langle x|u\rangle} \Rightarrow (x,u)\in\text{gra}\,A\ ✓\ \text{[From (ii)]}$$

$\leftrightarrow \langle x_n|y_n\rangle \to \langle x|u\rangle$ ✓ ■

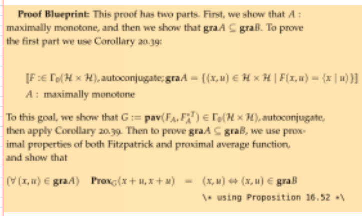
Theorem 20.53. (Extending a monotone operator to maximally monotone operator */

$[A:\mathcal{H}\to 2^{\mathcal{H}}$, monotone, $\text{gra}\,A\neq\emptyset$

$G = \text{pav}(F_A, F_A^{*\intercal})$

$B:\mathcal{H}\to 2^{\mathcal{H}}$, $\text{gra}\,B = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid G(x,u) = \langle x|u\rangle\}]$

$B$: maximally monotone extension of $A$



**Proof Blueprint:** This proof has two parts. First, we show that $A$ : maximally monotone, and then we show that $\mathbf{gra}A \subseteq \mathbf{gra}B$. To prove the first part we use Corollary 20.39:

$[F :\in \Gamma_0(\mathcal{H}\times\mathcal{H})$, autoconjugate; $\mathbf{gra}A = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F(x,u) = \langle x \mid u\rangle\}]$
$A$ : maximally monotone

To this goal, we show that $G := \mathbf{pav}(F_A, F_A^{*\intercal}) \in \Gamma_0(\mathcal{H}\times\mathcal{H})$, autoconjugate, then apply Corollary 20.39. Then to prove $\mathbf{gra}A \subseteq \mathbf{gra}B$, we use proximal properties of both Fitzpatrick and proximal average function, and show that

$(\forall(x,u)\in\mathbf{gra}A)\quad \mathbf{Prox}_G(x+u, x+u) \;=\; (x,u) \Leftrightarrow (x,u)\in\mathbf{gra}B$

`\* using Proposition 16.52 *\`

Proof:

Proposition 20.47 (ii) : Fitzpatrick function of a monotone operator is $\Gamma_0$ (CCP)

i.e., $F_A\in\Gamma_0(\mathcal{H}\times\mathcal{H})$ . . . (1)

$\Rightarrow F_A^*\in\Gamma_0(\mathcal{H}\times\mathcal{H})$ /* Fenchel-Moreau corollary 13.33 */

$\Rightarrow F_A^{*T} \in \Gamma_0(\mathcal{H}\times\mathcal{H})$ /* $\square^T(x,y) = \square(y,x)$ so convexity stays the same as only the variables are being reindexed ! */ $\quad \ldots (2)$

From (1), (2):

$G = \text{pav}(F_A, F_A^{*T}) \in \Gamma_0(\mathcal{H}\times\mathcal{H})$

$G^* = \text{pav}(F_A, F_A^{*T})^* = \text{pav}(F_A^*, F_A^{*T*})$

/* Using corollary 14.8 (i), (ii) : $[f, g \in \Gamma_0(\mathcal{H})]$ | $\text{pav}(f,g) \in \Gamma_0(\mathcal{H})$ ; $(\text{pav}(f,g))^* = \text{pav}(f^*, g^*)$ */

$= \text{pav}(F_A^*, F_A^{T**})$ /* $\triangleright F_A \in \Gamma_0(\mathcal{H}) \Rightarrow F_A^{**} = F_A$ (Fenchel Moreau corollary)

Proposition 13.30. $F \in \Gamma(\mathcal{H}\times\mathcal{H}) \Rightarrow F^{*T} = F^{T*}$

$\because F_A \in \Gamma_0(\mathcal{H}\times\mathcal{H}) \subseteq \Gamma(\mathcal{H}\times\mathcal{H}) \Rightarrow F_A^{*T} = F_A^{T*}$ */

$= \text{pav}(F_A^*, F_A^T)$ /* $F_A \in \Gamma_0(\mathcal{H}\times\mathcal{H})$

$\Rightarrow F_A^T \in \Gamma_0(\mathcal{H}\times\mathcal{H}) \Rightarrow F_A^{T**} = F_A^T$ /* Fenchel Moreau corollary */

$= \text{pav}(F_A^T, F_A^*)$ /* $\text{pav}(f,g) = \text{pav}(g,f)$ */

$= \text{pav}(F_A^T, F_A^{*TT})$ /* transpositioning a bivariate function twice brings back the original function */

$= \text{pav}(\boxed{F_A}^T, \boxed{F_A^{*T}}^T)$

$= \underbrace{\text{pav}(F_A, F_A^{*T})}_{G}{}^T$ /* Proposition 14.10. $[F, G \in \Gamma_0(\mathcal{H}\times\mathcal{H})]$ $(\text{pav}(F,G))^T = \text{pav}(F^T, G^T)$ */

$= G^T$

$\therefore G^* = G^T$ /* A bivariate function F is autoconjugate if $F^* = \overset{\text{transposition}}{F^T}$ */

$\leftrightarrow G$: autoconjugate $\quad \ldots (3)$

/* Corollary 20.39.

$[F \in \Gamma_0(\mathcal{H}\times\mathcal{H})$, autoconjugate

A: $\text{gra}\, A = \{(x,u) \in \mathcal{H}\times\mathcal{H} \mid F(x,u) = \langle x, u\rangle\}]$

A: maximally monotone */

given : $\text{gra}\, B = \{(x,u) \in \mathcal{H}\times\mathcal{H} : G(x,u) = \langle x | u\rangle\} \ldots (4)$

(3), (4), corollary 20.39 $\Rightarrow$

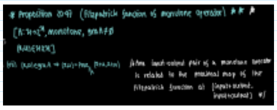
B: maximally monotone . . . (8)

define $L : \mathcal{H}\times\mathcal{H} \to \mathcal{H}\times\mathcal{H} : (x,u) \mapsto (u,x)$ // Another way of putting it is $L = \text{Id}^T : (x,y) \mapsto \text{Id}^T(x,y) = (y,x)$

$(x,u) \in \text{gra}\, A$

$\Rightarrow (x,u) = \text{Prox}_{F_A}(x+u, x+u)$

$\ldots (5)$



now, $\forall (x,u) \in \text{gra}\, A$

$\text{Prox}_G(x+u, x+u)$ /* $G = \text{pav}(F_A, F_A^{*T})$ */

$= \text{Prox}_{\text{pav}(F_A, F_A^{*T})}(x+u, x+u)$

$= \frac{1}{2}\text{Prox}_{F_A}(x+u, x+u) + \frac{1}{2}\text{Prox}_{F_A^{*T}}(x+u, x+u)$

/* Corollary 14.8. (iv)

$[f, g \in \Gamma_0(\mathcal{H}),\ q = \frac{1}{2}\|\cdot\|^2]$

$\text{Prox}_{\text{pav}(f,g)} = \frac{1}{2}\text{Prox}_f + \frac{1}{2}\text{Prox}_g$ */

$= \frac{1}{2}\text{Prox}_{F_A}(x+u, x+u) + \frac{1}{2}\underbrace{(\text{Id} - L\,\text{Prox}_{F_A})}(x+u, x+u)$

/* Proposition 16.53.

