Part 1

8:53 AM

Theorem 19.1. ⟦ $f\in\Gamma_0(\mathcal{H})$; $g\in\Gamma_0(\mathcal{K})$; $L\in\mathcal{B}(\mathcal{H},\mathcal{K})$; $\operatorname{dom} g\cap L(\operatorname{dom} f)\neq\emptyset$;

$\mu=\inf(f+g\circ L)(\mathcal{H})$; $\mu^*=\inf(f^*\circ L^*+g^{*\vee})(\mathcal{K})$; $x\in\mathcal{H}$; $v\in\mathcal{K}$ ⟧

The following are equivalent:

(i) x: primal solution; v: dual solution; $\mu=-\mu^*$

(ii) $L^*v\in\partial f(x)$, $-v\in\partial g(Lx)$;

(iii) $x\in\partial f^*(L^*v)\cap L^{-1}(\partial g^*(-v))$

Proof:

recall:

- Theorem 16.23. ⟦ $f\in\Gamma_0(\mathcal{H})$; $x\in\mathcal{H}$; $u\in\mathcal{H}$ ⟧ The following are equivalent:

(i) $(x,u)\in\operatorname{gra}\partial f$ ⇔

(ii) $(u,-1)\in N_{\operatorname{epi} f}(x,f(x))$ ⇔

(iii) $f(x)+f^*(u)=\langle x|u\rangle$ ⇔

(iv) $(u,x)\in\operatorname{gra}\partial f^*$

(i)⇔(ii):

x: primal solution, v: dual solution, $\mu=-\mu^*$

⇔ $f(x)+g(Lx)=\mu=-\mu^*=-\left(f^*(L^*v)+g^*(-v)\right)$

⇔ $f(x)+f^*(L^*v)+g(Lx)+g^*(-v)=0$

⇔ $\left(f(x)+f^*(L^*v)-\langle x|L^*v\rangle\right)+\left(g(Lx)+g^*(-v)+\underbrace{\langle x|L^*v\rangle}_{\langle Lx|v\rangle=-\langle Lx|-v\rangle}\right)=0$

⇔ $\underbrace{\left(f(x)+f^*(L^*v)-\langle x|L^*v\rangle\right)}_{(\zeta)}+\underbrace{\left(g(Lx)+g^*(-v)-\langle Lx|-v\rangle\right)}_{(\xi)}=0$ /* using Fenchel-Moreau inequality:

* Proposition 13.13. (Fenchel-Young inequality)

⟦ $f:\mathcal{H}\to\,]-\infty,+\infty]$, proper ⟧ /* wow! see how general the function is, infact any sensible function would satisfy this! */

$\forall_{x\in\mathcal{H}}\forall_{u\in\mathcal{H}}\quad f(x)+f^*(u)\geq\langle x|u\rangle$

so both term (ζ), (ξ) are non-negative, whose addition are zero, so both of them will be 0 seperately */

⇔ $f(x)+f^*(L^*v)-\langle x|L^*v\rangle=0\ \wedge\ g(Lx)+g^*(-v)-\langle Lx|-v\rangle=0$

⇔ $L^*v\in\partial f(x)\ \wedge\ -v\in\partial g(Lx)$ /* using (·) */

(ii)⇔(iii)

(ii): $L^*v\in\partial f(x)\ \wedge\ -v\in\partial g(Lx)$

Corollary 16.24. ⟦ $f\in\Gamma_0(\mathcal{H})$ ⟧ $(\partial f)^{-1}=\partial f^*$

$\underbrace{(\partial f)^{-1}}_{(\partial f)^*}(L^*v)\ni x$ ⇕ $x\in(\partial f)^*(L^*v)$

$\underbrace{(\partial g)^{-1}}_{(\partial g)^*}(-v)\ni Lx$ ⇔ $(\partial g)^*(-v)\ni Lx$ ⇔ $L^{-1}(\partial g)^*(-v)\ni x$

⇔ $x\in\partial f^*(L^*v)\cap L^{-1}(\partial g^*(-v))$

∎

∎ Proposition 19.4.

