Part 1

8:42 AM

Proposition 18.1.

$[\![f:\mathcal{H}\to]-\infty,+\infty],\ \text{convex};\ x\in \operatorname{cont} f]\!]$

f: Frechet differentiable at $x \Leftrightarrow$

$\forall_{\epsilon\in\mathbb{R}_{++}}\ \exists_{\eta\in\mathbb{R}_{++}}\ \forall_{y\in\mathcal{H}:\|y\|=1}\quad f(x+\eta y)+f(x-\eta y)-2f(x)\leq \eta\epsilon$

Proof: Assume $\mathcal{H}\neq\{0\}$ else it is trivial

($\Rightarrow$)

f: Frechet differentiable at x

/*

* (Frechet differentiability of a function)

$[\,f$: real valued; $\mathcal{H}\to]-\infty,+\infty]$; $x\in\operatorname{dom} f$; $f'(x;\cdot)=\langle\nabla f(x)|\cdot\rangle$: linear, continuous on $\mathcal{H}\,]$

f: Frechet differentiable at $x \overset{\text{def}}{\Leftrightarrow} \lim_{0\neq y\to 0}\dfrac{f(x+y)-f(x)-\langle\nabla f(x)|y\rangle}{\|y\|}=0$

Definition 1.23.0. (Functional limit)

$[\,\mathcal{X}$: Hausdorff space; $f:\mathcal{X}\to[-\infty,+\infty]$; $x\in\mathcal{X}\,]$

$\lim_{y\to x} f(y)=\ell \overset{\text{def}}{\Leftrightarrow} \forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{y\in\mathcal{X}:\, d(x,y)\in]0,\delta[}\ |f(y)-\ell|<\epsilon \Leftrightarrow \forall_{V_\epsilon(\ell)}\ \exists_{V_\delta(x)}\ \forall_{y\in V_\delta(x)}\ f(y)\in V_\epsilon(\ell)$

$\overset{\text{def}}{\Leftrightarrow} \forall_{(x_a)_{a\in A}:\text{net in }\mathcal{X},\, x_a\neq x,\, x_a\to x}\ \lim f(x_a)=\ell$

*/

define $g_x(y)=\dfrac{f(x+y)-f(x)-\langle\nabla f(x)|y\rangle}{\|y\|}=0$

$\therefore \lim_{0\neq y\to 0} g_x(y)=0 \Leftrightarrow \forall_{\tilde\epsilon>0}\ \exists_{\tilde\delta>0}\ \forall_{y:\|y-x\|<\tilde\delta}\ |g_x(y)-0|<\tilde\epsilon$

$\Leftrightarrow \forall_{\tilde\epsilon>0}\ \exists_{\tilde\delta>0}\ \forall_{y:\|y-x\|<\tilde\delta}\ \left|\dfrac{f(x+y)-f(x)-\langle\nabla f(x)|y\rangle}{\|y\|}-0\right| = \underbrace{\left|\dfrac{f(x+y)-f(x)-\langle\nabla f(x)|y\rangle}{\|y\|}\right|}_{\parallel\ \frac{f(x+y)-f(x)-\langle\nabla f(x)|y\rangle}{\|y\|}} < \tilde\epsilon$ /* because $f(\cdot)$: convex, $\forall_y\ f(x+y)-f(x)-\langle\nabla f(x)|y\rangle\geq 0$ */

$\Leftrightarrow \forall_{\tilde\epsilon>0}\ \exists_{\tilde\delta>0}\ \forall_{y:\|y-x\|<\tilde\delta}\ \dfrac{f(x+y)-f(x)-\langle\nabla f(x)|y\rangle}{\|y\|}<\tilde\epsilon$

(incomplete)

Proposition 18.6.

$[\![f,g\in\Gamma_0(\mathcal{H});\ x,y\in\mathcal{H};\ x\in\operatorname{dom}\partial(f\Box g);\ (f\Box g)(x)=f(y)+g(x-y);\ f$: Gateaux differentiable at $y]\!]$

$\partial(f\Box g)(x)=\{\nabla f(y)\}$

Proof: recall

/* Proposition 16.48.