$[L : \mathcal{H}\times\mathcal{H} \to \mathcal{H}\times\mathcal{H} : (x,u) \mapsto (u,x)$

$F \in \Gamma_0(\mathcal{H})]$

$\text{Prox}_{F^{*T}} = \text{Id} - L\,\text{Prox}_F L$ */

$= \frac{1}{2}\overbrace{\text{Prox}_{F_A}(x+u, x+u)}^{(x,u)}$ /* From (5) */

$+ \frac{1}{2}(x+u, x+u) - \frac{1}{2}L\,\underbrace{\text{Prox}_{F_A}\ L\,(x+u, x+u)}_{=(x,u)}$ /* From (5) */

$= \frac{1}{2}(x,u) + \frac{1}{2}(x+u, x+u) - \frac{1}{2}\underbrace{L(x,u)}_{(u,x)}$

$= \frac{1}{2}(x+x+u-u,\ u+x+u-x)$

$= (x,u)$

$\Leftrightarrow \text{Prox}_G(x+u, x+u) = (x,u)$

$\leftrightarrow G(x,u) = \langle x | u\rangle$

/* Proposition 16.52.

$[G$ : autoconjugate, $\in \Gamma_0(\mathcal{H}\times\mathcal{H})]$

$(x,u) = \text{Prox}_G(x+u, x+u) \leftrightarrow G(x,u) = \langle x | u\rangle$ */

$\leftrightarrow (x,u) \in \text{gra}\, B$

/* By definition of B */

$\ldots (6)$

$\therefore \forall_{(x,u) \in \text{gra}\, A}\ (x,u) \in \text{gra}\, B$

$\leftrightarrow \text{gra}\, A \subseteq \text{gra}\, \underbrace{B}_{\text{maximally monotone (from (8))}} \ldots (7)$

$\therefore$ B: maximally monotone extension of A

■

# Part 3

9:18 PM

Theorem 20·38·

$[\ F: \mathcal{H}\times\mathcal{H} \to ]-\infty,+\infty]$, convex function

$F^*$: proper, $F^* \geq \langle\cdot|\cdot\rangle \Leftrightarrow F^*(a,b) \geq \langle a|b\rangle$

$A: \mathcal{H}\to 2^{\mathcal{H}}$, $\text{gra}A = \{(x,u)\in\mathcal{H}\times\mathcal{H} : F^*(u,x)=\langle x|u\rangle\}\ ]$

i) A: monotone

ii) $F\geq\langle\cdot|\cdot\rangle \Rightarrow$ A: maximally monotone.

Proof:

$$\forall_{(x,u),(y,v)\in\text{gra }A} \quad \langle x|u\rangle+\langle y|v\rangle = F^*(u,x)+F^*(v,y)$$

$$\geq 2F^*\left(\tfrac{1}{2}(u,x)+\tfrac{1}{2}(v,y)\right)$$

$$= 2F^*\left(\tfrac{1}{2}\begin{bmatrix}u\\x\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}v\\y\end{bmatrix}\right)$$

$$= 2F^*\left(\tfrac{1}{2}\begin{bmatrix}u+v\\x+y\end{bmatrix}\right)$$

$$= 2F^*\left(\tfrac{1}{2}(u+v),\tfrac{1}{2}(x+y)\right)$$

$$\geq 2\cdot\left(\tfrac{1}{2}(u+v)\,\middle|\,\tfrac{1}{2}(x+y)\right) \quad /* \because F^*\geq\langle\cdot|\cdot\rangle */$$

$$= \cancel{2}\cdot\tfrac{1}{\cancel{2}}\cdot\tfrac{1}{2}\ \langle u+v\,|\,x+y\rangle$$

/* conjugate function is always convex

$\Rightarrow F^*(\cdot,\cdot)$ : convex

$$\Rightarrow F^*\left(\underset{\in[0,1]}{\theta}\begin{bmatrix}u\\x\end{bmatrix}+(1-\theta)\begin{bmatrix}v\\y\end{bmatrix}\right)\leq\theta F^*\left(\begin{bmatrix}u\\x\end{bmatrix}\right)+(1-\theta)F^*\left(\begin{bmatrix}v\\y\end{bmatrix}\right)$$

$$\text{set } \theta:=\tfrac{1}{2} \Rightarrow F^*\left(\tfrac{1}{2}(u,x)+\tfrac{1}{2}(v,y)\right)\leq\tfrac{1}{2}F^*(u,x)+\tfrac{1}{2}F^*(v,y)$$

$$F^*(u,x)+F^*(v,y)\geq 2F^*\left(\tfrac{1}{2}(u,x)+\tfrac{1}{2}(v,y)\right)$$ */

$$\Rightarrow \quad \langle x|u\rangle+\langle y|v\rangle \geq \tfrac{1}{2}\left(\langle u|x\rangle+\langle v|y\rangle+\langle v|x\rangle+\langle u|y\rangle\right)$$

$$\Leftrightarrow \quad \tfrac{1}{2}\langle u|x\rangle+\tfrac{1}{2}\underbrace{\langle y|v\rangle}_{=\langle v|y\rangle}\geq\tfrac{1}{2}\langle v|x\rangle+\tfrac{1}{2}\langle u|y\rangle$$

$$\Leftrightarrow \quad \langle u-v|x\rangle+\langle u-v|y\rangle\geq 0$$

$$\Leftrightarrow \quad \langle u-v\,|\,x-y\rangle=\langle x-y\,|\,u-v\rangle\geq 0$$

$\therefore$ So, we have shown $\forall_{(x,u)}\left((x,u)\in\text{gra}A \Rightarrow \forall_{(y,v)\in\text{gra}A}\ \langle x-y|u-v\rangle\geq 0\right)$

$\Leftrightarrow$ A: monotone ■

ii)

**Proof blueprint:** In (i), we have already proven that A is monotone, in this part we want to prove that it is maximal too.

Given: $\forall(x,u)\in\text{gra}A\ \ \langle z-x \mid w-u\ \rangle\geq 0$

Goal: $(z,w)\in\text{gra}A$

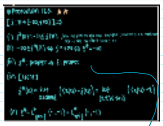
So the given is true for any $(x,u)\in\text{gra}A$, so the antecedent would still be true for a specially designed $(x,u)\in\text{gra}A$.

To do this, we set a very specific $(x,u):=(y,v)\in\text{gra}A : F^*(v,y)=\langle v|y\rangle$, and using this we show that $(z,w)\in\text{gra}A$. To do this we first prove that, $F^*\geq\langle\cdot\mid\cdot\rangle$. Then we proceed according to the blueprint.

given $F^*$: proper $\overset{\text{def}}{\Rightarrow} -\infty\notin f^*(\mathcal{H}) \Rightarrow f$: posses a continuous affine minorant $\overset{\text{def}}{\Leftrightarrow} \exists_{u\in\mathcal{H},\eta\in\mathbb{R}}\ f\geq\overbrace{\langle u|\cdot\rangle+\eta}^{\text{continuous affine term}}$

$F^*$: convex /*all the time*/

/*



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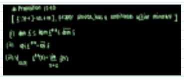


now given, $F\geq\langle\cdot|\cdot\rangle$

$\Leftrightarrow \forall_{(x,u)\in\mathcal{H}}\ F(x,u)\geq\langle x|u\rangle \quad \ldots(a)$

now, F: proper as $F^*$: proper by given, F: convex, $F:\mathcal{H}\times\mathcal{H}\to]-\infty,+\infty]$ ] Prop 13·40 (ii)

$$\Rightarrow\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\ F^{**}(x,u)=\underline{\lim}_{(y,v)\to(x,u)}F(y,v)$$

$$\geq\underline{\lim}_{(y,v)\to(x,u)}\langle y|v\rangle \quad /* \text{From (a)} */$$