⟦ $\phi\in\Gamma_0(\mathcal{H})$; $\psi\in\Gamma_0(\mathcal{K})$; $z\in\mathcal{H}$; $r\in\mathcal{K}$; $L\in\mathcal{B}(\mathcal{H},\mathcal{K})$; $r\in\operatorname{sri}(\operatorname{dom}\psi-L\operatorname{dom}\psi)$

- Consider the problem: $\min_{x\in\mathcal{H}}\ \phi(x)+\psi(Lx-r)+\frac{1}{2}\|x-z\|^2$ (19.6)
- together with the problem:

$$\left(\min_{v\in\mathcal{K}}\ {}^{1}(\phi^*)(L^*v+z)+\psi^*(-v)-\langle v|r\rangle\right)\quad(19.7)$$

$$=\left(\min_{v\in\mathcal{K}}\ \frac{1}{2}\|L^*v+z\|^2-{}^{1}\phi(L^*v+z)+\psi^*(-v)-\langle v|r\rangle\right)\quad(19.8)$$

- v: solution to (19.7) ⟧ ⇒

(19.6) has unique solution: $x=\operatorname{Prox}_\phi(L^*v+z)$

Part 2

8:01 AM

Corollary 19.6.

$[\ \varphi \in \Gamma_0(\mathcal{H})\ ;\ z \in \mathcal{H}\ ;\ r \in \mathcal{K}\ ;\ L \in \mathcal{B}(\mathcal{H},\mathcal{K}),\ \|L\| \leq 1\ ;\ -r \in \operatorname{sri}(L(\operatorname{dom}\varphi))\ ;$

$T : \mathcal{K} \to \mathcal{K} : v \mapsto r + v - L(\operatorname{Prox}_\varphi(L^* v + z))\ ] \Rightarrow$

- T: firmly nonexpansive
- $\operatorname{Fix} T \neq \emptyset$
- $\forall_{v \in \operatorname{Fix} T}\ x = \operatorname{Prox}_\varphi(L^* v + z)$: unique solution to $\min\limits_{\substack{x \in \mathcal{H}:\\ Lx = r}} \varphi(x) + \frac{1}{2}\|x - z\|^2$

Proof:

$S = \{v \in \mathcal{K} \mid L(\operatorname{Prox}_\varphi(L^* v + z)) = r\}$

recall: set $\psi = \iota_{\{0\}} : x \mapsto \begin{cases} 0, & \text{if } x = 0 \\ +\infty, & \text{else} \end{cases}$, this has conjugate function: $\psi^*(u) = \sup\limits_{\substack{x \in \operatorname{dom} f\\ = \{0\}}} (\langle x | u\rangle - \psi(x)) = 0$

**

Proposition 19.4. $[\ \varphi \in \Gamma_0(\mathcal{H})\ ;\ \psi \in \Gamma_0(\mathcal{K})\ ;\ z \in \mathcal{H}\ ;\ r \in \mathcal{K}\ ;\ L \in \mathcal{B}(\mathcal{H},\mathcal{K})\ ;\ r \in \operatorname{sri}(\operatorname{dom}\psi - L(\operatorname{dom}\varphi))\ ;$

Consider the problem:

$$\min_{x \in \mathcal{H}} \varphi(x) + \underbrace{\psi(Lx - r)}_{\iota(Lx - r)} + \frac{1}{2}\|x - z\|^2 \qquad (19.6)$$