$[\![f,g\in\Gamma_0(\mathcal{H});\ x\in\operatorname{dom}(f\Box g);\ y\in\mathcal{H}]\!]\Rightarrow$

(i) $\underbrace{(f\Box g)(x)=f(y)+g(x-y)}_{\text{given in our case}} \Rightarrow \partial(f\Box g)(x)=\partial f(y)\cap\partial g(x-y)$

(ii) $\partial f(y)\cap\partial g(x-y)\neq\emptyset \Rightarrow (f\Box g)(x)=f(y)+g(x-y)$ */

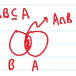
**

Proposition 17.26. $[\,f:\mathcal{H}\to]-\infty,+\infty]$, proper, convex; $x\in\operatorname{dom} f\,]$

(i) f: Gateaux differentiable at $x \Rightarrow \partial f(x)=\{\nabla f(x)\}$

(ii) $(x\in\operatorname{cont} f;\ \partial f(x)=\{u\}) \Rightarrow (f$: Gateaux differentiable at x; $u=\nabla f(x))$

$\partial(f\Box g)(x)=\partial f(y)\cap\partial g(x-y)=\{\nabla f(y)\}\cap\partial g(x-y)\subseteq\{\nabla f(y)\}$ // $\because A\cap B\subseteq A$



/* given: f: Gateaux differentiable at y

$\therefore \partial f(y)=\{\nabla f(y)\}$ */

given, $\partial(f\Box g)(x)\neq\emptyset$, $\{\nabla f(y)\}\subseteq\partial(f\Box g)(x)$ // [illegible]?

$\therefore \partial(f\Box g)(x)=\{\nabla f(y)\}$

∎

Corollary 18.8.

$[\![f,g\in\Gamma_0(\mathcal{H});\ f$: real valued, supercoercive, Frechet differentiable on $\mathcal{H}]\!]\Rightarrow f\Box g$: Frechet differentiable on $\mathcal{H}$.

Proof:

recall:

Proposition 12.14. (Sufficient condition for exact infimal convolution)

$[\![\, f, g \in \Gamma_0(\mathcal{H})$

One of the following holds:

(i) f: supercoercive

(ii) f: coercive, bounded below $]\!]$

$f \square g = f \boxdot g \in \Gamma_0(\mathcal{H})$

/* f: supercoercive $\overset{\text{def}}{\Leftrightarrow} \lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty$

f: coercive $\overset{\text{def}}{\Leftrightarrow} \lim_{\|x\| \to +\infty} f(x) = +\infty$ */

so, $f \square g = f \boxdot g \in \Gamma_0(\mathcal{H})$, as f: supercoercive

take $x \in \mathcal{H}$

/* Proposition 12.6. (ii): $[\![\, f, g \in \mathcal{H} \to]-\infty, +\infty] \,]\!]$ $\operatorname{dom}(f \square g) = \operatorname{dom} f + \operatorname{dom} g$

as f: real valued $\Rightarrow$ $\forall y \in \mathcal{H}$ $f(y) \in \mathbb{R}$ $\Rightarrow$ $\operatorname{dom} f = \{\tilde{x} \in \mathcal{H} \mid f(\tilde{x}) < +\infty\} = \mathcal{H}$

$\operatorname{dom}(f \square g) = \overbrace{\operatorname{dom} f}^{\mathcal{H}} + \operatorname{dom} g = \mathcal{H}$

$\therefore \exists_{y \in \mathcal{H}}$ $y = \underset{\tilde{x} \in \mathcal{H}}{\operatorname{argmin}}\ f(\tilde{x}) + g(x - \tilde{x})$

$(f \square g)(x) = (f \boxdot g)(x) = f(y) + g(x - y)$

Proposition 18.7. $[\![\, f, g \in \Gamma_0(\mathcal{H})$; $x, y \in \mathcal{H}$; $f \square g \in \Gamma_0(\mathcal{H})$, $(f \square g)(x) = f(y) + g(x-y)$;

f: Gateaux differentiable at $y \,]\!] \Rightarrow$

- $x \in \operatorname{cont}(f \square g)$
- $(f \square g)$: Gateaux differentiable at x
- $\nabla (f \square g)(x) = \nabla f(y)$
- f: Frechet differentiable at $y \Rightarrow$ $\boxed{f \square g \text{: Frechet differentiable at } x}$ ☺

$\therefore f \square g$: Frechet differentiable on $\mathcal{H}$.

■

Corollary 18.11.