/* now $\langle\cdot|\cdot\rangle \Rightarrow \underline{\lim}_{(y,v)\to(x,u)}\langle y|v\rangle=\lim_{(y,v)\to(x,u)}\langle y|v\rangle=\langle x|u\rangle$ */

$$=\langle x|u\rangle$$

$$\Rightarrow\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\ F^*(x,u)\geq\langle x|u\rangle \quad \ldots(b)$$

$$\therefore F^*\geq\langle\cdot|\cdot\rangle \quad \ldots(c)$$

We want to prove that:

$$A:\text{maximally monotone}\overset{\text{def}}{\Leftrightarrow}\forall_{(z,w)}\left((z,w)\in\text{gra}A\Leftrightarrow\forall_{(x,u)\in\text{gra}A}\ \langle z-x|w-u\rangle\geq 0\right)$$

now in (i), we have proven that A: monotone, so $\Rightarrow$ direction is already proven

so, we want to prove the other direction. i.e., $(\Leftarrow)$

$$\forall_{(z,w)}\left(\underbrace{(z,w)\in\text{gra}A}_{?}\Leftarrow\underbrace{\forall_{(x,u)\in\text{gra}A}\ \langle z-x|w-u\rangle\geq 0}\right)$$

(goal (ii))

so our antecedent is:

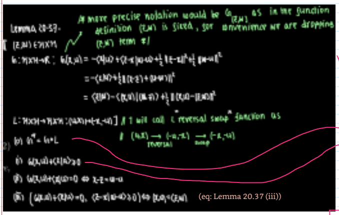
$\forall_{(z,w)\in\mathcal{H}\times\mathcal{H}}\ \forall_{(x,u)\in\text{gra}A}\ \langle z-x|w-u\rangle\geq 0 \quad \ldots(\text{antecedent})$, goal: $(z,w)\in\text{gra}A$

(eq: antecedent prop 20.38)

so in given-goal form: $\forall(z,w)$

| Given | Goal |
|---|---|
| $(z,w)\in\mathcal{H}\times\mathcal{H}$ | $(z,w)\in\text{gra }A$ |
| $\forall\ (x,u)\in\text{gra }A$ | |
| $\langle z-x\vert w-u\rangle\geq 0$ | |

So, from the proofs perspective (z,w) is fixed



(eq: Lemma 20.37 (iii))

define G and L like this:

$$\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\ G(x,u)+\langle x|u\rangle\geq 0 \quad \ldots(\varphi)$$

by adding them:

$$(\forall(x,u)\in\mathcal{H}\times\mathcal{H})\quad F^*(x,u)+G(x,u)\geq 0$$

$$\Leftrightarrow \quad F^*+G\geq 0 \quad \ldots(d)$$

we will apply this to our case:

true for any function

$\mathcal{H}:=\mathcal{H}\times\mathcal{H}$

$f:=F^*$ ☑ $\in\Gamma_0(\mathcal{H}\times\mathcal{H})$ /* as $F^*$: proper, $\overbrace{F^*\in\Gamma(\mathcal{H}\times\mathcal{H})}\Rightarrow F^*\in\Gamma(\mathcal{H}\times\mathcal{H})$ */ ☑

☑ $g:=G\in\Gamma_0(\mathcal{H}\times\mathcal{H})$ /* by definition */, $\underset{\checkmark}{F^*+G\geq 0}$ ☑, $G^*=G\circ L$, $L(x,u)=(-u,-x)$ /* this L is clearly $\in\mathcal{B}(\mathcal{H}\times\mathcal{H},\mathcal{H}\times\mathcal{H})$ think about the matrix case */

final antecedent to check:

$0\in\text{sri}\,(\text{dom}\,F^*-\text{dom}\,G)$

now $\text{dom}\,G=\mathcal{H}\times\mathcal{H}$, $\text{dom}\,F^*=\mathcal{H}\times\mathcal{H}$ $\Rightarrow$ $\text{dom}\,F^*-\text{dom}\,G=\mathcal{H}\times\mathcal{H}$ $\Rightarrow$ $\text{sri}\,(\text{dom}\,F^*-\text{dom}\,G)=\mathcal{H}\times\mathcal{H}\ni 0$

$\therefore\ 0\in\text{sri}(\text{dom}\,F^*-\text{dom}\,G)$ ☑

so we can apply the consequent !

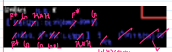
which is:

$G(z,v)$

$\Rightarrow \operatorname{sri}(\operatorname{dom} F^* - \operatorname{dom} G) = \mathcal{H}\times\mathcal{H} \ni 0$

$\therefore 0 \in \operatorname{sri}(\operatorname{dom} F^* - \operatorname{dom} G)$ ☑

so we can apply the consequent !



$\exists_{(v,y)\in\mathcal{H}\times\mathcal{H}}$ $F^*(v,y) + G(-L(v,y)) \leq 0$

which is:

$\exists_{(v,y)\in\mathcal{H}\times\mathcal{H}}$ $F^*(v,y) + G(\overbrace{-L\overbrace{(v,y)}^{(y,-v)}}^{G(y,v)}) \leq 0 \Leftrightarrow F^*(v,y) + G(y,v) \leq 0$

now given : $\langle\cdot|\cdot\rangle \leq F^*$

$\underbrace{\langle v|y\rangle}_{\langle y|v\rangle} + G(y,v) \leq F^*(v,y) + G(y,v) \leq 0$

$\Rightarrow \exists_{(v,y)\in\mathcal{H}\times\mathcal{H}}$ $0 \leq G(y,v) + \langle y|v\rangle \leq F^*(v,y) + G(y,v) \leq 0$

$\exists_{(v,y)\in\mathcal{H}\times\mathcal{H}}$ $G(y,v) + \langle y|v\rangle \geq 0$

$\therefore$ (20.27)

$\Leftrightarrow \exists_{(v,y)\in\mathcal{H}\times\mathcal{H}}$ $\boxed{G(y,v) + \langle y|v\rangle = 0 \ \wedge\ F^*(v,y) + G(y,v) = 0}$

$\Downarrow$ $G(y,v) = -\langle y|v\rangle$ $\qquad$ $\Downarrow$ $F^*(v,y) = -G(y,v) = \langle y|v\rangle$

$\therefore \exists_{(v,y)\in\mathcal{H}\times\mathcal{H}}$ $F^*(v,y) = \langle y|v\rangle$ $\quad$ /* recall : $\operatorname{gra} A = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F^*(u,x) = \langle x|u\rangle\}$

so. $(y,v) \in \operatorname{gra} A$ */

$\Updownarrow$

$(y,v) \in \operatorname{gra} A$ $\quad$ /* note that this is coming from the definition of gra A */

From (eq: antecedent prop 20.38) $\forall_{(z,w)\in\mathcal{H}\times\mathcal{H}}\ \forall_{(x,u)} \big((x,u)\in\operatorname{gra} A \Rightarrow \langle z-x|w-u\rangle \geq 0\big)$

$\langle z-y|w-v\rangle \geq 0$ . . . (20.29)

from (20.27), (20.29) and (eq: Lemma 20.37 (iii)) $\therefore$

$(z,w) = (y,v)$

but we have shown that : $(z,w) \in \operatorname{gra} A$ $\therefore$ (goal (ii)) reached

$\therefore$ A : maximally monotone.