$= \min\limits_{\substack{x \in \mathcal{H}\\ Lx = r}} \varphi(x) + \frac{1}{2}\|x - z\|^2$: which is same as the problem $(\cdot)$

together with the problem:

$$\left(\min_{v \in \mathcal{K}} {}^1(\varphi^*)(L^* v + z) + \psi^*(-v) - \langle v | r\rangle\right) \qquad (19.7)$$

$$= \left(\min_{v \in \mathcal{K}} \frac{1}{2}\|L^* v + z\|^2 - {}^1\varphi(L^* v + z) + \psi^*(-v) - \langle v | r\rangle\right) \qquad (19.8)$$

v : solution to (19.7) $]$ (19.6) has unique solution: $x = \operatorname{Prox}_\varphi(L^* v + z)$

becomes

$$\min_{v \in \mathcal{K}} {}^1(\varphi^*)(L^* v + z) + \underbrace{\psi^*(-v)}_{0} - \langle v | r\rangle$$

$$= \min_{v \in \mathcal{K}} \underbrace{{}^1(\varphi^*)(L^* v + z) - \langle v | r\rangle}_{\text{gradient} = 0 \text{ at optimal } v} \qquad (0)$$

/*

* Some important implications of Moreau's theorem:

- $(f \boxdot \frac{1}{2}\|\cdot\|^2) + (f^* \boxdot \frac{1}{2}\|\cdot\|^2) = \frac{1}{2}\|\cdot\|^2$
- $\operatorname{Prox}_f + \operatorname{Prox}_{f^*} = \operatorname{Id}$
- $\operatorname{Prox}_f = \operatorname{Id} - \nabla(f \boxdot \frac{1}{2}\|\cdot\|^2)^* = \nabla(f^* \boxdot \frac{1}{2}\|\cdot\|^2)$

*/

i.e., $0 = \nabla_v \left[{}^1(\varphi^*)(L^* v + z) - \langle v | r\rangle\right]$

$= \underbrace{L(\operatorname{Prox}_\varphi(L^* v + z)) - r}_{\text{firmly nonexpansive}}$

/* ${}^1(f^*) = (f + \frac{1}{2}\|\cdot\|^2)^* = f^* \boxdot \frac{1}{2}\|\cdot\|^2$

$\nabla({}^1(f^*)) = \nabla(f^* \boxdot \frac{1}{2}\|\cdot\|^2) = \operatorname{Prox}_f$ */

so. $\nabla_v {}^1(\varphi^*)(v) = \operatorname{Prox}_\varphi(\cdot)$

$\Rightarrow \nabla_v {}^1(\varphi^*)(L^* v + z) = L \operatorname{Prox}_\varphi(L^* v + z)$

also $\nabla_v \langle v | r\rangle = r$ */

$\therefore T : v \mapsto r + v - L(\operatorname{Prox}_\varphi(L^* v + z))$: firmly nonexpansive

/* from

Exercise 4.7 Let $\mathcal{K}$ be a real Hilbert space, let $T : \mathcal{K} \to \mathcal{K}$ be firmly nonexpansive, let $L \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be such that $\|L\| \leq 1$, let $\bar{x} \in \mathcal{H}$, and let $\bar{z} \in \mathcal{K}$. Show that $x \mapsto \bar{x} + L^* T(\bar{z} + Lx)$ is firmly nonexpansive. */

$v \in \operatorname{Fix} T \Leftrightarrow r + v - L(\operatorname{Prox}_\varphi(L^* v + z)) = v$

$\Leftrightarrow L(\operatorname{Prox}_\varphi(L^* v + z)) = v$

$\Leftrightarrow v \in S$

$\therefore \operatorname{Fix} T = S =$ solution set of (0)

$\therefore \forall_{v \in \operatorname{Fix} T}\ x = \operatorname{Prox}_\varphi(L^* v + z)$: unique solution to $\min\limits_{\substack{x \in \mathcal{H}:\\ Lx = r}} \varphi(x) + \frac{1}{2}\|x - z\|^2$

∎

Parametric Duality:

Proposition 19.11.