$[\![\, f \in \Gamma_0(\mathcal{H})$, $\operatorname{dom} \partial f = \operatorname{int} \operatorname{dom} f \,]\!]$

f: Gateaux differentiable on $\operatorname{int} \operatorname{dom} f \Leftrightarrow$

f^*: strictly convex on every nonempty convex subset of $\operatorname{dom} \partial f^*$

$\Rightarrow$

$\operatorname{dom} \partial f^* = \nabla f(\operatorname{int} \operatorname{dom} f)$

Proof: $(\Leftarrow)$

/* Proposition 18.9. given

Proposition 18.9. $[\![\, f \in \Gamma_0(\mathcal{H})$; f^*: strictly convex on every nonempty convex subset of $\operatorname{dom} \partial f^* \,]\!] \Rightarrow f$: Gateaux differentiable on $\operatorname{int} \operatorname{dom} f$.

*/

$\therefore f$: Gateaux differentiable on $\operatorname{int} \operatorname{dom} f$ (0)

$(\Rightarrow)$

given: f: Gateaux differentiable on $\operatorname{int} \operatorname{dom} f$

take $\forall x \in \operatorname{int} \operatorname{dom} f$ $\therefore f$: Gateaux differentiable at x

Proposition 17.26. $[\![\, f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex; $x \in \operatorname{dom} f \,]\!]$

(i) f: Gateaux differentiable at $x \Rightarrow$ $\boxed{\partial f(x) = \{\nabla f(x)\}}$

(ii) $(x \in \operatorname{cont} f$; $\partial f(x) = \{u\}) \Rightarrow (f$: Gateaux differentiable at x; $u = \nabla f(x))$

$\therefore \partial f(x) = \{\nabla f(x)\}$ $\forall x \in \operatorname{int} \operatorname{dom} f$

so $\nabla f(\operatorname{int} \operatorname{dom} f) = \partial f(\operatorname{int} \operatorname{dom} f)$

Proposition 17.26. [$f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex ; $x \in \operatorname{dom} f$]
(i) f: Gateaux differentiable at $x \Rightarrow \boxed{\partial f(x) = \{\nabla f(x)\}}$
(ii) ($x \in \operatorname{cont} f$; $\partial f(x) = \{u\}$) $\Rightarrow$ (f: Gateaux differentiable at x ; $u = \nabla f(x)$)

$\therefore \partial f(x) = \{\nabla f(x)\}$ $\forall x \in \operatorname{int} \operatorname{dom} f$

So $\nabla f(\operatorname{int} \operatorname{dom} f) = \partial f(\operatorname{int} \operatorname{dom} f)$

$= \partial f(\operatorname{dom} \partial f)$ /* given: $\operatorname{dom} \partial f = \operatorname{int} \operatorname{dom} f$ */

$= \operatorname{ran} \partial f$ /* For any T: set valued operator

$T(\operatorname{dom} T) = \operatorname{ran} T$

$T\{x : T(x) \neq \emptyset\} = \bigcup_{x \in \operatorname{dom} T} Tx$

$= \bigcup_{x \in \operatorname{dom} T} Tx \cup \bigcup_{x \notin \operatorname{dom} T} \overbrace{Tx}^{\emptyset}$

$= \bigcup_{x \in X} Tx = \operatorname{ran} T$ */

$= \operatorname{dom} (\partial f)^{-1}$ /* For any set-valued operator: $\operatorname{ran} T = \operatorname{dom}(T^{-1})$ */

$= \operatorname{dom} \partial f^*$ /*

Corollary 16.24. $\underbrace{[f \in \Gamma_0(\mathcal{H})]}_{\text{given}}$ $(\partial f)^{-1} = \partial f^*$

*/

$\therefore f$: Gateaux differentiable on $\operatorname{int} \operatorname{dom} f \Rightarrow \nabla f(\operatorname{int} \operatorname{dom} f) = \operatorname{dom} \partial f^*$ (1)

also from given of ($\Rightarrow$)

Proposition 18.10. [$f \in \Gamma_0(\mathcal{H})$, Gateaux differentiable on $\operatorname{int} \operatorname{dom} f$] $\Rightarrow$
f^*: strictly convex on every nonempty convex subset of $\nabla f(\operatorname{int} \operatorname{dom} f)$

$\therefore f$: Gateaux differentiable on $\operatorname{int} \operatorname{dom} f$
$\Rightarrow f^*$: strictly convex on every nonempty convex subset of

$\nabla f(\operatorname{int} \operatorname{dom} f) = \operatorname{dom} \partial f^*$ (from (1))

... (2)

So the equivalence is established from (0), (2)

the implication follows from (1)

∎

Corollary 18.17.
[$L \in \mathcal{B}(\mathcal{H})$, self-adjoint, positive ; $x \in \mathcal{H}$] $\Rightarrow$
$\|L\| \langle Lx | x \rangle \geq \|Lx\|^2$