■

Proposition 20.47.

$[\![A : \mathcal{H}\to 2^{\mathcal{H}}, \text{monotone} : \operatorname{gra} A \neq \emptyset$

$(x,u)\in\mathcal{H}\times\mathcal{H}]\!]$

(i) $(x,u)\in\operatorname{gra} A \Rightarrow F_A(x,u) = \langle x|u\rangle$

(ii) $F_A = (\iota_{\operatorname{gra} A^{-1}} + \langle\cdot|\cdot\rangle) \in \Gamma_0(\mathcal{H}\times\mathcal{H})$

(iii) $F_A(x,u) \leq \langle x|u\rangle \leftrightarrow \{(x,u)\}\cup \operatorname{gra} A$ : monotone

(iv) $F_A(x,u) \leq F_A^*(u,x)$

(v) $(x,u)\in\operatorname{gra} A \Rightarrow F_A^*(u,x) = \langle x|u\rangle$

(vi) $F_A(x,u) = F_{A^{-1}}(u,x)$

(vii) $\gamma\in\mathbb{R}_{++} \Rightarrow F_{\gamma A}(x,u) = \gamma F_A(x, u/\gamma)$

(viii) $(x,u)\in\operatorname{gra} A \Rightarrow (x,u) = \operatorname{Prox}_{F_A}(x+u, x+u)$

Proof:

(i)

A : monotone $\overset{\text{def}}{\Leftrightarrow} \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \big((x,u)\in\operatorname{gra} A \Rightarrow \forall_{(y,v)\in\operatorname{gra} A} \langle x-y|u-v\rangle \geq 0$

$\leftrightarrow \inf_{(y,v)\in\operatorname{gra} A} \langle x-y|y-v\rangle \geq 0$

clearly $(y,v) := (x,u)$ will yield the infimum 0.

$\rightarrow \inf_{(y,v)\in\operatorname{gra} A} \langle x-y|u-v\rangle = 0$ . . . (eq: 1)

now. $F_A(x,u) \overset{\text{def}}{=} \langle x|u\rangle - \inf_{(y,v)\in\operatorname{gra} A} \langle x-y|u-v\rangle$

$= \langle x|u\rangle$ ☺

(ii)

By definition,

$F_A(x,u) \overset{\text{def}}{=} \sup_{(y,v)\in\operatorname{gra} A} (\langle y|v\rangle + \langle x|v\rangle - \langle y|v\rangle)$

now consider

$(\iota_{\operatorname{gra} A^{-1}} + \langle\cdot|\cdot\rangle)^*(u,x)$ $\quad$ /* $f^*(u) = \sup_{x\in\mathcal{H}} (-f(x) + \langle x|u\rangle)$ */

$= \sup_{(v,y)\in\mathcal{H}\times\mathcal{H}} \big(-(\iota_{\operatorname{gra} A^{-1}} + \langle\cdot|\cdot\rangle)(v,y) + \underbrace{\langle (v,y)|(u,x)\rangle}\big)$

$\langle \begin{bmatrix} y \\ v\end{bmatrix} | \begin{bmatrix} u \\ x\end{bmatrix}\rangle = \langle y|u\rangle + \langle v|x\rangle$ $\quad$ /* e.g., in finite dimension: $\langle (y,v)|(x,u)\rangle = \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ v_1 \\ \vdots \\ v_n\end{bmatrix}^T \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ u_1 \\ \vdots \\ u_n\end{bmatrix} = (y_1x_1 + \ldots + y_mx_m) + (v_1u_1 + \ldots + v_nu_n) = y^Tx + u^Tv = \langle y|x\rangle + \langle u|v\rangle$ */

. . . (note: 1)

$= \sup_{(v,y)\in\mathcal{H}\times\mathcal{H}} \big(-\iota_{\operatorname{gra} A^{-1}}(v,y) - \langle v|y\rangle + \langle y|u\rangle + \langle v|x\rangle\big)$

$= -\inf_{(v,y)\in\mathcal{H}\times\mathcal{H}} \big(\iota_{\operatorname{gra} A^{-1}}(y,v) - (\langle y|u\rangle + \langle v|x\rangle - \langle v|y\rangle)\big)$ $\quad$ /* $\sup(\cdot) = -\inf(-\cdot)$ */

$= -\inf_{(v,y)\in\operatorname{gra} A^{-1}} \big(-(\langle y|u\rangle + \langle v|x\rangle - \langle v|y\rangle)\big)$

$= \sup_{\substack{(v,y)\in\operatorname{gra} A^{-1} \\ \leftrightarrow (y,v)\in\operatorname{gra} A}} \big(\langle y|u\rangle + \langle v|x\rangle - \langle v|y\rangle\big)$

$= F_A(x,u)$

$\therefore F_A = (\iota_{\operatorname{gra} A^{-1}} + \langle\cdot|\cdot\rangle)^* \in \Gamma(\mathcal{H}\times\mathcal{H})$ /* Any conjugate function is lower-semicontinuous and convex */ . . . (eq: 2)

yes, any

now, we have to prove properness, i.e., $\operatorname{dom} F_A \neq \emptyset$, $-\infty \in F_A(\mathcal{H})$

now in (i) : for A : monotone $F_A(x,u) = \langle x|u\rangle$ when $(x,u)\in\operatorname{gra}A$.

now $\operatorname{gra} A \neq \emptyset$ by given, so $\exists (\tilde{x},\tilde{u})\in\operatorname{gra} A \Rightarrow F_A(\tilde{x},\tilde{u}) = \langle\tilde{x}|\tilde{u}\rangle = \text{finite} \Rightarrow \operatorname{dom} F_A \neq \infty$ . . . (eq: 3)

and similarly, $-\infty \notin F_A(\operatorname{dom} F_A)$, and when $(x,u)\in\mathcal{H}\times\mathcal{H}\setminus\operatorname{dom} F_A$ we have $F_A(x,u) = +\infty \neq -\infty$, so combining both we have $-\infty \in F_A(\mathcal{H}\times\mathcal{H})$. So from this and eq: 3 we have $F_A$ is proper.

$F_A$ : proper, lower-semicontinuous and convex $\leftrightarrow F_A \in \Gamma_0(\mathcal{H}\times\mathcal{H})$

☺

(iii)

$F_A(x,u) \overset{\text{def}}{=} \langle x|u\rangle - \inf_{(y,v)\in\operatorname{gra} A} \langle x-y|u-v\rangle$

$\leftrightarrow F_A(x,u) - \langle x|u\rangle = \underbrace{-\inf_{(y,v)\in\operatorname{gra} A} \langle x-y|u-v\rangle}$

$\geq 0 \leftrightarrow \{(x,u)\}\cup\operatorname{gra} A$ : monotone

$\leftrightarrow \forall_{(\tilde{x},\tilde{u})\in\mathcal{H}} \big((\tilde{x},\tilde{u})\in\operatorname{gra}(\{(x,u)\}\cup\operatorname{gra} A) \Rightarrow \forall_{(y,v)\in\operatorname{gra}(\{(x,u)\}\cup\operatorname{gra} A)} \langle\tilde{x}-y|\tilde{u}-v\rangle \geq 0$ */

$\leftrightarrow$

$F_A(x,u) - \langle x|u\rangle \leq 0 \Leftrightarrow \{(x,u)\}\cup\operatorname{gra} A$ : monotone