⟦ $F : \mathcal{H} \times \mathcal{K} \to]-\infty, +\infty]$;

$\vartheta : \mathcal{K} \to [-\infty, +\infty] : y \mapsto \inf F(\mathcal{H}, y)$, the associated value function ⟧ $\Rightarrow$

(i) $\vartheta^* = F^*(0, \cdot)$

(ii) $-\inf F^*(0, \mathcal{K}) = \vartheta^{**}(0) \leq \vartheta(0) = \inf F(\mathcal{H}, 0)$

Proof: Assume F : proper

/* recall

(Definitions for parametric duality)

Definition 19.10 · ⟦ $F : \mathcal{H} \times \mathcal{K} \to]-\infty, +\infty]$ ⟧

Primal problem : $\min_{x \in \mathcal{H}} F(x, 0)$: solution to this is called the primal solution

Dual problem : $\min_{v \in \mathcal{K}} F^*(0, v)$: " " " " " dual solution

Value function : $\vartheta : \mathcal{K} \to [-\infty, +\infty] : y \mapsto \inf F(\mathcal{H}, y)$ // ϑ : vartheta

*/

(i) /* *Proposition 13.28 : **

⟦ $\mathcal{K}$: real Hilbert space

$F : \mathcal{H} \times \mathcal{K} \to]-\infty, +\infty]$, proper

$f : \mathcal{H} \to [-\infty, +\infty] : x \mapsto \inf F(x, \mathcal{K})$ ⟧

$f^* = F^*(\cdot, 0)$ */

in our case, we have $\vartheta : \mathcal{K} \to [-\infty, +\infty] : y \mapsto \inf F(\mathcal{H}, y)$

so. $\vartheta^* = F^*(0, \cdot)$

(ii) *Proposition 12.9. **

⟦ $f : \mathcal{H} \to [-\infty, +\infty]$ ⟧ $\Rightarrow$

(i) $f^*(0) = -\inf f(\mathcal{H})$ [illegible]

(ii) $-\infty \in f^*(\mathcal{H}) \Leftrightarrow f \equiv +\infty \Leftrightarrow f^* \equiv -\infty$

(iii) f^* : proper $\Rightarrow$ f : proper

(iv) [$u \in \mathcal{H}$]

$f^*(u) = \sup_{x \in \operatorname{dom} f} (\langle x | u \rangle - f(u)) = \sup_{(x, \xi) \in \operatorname{epi} f} (\langle x | u \rangle - \xi)$

(v) $f^* = \iota^*_{\operatorname{gra} f}(\cdot, -1) = \iota^*_{\operatorname{epi} f}(\cdot, -1)$

from (i) we have: $\vartheta^*(x) = F^*(0, x)$

so, $\vartheta^*(0) = -\inf \vartheta^*(\mathcal{K})$

$= -\inf_{x \in \mathcal{K}} F^*(0, x) = -\inf F^*(0, \mathcal{K})$ . . . (1)

again : by definition :

$\vartheta(y) = \inf F(\mathcal{H}, y)$

$\Rightarrow \vartheta(0) = \inf F(\mathcal{H}, 0)$ . . . (2)

/*

*Proposition 13.14. ***

⟦ $f, g : \mathcal{H} \to [-\infty, +\infty]$ ⟧ /* these properties are actually quite useful */

(i) $f^{**} \leq f$ /* Biconjugate is always a lower bound */

(ii) $f \leq g \Rightarrow (f^* \geq g^*, f^{**} \leq g^{**})$ /* Conjugation flips the relation between two functions */

(iii) $f^{***} = f^*$ /* Conjugating thrice and once is the same */

(iv) $(\breve{f})^* = f^*$ /* $\breve{f} = \sup \{ g \in \Gamma(\mathcal{H}) : g \leq f \}$: lower semicontinuous convex envelope of f */

* Conjugating a function and its lower continuous convex envelope gives the same function */

*/

apply to ϑ at 0

$\vartheta^{**}(0) \leq \vartheta(0)$

$\Rightarrow -\inf F^*(0, \mathcal{K}) = \vartheta^{**}(0) \leq \vartheta(0) = \inf F(\mathcal{H}, 0)$

∎

Proposition 19.14.