Proof:

Denote: $f: \mathcal{H} \to \mathbb{R} : y \mapsto \frac{1}{2} \langle Ly | y \rangle$

recall:

** set $u := 0$, $\tilde{L} := \frac{L}{2}$

Example 2.46. [$L \in \mathcal{B}(\mathcal{H})$; $u \in \mathcal{H}$; $x \in \mathcal{H}$; $f: \mathcal{H} \to \mathbb{R}$, $y \mapsto \langle \tilde{L}y | y \rangle - \langle y | u \rangle$] $\Rightarrow$
f: twice Frechet differentiable on $\mathcal{H}$; $\nabla f(x) = (\tilde{L} + \tilde{L}^*) x - u$; $\nabla^2 f(x) = \tilde{L} + \tilde{L}^*$

f: twice Frechet differentiable on $\mathcal{H}$.

$\nabla f(x) = (\tilde{L} + \tilde{L}^*) x - u$

$= 2\tilde{L} x$ // L: self adjoint.

$= Lx$

So, $\|\nabla f(x) - \nabla f(y)\| = \|Lx - Ly\| = \|L(x-y)\|$ // L: linear

$\leq \|L\| \, \|x - y\|$ // $\because$ any linear T: $\forall_x \; \|T\| \|x\| \geq \|Tx\|$

$\therefore \begin{cases} \nabla f: \text{Lipschitz continuous with constant } \|L\| = \beta \\ f: \text{Frechet differentiable on } \mathcal{H} \end{cases}$

Theorem 18.15. [$f \in \Gamma_0(\mathcal{H})$; $\beta \in \mathbb{R}_{++}$; $h = f^* - \frac{1}{\beta} q$: $q = \frac{1}{2}\|\cdot\|^2$]
The following are equivalent:
(i) $\boxed{f: \text{Frechet differentiable on } \mathcal{H} \wedge \nabla f: \beta\text{-Lipschitz continuous}}$ $\Leftrightarrow$
(ii) f: Frechet differentiable on $\mathcal{H}$ $\wedge$
$\forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} \; \langle x - y | \nabla f(x) - \nabla f(y) \rangle \leq \beta \|x - y\|^2$ $\Leftrightarrow$
(iii) (descent lemma)
f: Frechet differentiable on $\mathcal{H}$ $\wedge$

(ii) f: Frechet differentiable on $\mathcal{H}$ $\wedge$

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \quad \langle x-y \mid \nabla f(x)-\nabla f(y)\rangle \leq \beta\|x-y\|^2 \Leftrightarrow$

(iii) (descent lemma)

f: Frechet differentiable on $\mathcal{H}$ $\wedge$

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \quad f(y)\leq f(x)+\langle y-x \mid \nabla f(x)\rangle+\frac{\beta}{2}\|x-y\|^2 \Leftrightarrow$

(iv) f: Frechet differentiable on $\mathcal{H}$ $\wedge$

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \quad f^*(\nabla f(y)) \geq f^*(\nabla f(x))+\langle x \mid \nabla f(y)-\nabla f(x)\rangle+\frac{1}{2\beta}\|\nabla f(x)-\nabla f(y)\|^2 \Leftrightarrow$

(v) $\boxed{f\text{: Frechet differentiable on } \mathcal{H} \wedge \nabla f : \tfrac{1}{\beta} \text{ cocoercive}} \Leftrightarrow$

(vi) $\beta q - f = \frac{\beta}{2}\|\cdot\|^2 - f$: convex /* this is quite amazing, because in generally this is subtraction of two convex functions */ $\Leftrightarrow$

(vii) $f^* - \frac{1}{\beta} q$: convex /* same as saying f^* : $\frac{1}{\beta}$ strongly convex */ $\Leftrightarrow$

(viii) $h\in\Gamma_0(\mathcal{H}) \wedge f = {}^{(\frac{1}{\beta})}(h^*) = \beta q - {}^{\beta}h \circ \beta\mathrm{Id} \Leftrightarrow$

(ix) $h\in\Gamma_0(\mathcal{H}) \wedge \nabla f = \mathrm{Prox}_{\beta h} \circ \beta\mathrm{Id} = \beta(\mathrm{Id}-\mathrm{Prox}_{h^*/\beta})$

$\nabla f : \left[\frac{1}{\beta}\right]$ cocoercive $=$
$\quad = [1/\|L\|]$

* Definition 4.4. (β-cocoercive / β-inverse strongly monotone)