(iv)

$$F_A(x,u) \overset{\text{def}}{=} \sup_{(y,v)\in \operatorname{gra} A} \left( \langle y|u\rangle + \langle x|v\rangle - \langle y|v\rangle \right)$$

/* as, $(y,v)\in \operatorname{gra} A \Rightarrow F_A(y,v) = \langle y|v\rangle$ from (i) */

$$= \sup_{(y,v)\in \operatorname{gra} A} \left( \langle y|u\rangle + \underset{\langle v|x\rangle}{\underset{\shortparallel}{\langle x|v\rangle}} - F_A(y,v) \right)$$

$$\leq \sup_{(y,v)\in \mathcal{H}\times\mathcal{H}} \left( \langle y|u\rangle + \langle v|x\rangle - F_A(y,v) \right)$$ /* using the fact that optimizing over a larger set gives better objective value */

$$= \sup_{(y,v)\in \mathcal{H}\times\mathcal{H}} \left( \langle (y,v) \,|\, (u,x)\rangle - F_A(y,v) \right)$$ /* using (note: 1) */

$$= F_A^*(u,x)$$

$$\therefore \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \quad F_A(x,u) = F_A^*(u,x)$$

:)

(v)

By (ii):

$$\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \quad F_A(x,u) = \left(\iota_{\operatorname{gra} A^{-1}} + \langle\cdot|\cdot\rangle\right)^*(x,u)$$

$$\Rightarrow F_A^* = \left(\iota_{\operatorname{gra} A^{-1}} + \langle\cdot|\cdot\rangle\right)^{**}$$

$$\leq \iota_{\operatorname{gra} A^{-1}} + \langle\cdot|\cdot\rangle$$ /* Biconjugate is always a lower bound (minorant) of the original function */

in (iv) we have shown:

$$\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \quad F_A(x,u) \leq F_A^*(u,x) \leq \left(\iota_{\operatorname{gra} A^{-1}} + \langle\cdot|\cdot\rangle\right)(u,x) = \underset{\iota_{\operatorname{gra} A}(x,u)}{\underset{\shortparallel}{\iota_{\operatorname{gra} A^{-1}}(u,x)}} + \langle u|x\rangle$$

now $(x,u)\in \operatorname{gra} A \Rightarrow$ $\left|\begin{array}{l} \iota_{\operatorname{gra} A}(x,u) = 0 \\ F_A(x,u) = \langle x|u\rangle \quad \text{(from (i))} \end{array}\right.$

$$\forall_{(x,u)\in \operatorname{gra} A} \quad F_A(x,u) \leq F_A^*(u,x) \leq \langle u|x\rangle = F_A(x,u)$$

$$\Leftrightarrow \forall_{(x,u)\in \operatorname{gra} A} \quad F_A(x,u) = F_A^*(u,x)$$ :)

(vi), (vii): direct application of the definition: $F_A(x,u) = \sup_{(y,v)\in \operatorname{gra} A} \left( \langle y|u\rangle + \langle x|v\rangle - \langle y|v\rangle \right)$

(viii)

in (i): $\forall_{(x,u)\in \operatorname{gra} A} \quad F_A(x,u) = \langle x|u\rangle$

in (v): $\forall_{(x,u)\in \operatorname{gra} A} \quad F_A^*(u,x) = \langle x|u\rangle$

---

$$\forall_{(x,u)\in \operatorname{gra} A} \quad F_A\underbrace{(x,u)}_{a} + F_A^*\underbrace{(u,x)}_{b} = 2\langle x|u\rangle = \langle \underbrace{(x,u)}_{a} \,|\, \underbrace{(u,x)}_{b}\rangle$$ /* clever rewriting using (note. 1) */

/* recall Theorem 16.23.:

$[f\in\Gamma_0(\mathcal{H}), x\in\mathcal{H}, u\in\mathcal{H}]$ $(x,u)\in \operatorname{gra}\partial f \Leftrightarrow f(x) + f^*(u) = \langle x|u\rangle$ */

now in (ii) we have shown $F_A\in\Gamma_0(\mathcal{H}\times\mathcal{H})$

$$F_A(a) + F_A(b) = \langle a|b\rangle$$

$$\Leftrightarrow (a,b)\in \operatorname{gra}\partial F_A$$

$$\Leftrightarrow \partial F_A(\underset{(x,u)}{\underset{\shortparallel}{a}}) \ni \underset{(u,x)}{\underset{\shortparallel}{b}}$$

$$\Leftrightarrow \partial F_A(x,u) \ni (u,x)$$

$$\Leftrightarrow \underbrace{(x,u) + \partial F_A(x,u)}_{(\mathrm{Id}+\partial F_A)(x,u)} \ni (u,x) + (x,u) = \begin{bmatrix} u \\ x \end{bmatrix} + \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} u+x \\ x+u \end{bmatrix} = (u+x, x+u)$$

$$\Leftrightarrow (\mathrm{Id}+\partial F_A)(x,u) \ni (x+u, x+u)$$

$$\Leftrightarrow (x,u) \in \underset{\operatorname{Prox}_{F_A}}{\underset{\shortparallel}{(\mathrm{Id}+\partial F_A)^{-1}}}(x+u, x+u)$$ /* $\operatorname{Prox}_f = (\mathrm{Id}+\partial f)^{-1}$ */

$$\Leftrightarrow (x,u) \in \operatorname{Prox}_{F_A}(x+u, x+u)$$

$$\Leftrightarrow (x,u) = \operatorname{Prox}_{F_A}(x+u, x+u)$$ /* Because $\operatorname{Prox}_{\square}$ is a function */

■

Proposition 20.50.

[ C,D: closed affine subspaces of $\mathcal{H}$: $D-D = (C-C)^{\perp}$

A: maximally monotone

$(x_n, y_n)\subseteq \operatorname{gra} A$

$(x,u)\in\mathcal{H}\times\mathcal{H}$ ]

$$\left.\begin{array}{l} x_n \rightharpoonup x \\ u_n \rightharpoonup u \\ x_n - P_C x_n \to 0 \\ u_n - P_D u_n \to 0 \end{array}\right] \Rightarrow \left|\begin{array}{l} (x,u)\in (C\times D)\cap \operatorname{gra} A \\ \langle x_n|y_n\rangle \to \langle x|u\rangle \end{array}\right.$$

Proof: // This proof is a beast, though the proof blueprint is relatively simple, uses several facts about affine space and some of the previous results.