[$F \in \Gamma_0(\mathcal{H}, \mathcal{K})$; $(x, v) \in \mathcal{H} \times \mathcal{K}$] The following are equivalent:

(i) (x: primal solution,

v: dual solution,

$\inf F(\mathcal{H}, 0) = -\inf F^*(0, \mathcal{K}) \in \mathbb{R}$) $\Leftrightarrow$

(ii) $F(x, 0) + F^*(0, v) = 0$ $\Leftrightarrow$

(iii) $(0, v) \in \partial F(x, 0)$ $\Leftrightarrow$

(iv) $(x, 0) \in \partial F^*(0, v)$

Proof:

(i) $\Rightarrow$ (ii):

given (x: primal solution $\leftrightarrow$ $x = \operatorname{argmin}_{\tilde{x}} F(\tilde{x}, 0)$

v: dual solution $\leftrightarrow$ $v = \operatorname{argmin}_{\tilde{v}} F^*(0, \tilde{v})$

$\inf F(\mathcal{H}, 0) = -\inf F^*(0, \mathcal{K}) \in \mathbb{R}$)

so, $F(x, 0) = \inf F(\mathcal{H}, 0) = -\inf F^*(0, \mathcal{K})$

$= -F^*(0, v) \in \mathbb{R}$

$\Rightarrow F(x, 0) + F^*(0, v) = 0$

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv):

/* recall Theorem 16.23.

Theorem 16.23. [$f \in \Gamma_0(\mathcal{H})$; $x \in \mathcal{H}$; $u \in \mathcal{H}$] The following are equivalent:

(i) $(x, u) \in \operatorname{gra} \partial f$ $\Leftrightarrow$

(ii) $(u, -1) \in N_{\operatorname{epi} f}(x, f(x))$ $\Leftrightarrow$

(iii) $f(x) + f^*(u) = \langle x | u \rangle$ $\Leftrightarrow$

(iv) $(u, x) \in \operatorname{gra} \partial f^*$

*/

$F(x, 0) + F^*(0, v) = 0$

$= \langle (x, 0) | (0, v) \rangle$

$\leftrightarrow ((x, 0), (0, v)) \in \operatorname{gra} \partial F$

$\leftrightarrow \partial F(x, 0) \ni (0, v)$ /* this is (iii) */

$\leftrightarrow ((0, v), (x, 0)) \in \operatorname{gra} \partial F^*$

$\leftrightarrow \partial F^*(0, v) \ni (x, 0)$ /* this is (iv) */

Finally: (iii) $\Rightarrow$ (i) ...

Finally: (iv) $\Rightarrow$ (i)

but (iv) $\Leftrightarrow$ (ii) } so we will prove (ii) $\Rightarrow$ (i)

(ii) says

$$F(x,0) + F^*(0,v) = 0$$

/*

Proposition 19.11. ⟦ $F : \mathcal{H} \times \mathcal{K} \to]-\infty, +\infty]$; $\vartheta : \mathcal{K} \to [-\infty, +\infty] : y \mapsto \inf F(\mathcal{H}, y)$, the associated value function ⟧ $\Rightarrow$

(i) $\vartheta^* = F^*(0, \cdot)$

(ii) $\underbrace{-\inf F^*(0, \mathcal{K})}_{\text{optimal value of the dual problem}} = \vartheta^{**}(0) \leq \vartheta(0) = \underbrace{\inf F(\mathcal{H}, 0)}_{\text{optimal value of the primal problem}}$

*/

$$-F^*(0,v) \leq -\inf F^*(0,\mathcal{K}) \leq \inf F(\mathcal{H},0) \leq F(x,0) \quad \ldots (0)$$

so v: dual solution,

x: primal solution, and

$$F(x,0) + F^*(0,v) \geq 0$$ but

we already have

$$F(x,0) + F^*(0,v) = 0$$

so, inequality (0) collapses: and we have:

$$\inf F(\mathcal{H},0) = -\inf F^*(0,\mathcal{K}) \in \mathbb{R}$$

■