[D: nonempty subset of $\mathcal{H}$, $T: D\to\mathcal{H}$, $\beta\in\mathbb{R}_{++}$]

T: β-cocoercive (β-inverse strongly monotone) $\overset{\text{def}}{\Leftrightarrow}$ βT: firmly nonexpansive $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{x\in D}\forall_{y\in D} \ \langle x-y \mid Tx-Ty\rangle \geq \beta\|Tx-Ty\|^2$

$$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \quad \underbrace{\langle x-y \mid \underbrace{\nabla f(x)-\nabla f(y)}_{Lx-Ly}\rangle}_{\langle x-y \mid Lx-Ly\rangle} \geq \frac{1}{\|L\|} \underbrace{\|\nabla f(x)-\nabla f(y)\|^2}_{\|Lx-Ly\|^2}$$

$$= \langle x-y \mid L(x-y)\rangle$$

now set $y=0$ then

$$\forall_{x\in\mathcal{H}} \quad \langle x \mid Lx\rangle \geq \frac{1}{\|L\|}\|Lx\|^2$$

$$\Leftrightarrow \forall_{x\in\mathcal{H}} \quad \|L\|\,\langle x \mid Lx\rangle \geq \|Lx\|^2$$

▨

Corollary 18.19.

[$(f_i)_{i\in I}$: finite family of functions in $\Gamma_0(\mathcal{H})$;

$(\alpha_i)_{i\in I}$: finite family of real numbers, $\subseteq\,]0,1]$, $\sum_{i\in I}\alpha_i = 1$;

$q = \frac{1}{2}\|\cdot\|^2$; $h = \left(\sum_{i\in I}\alpha_i\,(f_i^* \,\square\, q)\right)^* - q$]

$\mathrm{Prox}_h = \sum_{i\in I}\alpha_i\,\mathrm{Prox}_{f_i}$

Proof:

$f := \sum_{i\in I}\alpha_i\,(f_i^* \,\square\, q)$

now : $\forall_{i\in I} \ f_i\in\Gamma_0(\mathcal{H})$

$\Rightarrow \forall_{i\in I} \ f_i^*\in\Gamma_0(\mathcal{H})$ /*: *** Corollary 13.38 (An immediate consequence of Fenchel-Moreau theorem) [$f\in\Gamma_0(\mathcal{H})$] • $f^*\in\Gamma_0(\mathcal{H})$ • $f^{**}=f$ */

$\Rightarrow \forall_{i\in I} \ (f_i^* \,\square\, q) = (f_i^* \,\square\, \frac{1}{2}\|\cdot\|^2) = {}^{1}f_i^*$: ($\underbrace{\text{convex, real-valued} \Rightarrow \text{proper, continuous,}}_{\Downarrow\ \in\Gamma_0(\mathcal{H})}$ and Frechet differentiable on $\mathcal{H}$) /*:

* Proposition 12.15. /* Properties of power norm infimal convolution of a convex function */
[$f\in\Gamma_0(\mathcal{H})$
$\gamma\in\mathbb{R}_{++}$
$p\in\,]1,+\infty[$
$g_\gamma^p = f \,\square\, \frac{1}{\gamma p}\|\cdot\|^p : \mathcal{H}\to\,]-\infty,+\infty] : x\mapsto \inf_{y\in\mathcal{H}}\left(f(y)+\frac{1}{\gamma p}\|x-y\|^p\right)$]
• $g_\gamma^p = f \,\square\, \frac{1}{\gamma p}\|\cdot\|^p$: convex, real-valued, continuous, exact, infimum uniquely attained.
/* Moreau envelope satisfies all these properties */

* Proposition 12.29.
[$f\in\Gamma_0(\mathcal{H})$
$\gamma\in\mathbb{R}_{++}$]
${}^{\gamma}f : \mathcal{H}\to\mathbb{R}$, Frechet differentiable : ${}^{\gamma}f = f \,\square\, \frac{1}{2\gamma}\|\cdot\|^2$
$\nabla({}^{\gamma}f) = \frac{1}{\gamma}(\mathrm{Id}-\mathrm{Prox}_{\gamma f}) = \frac{1}{\gamma}\mathrm{Prox}_{(\gamma f)^*}$: $\frac{1}{\gamma}$ Lipschitz continuous */