$V := C - C$: linear subspace parallel to $C$

given, $x_n - P_C x_n \to 0 \Leftrightarrow P_C x_n - x_n \to 0$

$\Leftrightarrow P_C x_n \to x_n \rightharpoonup x$ // given $x_n \rightharpoonup x$

$\therefore P_C x_n \rightharpoonup x$ (eq:1)

given, C: closed affine subspaces of $\mathcal{H}$

$\Rightarrow$ C: closed convex set of $\mathcal{H}$ /* Affine sets are convex */

$\Leftrightarrow$ C: weakly sequentially closed $\underset{\text{convex}}{\downarrow}$ set of $\mathcal{H}$ /* For a convex set, all notions of closedness collapses */

recall the definition of a weakly sequentially closed set via sequences: C: weakly sequentially closed $\overset{\text{def}}{\Leftrightarrow} \forall_{(x_n)_{n\in\mathbb{N}}:\subseteq C,\, x_n\rightharpoonup x} \; x\in C$ ...(eq: 2)

As, $(P_C x_n)_{n\in\mathbb{N}}\subseteq C$ by definition of $P_C(\cdot)$, using (eq:1) and (eq:2) we have: $x\in C \Rightarrow C = x + \overset{V}{\widetilde{C-C}}$ /* C: affine subspace $\Rightarrow \forall_{x\in C} \; C = x + \underset{C-C}{\underbrace{V}}$ : page 1: Bauschke */

using similar logic, $\underbrace{u\in D}_{(\text{eq:2.5})} \Rightarrow D = u + D - D$

$= u + (C-C)^{\perp}$ /* given: $D-D = (C-C)^{\perp}$ */

$= u + V^{\perp}$

$$\therefore \text{so,} \quad \left.\begin{array}{l} C = x + V \\ D = u + V^{\perp} \end{array}\right\} \text{(eq:3)}$$

$\therefore P_C w = P_{x+V} w = x + P_V(w - x)$ /* Proposition 3.17. [ C: nonempty closed $\overset{\text{affine}}{\Downarrow}$ convex set of $\mathcal{H}$; $x,y\in\mathcal{H}$ ] $P_{y+C}x = y + P_C(x-y)$

$= x + P_V(w-x)$ /* V: linear subspace $\Rightarrow P_V(0) = 0$

$\Downarrow$

V: affine subspace too $\Rightarrow P_V$: affine operator $\Leftrightarrow \tilde{x} \mapsto P_V\tilde{x} - \overset{0}{\underset{}{\overset{\shortparallel}{P_V(0)}}}$ : linear

$\Leftrightarrow \tilde{x} \mapsto P_V\tilde{x}$ : linear $\Leftrightarrow P_V$ : linear */

$= x + P_V w - P_V(x)$

$= P_V w + \underbrace{(\mathrm{Id} - P_V)(x)}_{P_{V^{\perp}} x}$ /* Corollary 3.22. (v): [V: closed linear subspace of $\mathcal{H}$; $x\in\mathcal{H}$] $P_{V^{\perp}} = \mathrm{Id} - P_V$ */

$\Updownarrow$

$\forall_{x\in\mathcal{H}} \quad P_{V^{\perp}} x = x - P_V x \Leftrightarrow x = P_V x + P_{V^{\perp}} x$ ...(eq: 4)

$$\left.\begin{array}{ll} \Rightarrow & P_C w = P_V w + P_{V^{\perp}} x \\ \text{similarly,} & P_D w = P_{V^{\perp}} w + P_V u \end{array}\right\} \forall_{w\in\mathcal{H}} \quad \text{...(eq:5)}$$

/* Proposition 4.11.

[ C: closed affine subspace of $\mathcal{H}$ ] $P_C$: weakly continuous */

using this:

$P_C, P_D$ : weakly continuous // $T: D\to K$ : weakly continuous $\overset{\text{def}}{\Leftrightarrow} \forall_{(x_a)_{a\in A}:\subseteq D,\, x_a\rightharpoonup x} \; Tx_a \rightharpoonup Tx$

$\therefore \langle x_n|u_n\rangle = \langle P_V x_n + P_{V^{\perp}} x_n \,|\, P_V u_n + P_{V^{\perp}} u_n\rangle$

$= \langle P_V x_n|P_V u_n\rangle + \underbrace{\langle P_{V^{\perp}} x_n|P_V u_n\rangle}_{\overset{\shortparallel}{0}} + \overbrace{\langle P_V x_n|P_{V^{\perp}} u_n\rangle}^{0} + \langle P_{V^{\perp}} x_n|P_{V^{\perp}} u_n\rangle$ /* same logic */

/* as $P_{V^{\perp}} x_n\in V^{\perp}$ and $P_V u_n\in V$, $\langle P_{V^{\perp}} x_n|P_V u_n\rangle = 0$ */

$= \langle P_V x_n|P_V u_n\rangle + \langle P_{V^{\perp}} x_n|P_{V^{\perp}} u_n\rangle$

$\therefore \langle x_n | u_n \rangle = \langle P_V x_n + P_{V\perp} x_n | P_V u_n + P_{V\perp} u_n \rangle$

$= \langle P_V x_n | P_V u_n \rangle + \underbrace{\langle P_{V\perp} x_n | P_V u_n \rangle}_{\overset{\shortparallel}{0}} + \overbrace{\langle P_V x_n | P_{V\perp} u_n \rangle}^{0} + \langle P_{V\perp} x_n | P_{V\perp} u_n \rangle$ /* same logic */

/* as $P_{V\perp} x_n \in V^{\perp}$ and $P_V u_n \in V$, $\langle P_{V\perp} x_n | P_V u_n \rangle = 0$ */

$= \langle P_V x_n | \underbrace{P_V}_{\overset{\shortparallel}{(Id - P_{V\perp})}} u_n \rangle + \langle \underbrace{P_{V\perp}}_{\overset{\shortparallel}{(Id - P_V)}} x_n | P_{V\perp} u_n \rangle$

$= \langle P_V x_n | u_n - \underbrace{P_{V\perp} u_n}_{(P_D u_n \overset{\shortparallel}{-} P_V u)} \rangle + \langle x_n - \underbrace{P_V x_n}_{(P_C x_n - P_{V\perp} x)} | P_{V\perp} u_n \rangle$ /* using eq:5 */

$= \underbrace{\langle P_V x_n | u_n - P_D u_n \rangle}_{\downarrow \atop 0} + \underbrace{\langle P_V x_n | P_V u \rangle}_{\downarrow \atop \langle P_V x | P_V u \rangle} + \underbrace{\langle x_n - P_C x_n | P_{V\perp} u_n \rangle}_{\downarrow \atop 0} + \underbrace{\langle P_{V\perp} x | P_{V\perp} u_n \rangle}_{\downarrow \atop \langle P_{V\perp} x | P_{V\perp} u \rangle}$

$\to \langle P_V x | P_V u \rangle + \langle P_{V\perp} x | P_{V\perp} u \rangle$

$= \langle x | u \rangle$

/* subproof.

$\langle x | u \rangle = \langle P_V x + P_{V\perp} x | P_V u + P_{V\perp} u \rangle$ /* from (eq: 4) */

$= \langle P_V x | P_V u \rangle + \langle P_{V\perp} x | P_{V\perp} u \rangle$

$+ \underbrace{\langle P_V x | P_{V\perp} u \rangle}_{\overset{\shortparallel}{0}} + \underbrace{\langle P_{V\perp} x | P_V u \rangle}_{\overset{\shortparallel}{0}}$

/* as $P_V x \in V$, $P_{V\perp} u \in V^{\perp}$ */

$= \langle P_V x | P_V u \rangle + \langle P_{V\perp} x | P_{V\perp} u \rangle$ */

$\langle P_V x_n | u_n - P_D u_n \rangle \to 0$

subproof. $\underbrace{x_n \to x}_{\text{given}} \Rightarrow P_V x_n \to P_V x$, $\underbrace{u_n - P_D u_n \to 0}_{\text{given}}$, so $\langle P_V x_n | u_n - P_D u_n \rangle \to 0$ By using

similarly, $\langle x_n - P_C x_n | P_{V\perp} u_n \rangle \to 0$

subproof. given $\left[ \begin{array}{l} u_n \to u \\ x_n - P_C x_n \to 0 \end{array} \right] \Rightarrow P_{V\perp} u_n \to P_V u$

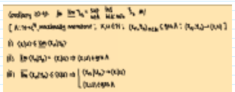
$\langle x_n - P_C x_n | P_{V\perp} u_n \rangle \to 0$

similarly, $P_V x_n \to P_V x$, $P_V u \to P_V u \Rightarrow \langle P_V x_n | P_V x \rangle \to \langle P_V x | P_V u \rangle$

and $\underbrace{P_{V\perp} x \to P_{V\perp} x}_{\text{they are the same !}}$, $P_{V\perp} u_n \to P_{V\perp} u \Rightarrow \langle P_{V\perp} x | P_{V\perp} u_n \rangle \to \langle P_{V\perp} x | P_{V\perp} u \rangle$

$\therefore \underbrace{\langle x_n | u_n \rangle \to \langle x | u \rangle}_{\downarrow \atop \text{one goal proved}} \leftrightarrow \lim \langle x_n | u_n \rangle = \langle x | u \rangle \Rightarrow \underline{\lim} \langle x_n | y_n \rangle = \langle x | u \rangle \Rightarrow (x, u) \in \operatorname{gra} A$ ... (eq: 6)

/* using Proposition 20.49.



So, using (eq: 2), (eq: 2.5) and (eq: 6) :

$(x, u) \in (C \times D) \cap \operatorname{gra} A$ : other goal proved

■

# Part 4

1:41 PM

Proposition 20.51.

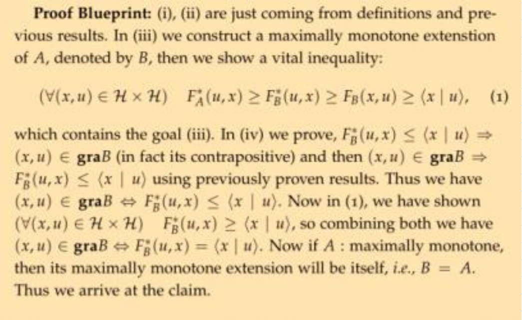
$[A:\mathcal{H}\to 2^{\mathcal{H}}, \text{monotone}, \operatorname{gra}A\neq\emptyset]$

(i) $F_A^* = (\iota_{\operatorname{gra}A^{-1}} + \langle\cdot|\cdot\rangle)^{**}$

(ii) $\operatorname{conv}\operatorname{gra}A^{-1} \subseteq \operatorname{dom}F_A^* \subseteq \overline{\operatorname{conv}}\operatorname{gra}A^{-1} \subseteq \overline{\operatorname{conv}}\operatorname{ran}A \times \overline{\operatorname{conv}}\operatorname{dom}A$

(iii) $F_A^* \geqslant \langle\cdot|\cdot\rangle$

(iv) $A$: maximally monotone $\Rightarrow$ $\operatorname{gra}A = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_A^*(u,x) = \langle x|u\rangle\}$



**Proof Blueprint:** (i), (ii) are just coming from definitions and previous results. In (iii) we construct a maximally monotone extension of $A$, denoted by $B$, then we show a vital inequality:

$$(\forall(x,u)\in\mathcal{H}\times\mathcal{H}) \quad F_A^*(u,x) \geq F_B^*(u,x) \geq F_B(x,u) \geq \langle x \mid u\rangle, \quad (1)$$

which contains the goal (iii). In (iv) we prove, $F_B^*(u,x) \leq \langle x \mid u\rangle \Rightarrow (x,u) \in \mathbf{gra}B$ (in fact its contrapositive) and then $(x,u) \in \mathbf{gra}B \Rightarrow F_B^*(u,x) \leq \langle x \mid u\rangle$ using previously proven results. Thus we have $(x,u) \in \mathbf{gra}B \Leftrightarrow F_B^*(u,x) \leq \langle x \mid u\rangle$. Now in (1), we have shown $(\forall(x,u)\in\mathcal{H}\times\mathcal{H}) \quad F_B^*(u,x) \geq \langle x \mid u\rangle$, so combining both we have $(x,u) \in \mathbf{gra}B \Leftrightarrow F_B^*(u,x) = \langle x \mid u\rangle$. Now if $A$ : maximally monotone, then its maximally monotone extension will be itself, *i.e.*, $B = A$. Thus we arrive at the claim.

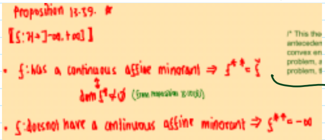
proof:

(i) /* Proposition 20.47.(ii): $[A:\mathcal{H}\to 2^{\mathcal{H}}, \text{monotone}, \operatorname{gra}A\neq\emptyset]$ $F_A = (\iota_{\operatorname{gra}A^{-1}} + \langle\cdot|\cdot\rangle)^*$ */

$\Rightarrow F_A^* = (\iota_{\operatorname{gra}A^{-1}} + \langle\cdot|\cdot\rangle)^{**}$ ⍨

(ii) /* recall: Proposition 9.8. (iv). $[f:\mathcal{H}\to[-\infty,+\infty]]$ $\operatorname{conv}\operatorname{dom}f \subseteq \operatorname{dom}\breve{f} \subseteq \overline{\operatorname{conv}}\operatorname{dom}f$ */

$\breve{f}$: largest lower semicontinuous convex function majorized by $f$.



Proposition 13.39. /*

$[f:\mathcal{H}\to]-\infty,+\infty]]$

- $f$: has a continuous affine minorant $\Rightarrow f^{**} = \breve{f}$
  $\Updownarrow$
  $\operatorname{dom}f^* \neq \emptyset$
- $f$: does not have a continuous affine minorant $\Rightarrow f^{**} = -\infty$

/* This the

antecede

convex en

problem,

problem,

define $G: (u,x) \mapsto (\iota_{\operatorname{gra}A^{-1}} + \langle\cdot|\cdot\rangle)(u,x)$

$$= \begin{cases} \langle\cdot|\cdot\rangle, & \text{if } (u,x)\in\operatorname{gra}A^{-1} \\ +\infty, & \text{if } (u,x)\notin\operatorname{gra}A^{-1}, \end{cases}$$

also $\operatorname{dom}G = \{(u,x)\in\mathcal{H}\times\mathcal{H} \mid G(x,u) < +\infty\} = \operatorname{gra}A^{-1}$

so, $\langle\cdot|\cdot\rangle \leq G \Leftrightarrow G$: has a continuous affine minorant $\Rightarrow G^{**} = \breve{G} = (\iota_{\operatorname{gra}A^{-1}} + \langle\cdot|\cdot\rangle)^{**} = F_A^*$

in proposition 9.8 (iv) set $f := G \Rightarrow$

$\operatorname{conv}\operatorname{dom}G \subseteq \operatorname{dom}\breve{G} \subseteq \overline{\operatorname{conv}}\operatorname{dom}G$

$\leftrightarrow$ $\operatorname{conv}\operatorname{gra}A^{-1} \subseteq \operatorname{dom}F_A^* \subseteq \overline{\operatorname{conv}}\operatorname{gra}A^{-1}$ . . . (eq:1)

$$\left.\begin{array}{l} \operatorname{dom}A = \{x\in\mathcal{H} \mid Ax\neq\emptyset\} \\ \operatorname{ran}A = \{u\in\mathcal{H} \mid \exists_{x\in\operatorname{dom}A} Ax\ni u\} \end{array}\right\} \Rightarrow \operatorname{gra}A \subseteq \operatorname{dom}A\times\operatorname{ran}A$$

$\leftrightarrow \operatorname{gra}A^{-1} \subseteq \operatorname{ran}A\times\operatorname{dom}A$

$\Rightarrow \overline{\operatorname{conv}}\operatorname{gra}A^{-1} \subseteq \overline{\operatorname{conv}}\operatorname{ran}A \times \overline{\operatorname{conv}}\operatorname{dom}A$ . . . (eq:2)

From (eq:1) and (eq:2) we have:

$\operatorname{conv}\operatorname{gra}A^{-1} \subseteq \operatorname{dom}F_A^* \subseteq \overline{\operatorname{conv}}\operatorname{gra}A^{-1} \subseteq \overline{\operatorname{conv}}\operatorname{ran}A \times \overline{\operatorname{conv}}\operatorname{dom}A$ ⍨

(iii)

Assume $B$: maximally monotone extension of $A$ //By the virtue of Theorem 20.21. which says that
// a monotone operator has a maximally monotone
//extension

$\operatorname{gra}A \subseteq \operatorname{gra}B$

now, $F_A(x,u) = \sup\limits_{(y,v)\in\operatorname{gra}A} (\langle y|u\rangle + \langle x|v\rangle - \langle y|v\rangle)$

$$\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \quad F_B(x,u) = \sup_{(y,v)\in\operatorname{gra}B} (\langle y|u\rangle + \langle x|v\rangle - \langle y|v\rangle) \geqslant \sup_{(y,v)\in\operatorname{gra}A} (\langle y|u\rangle + \langle x|v\rangle - \langle y|v\rangle)$$

$$= F_A(x,u)$$

$\leftrightarrow F_B \geqslant F_A$

$\Rightarrow F_B^* \leqslant F_A^*$ // taking conjugate flips the relationship between functions ∵ Proposition 13.14.(ii)
. . . (eq:3)

/* Proposition 20.47.(iv): $[A:\mathcal{H}\to 2^{\mathcal{H}}, \text{monotone}, \operatorname{gra}A\neq\emptyset]$ $\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \; F_A(x,u) \leqslant F_A^*(u,x)$

∵ $B$: monotone, $\operatorname{gra}B\neq\emptyset$ $\Rightarrow$ $\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \; F_B(x,u) \leqslant F_B^*(u,x)$

Proposition 20.47.(iv): $[A:\mathcal{H}\to 2^{\mathcal{H}}, \text{monotone}, \operatorname{gra} A \neq \emptyset]$ $\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\ F_A(x,u) \leqslant F_A^*(u,x)$

$\therefore B:\text{monotone}, \operatorname{gra} B \neq \emptyset \Rightarrow \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\ F_B(x,u) \leqslant F_B^*(u,x)$

$\forall_{(x,u)\in\mathcal{H}}\quad F_A^*(u,x) \geqslant F_B^*(u,x) \geqslant \underbrace{F_B(x,u) \geqslant \langle x|u\rangle}\ = \langle u|x\rangle \quad \ldots (eq.:4)$

// Fitzpatrick function of a maximally monotone operator has the inner product as its minorant Proposition 20.48

$\Rightarrow \forall_{(x,u)\in\mathcal{H}\times\mathcal{H}}\quad F_A^*(u,x) \geqslant \langle u|x\rangle$

(iv)

consider $(x,u)\notin \operatorname{gra} B$

$B:\text{maximally monotone} \Rightarrow (x,u)\in \operatorname{gra} B \Leftrightarrow F_B(x,u) = \langle x|u\rangle$ // Proposition 20.48

$\Leftrightarrow (x,u)\notin \operatorname{gra} B \Leftrightarrow F_B(x,u) \neq \langle x|u\rangle$

But $(eq.:4) \Rightarrow \quad F_B^*(u,x) \geqslant F_B(x,u) \geqslant \langle x|u\rangle$

$\Big\}\Rightarrow F_{B^*}(u,x) \geqslant F_B(x,u) > \langle x|u\rangle \quad \ldots (eq.:5)$

$\therefore (x,u)\notin \operatorname{gra} B \Rightarrow F_B^*(u,x) > \langle x|u\rangle$

$\Leftrightarrow F_B^*(u,x) \leqslant \langle x|u\rangle \Rightarrow (x,u)\in \operatorname{gra} B \quad \ldots (eq.:6)$

$\Rightarrow \left(F_B^*(u,x) = \langle x|u\rangle \Rightarrow (x,u)\in \operatorname{gra} B\right) \quad \ldots (eq.:6.5)$

now consider $(x,u)\in \operatorname{gra} B$, take $(y,v)\in\mathcal{H}\times\mathcal{H} \Rightarrow F_B(y,v) = \sup\limits_{(\tilde{x},\tilde{u})\in \operatorname{gra} B}\left(\langle y|\tilde{u}\rangle + \langle \tilde{x}|v\rangle - \langle \tilde{x}|\tilde{u}\rangle\right)$

$\geqslant \langle y|u\rangle + \langle x|v\rangle - \langle x|u\rangle$ // now: $\langle y|u\rangle + \langle x|v\rangle = y^Tu + x^Tv = y^Tu + v^Tx = [y^T\ v^T]\begin{bmatrix} u \\ x\end{bmatrix} = \begin{bmatrix} y \\ v\end{bmatrix}^T\begin{bmatrix} u \\ x\end{bmatrix} = \langle (y,v)|(u,x)\rangle$ *

$= \langle (y,v)|(u,x)\rangle - \langle x|u\rangle$

$\Rightarrow F_B(y,v) \geqslant \langle (y,v)|(u,x)\rangle - \langle x|u\rangle \quad \forall_{(y,v)\in\mathcal{H}\times\mathcal{H}}$

$\Leftrightarrow \langle x|u\rangle \geqslant \langle (y,v)|(u,x)\rangle - F_B(y,v) \quad \forall_{(y,v)\in\mathcal{H}\times\mathcal{H}}$

$\Leftrightarrow \langle x|u\rangle \geqslant \sup\limits_{(y,v)\in\mathcal{H}\times\mathcal{H}}\left(\langle (y,v)|(u,x)\rangle - F_B(y,v)\right)$ /* now: $f^*(u) = \sup\limits_{x\in\mathcal{H}}\left(\langle x|u\rangle - f^*(x)\right)$, so:

$\therefore \sup\limits_{(y,v)\in\mathcal{H}\times\mathcal{H}}\left(\langle (y,v)|(u,x)\rangle - F_B(y,v)\right) = F_B^*(u,x)$ */

$= F_B^*(u,x)$

$\therefore (x,u)\in \operatorname{gra} B \Rightarrow F_B^*(u,x) \leqslant \langle x|u\rangle \quad \ldots (eq.:7)$

But $F_B^* \geqslant \langle\cdot|\cdot\rangle$ for any monotone operator by (iii)

$\Big\}\Rightarrow \left((x,u)\in \operatorname{gra} B \Rightarrow F_B^*(u,x) = \langle x|u\rangle\right) \quad \ldots (eq.:7.5)$

$\overset{(eq.:6.5,\ 7.5)}{\therefore B = \text{maximally monotone} \Rightarrow \operatorname{gra} B = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_B^*(u,x) = \langle x|u\rangle\}}$

now recall that, $B :=$ maximal monotone extension of $A$

given in (iv): $A = \text{maximally monotone} \Rightarrow A = B$

$\therefore A:\text{maximally monotone} \Rightarrow \operatorname{gra} A = \{(x,u)\in\mathcal{H}\times\mathcal{H} \mid F_A^*(u,x) = \langle x|u\rangle\}$

■