$\Rightarrow \underbrace{\sum_{i\in I}\alpha_i\,(f_i^* \,\square\, q)}_{f}$: $\in\Gamma_0(\mathcal{H})$ and Frechet differentiable on $\mathcal{H}$ /*: * Proposition 8.15. The set of convex functions from $\mathcal{H}$ to $]-\infty,+\infty]$ is closed under addition and multiplication by strictly positive real numbers, i.e. $(f_i)_{i\in I}$ ... $\sum_{i\in I}\alpha_i f_i$: convex */

with $\nabla f = \nabla\left(\sum_{i\in I}\alpha_i\,(f_i^* \,\square\, q)\right)$

$= \sum_{i\in I}\alpha_i\,\nabla(\underbrace{f_i^* \,\square\, q}_{{}^{1}f_i^*})$

$= \sum_{i\in I}\alpha_i\,\nabla({}^{1}f_i^*)$

$= \sum_{i\in I}\alpha_i\,\mathrm{Prox}_{f_i^{**}}$

$= \sum_{i\in I}\alpha_i\,\mathrm{Prox}_{f_i}$ /* $f_i\in\Gamma_0(\mathcal{H}) \Rightarrow f_i^{**} = f_i$ */

$\therefore f := \sum_{i\in I}\alpha_i\,(f_i^* \,\square\, q)$: $\in\Gamma_0(\mathcal{H})$ and Frechet differentiable on $\mathcal{H}$ with $\nabla f = \sum_{i\in I}\alpha_i\,\mathrm{Prox}_{f_i}$ and 1-Lipschitz continuous . . . (1)

also, Prox_{f_i} : firmly nonexpansive $\Leftrightarrow \frac{1}{2}$ averaged /* * Proposition 12.27. *** [$f\in\Gamma_0(\mathcal{H})$] Prox_f : firmly nonexpansive; $\mathrm{Id}-\mathrm{Prox}_f$: firmly nonexpansive $\Leftrightarrow \mathrm{Prox}_{f^*}$: firmly nonexpansive */

$\therefore \nabla f = \sum_{i\in I}\alpha_i\,\mathrm{Prox}_{f_i}(\cdot)$: $\frac{1}{2}$ averaged (2) $\quad \left(\sum_{i\in I}\alpha_i = 1\right)$
$\Updownarrow$
firmly nonexpansive /* * Proposition 4.30. (Addition of averaged operators) /* A convex combination of α_i-averaged operators is $\max \alpha_i$-averaged */
[D: nonempty subset of $\mathcal{H}$
$(T_i)_{i\in I} : D\to\mathcal{H}$: (• finite family of nonexpansive operators • $\forall_{i\in I}$ T_i : α_i-averaged, $\alpha_i\in\,]0,1[$)
$(\omega_i)_{i\in I} \in\,]0,1]$: $\sum_{i\in I}\omega_i = 1$] $\Rightarrow \sum_{i\in I}\omega_i T_i$: $\left(\max_{i\in I}\alpha_i\right)$ averaged. */

now use .

$(T_i)_{i\in I}$: (• finite family of nonexpansive operators $: D \to \mathcal{H}$

• $\forall_{i\in I}$ T_i : $\underbrace{\alpha_i}_{\in]0,1[}$-averaged)

$\underbrace{(\omega_i)_{i\in I}}_{\in]0,1]}$: $\sum_{i\in I} \omega_i = 1$] $\Rightarrow$ $\sum_{i\in I} \omega_i T_i$: $\left(\max_{i\in I} \alpha_i\right)$: averaged.

*/

now use.

set : $f = \sum_{i\in I} \alpha_i (f_i^* \square q)$, $\beta := 1$,

Theorem 18.15 [$f \in \Gamma_0(\mathcal{H})$; $\beta \in \mathbb{R}_{++}$; $h = f^* - \frac{1}{\beta} q$; $q = \frac{1}{2}\|\cdot\|^2$]

The following are equivalent:

(i) f : Frechet differentiable on $\mathcal{H}$ $\wedge$ ∇f : β-Lipschitz continuous $\Leftrightarrow$ holds by (1)

(ii) f : Frechet differentiable on $\mathcal{H}$ $\wedge$

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $\langle x-y \mid \nabla f(x) - \nabla f(y)\rangle \le \beta \|x-y\|^2$ $\Leftrightarrow$

(iii) (descent lemma)

f : Frechet differentiable on $\mathcal{H}$ $\wedge$

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $f(y) \le f(x) + \langle y-x \mid \nabla f(x)\rangle + \frac{\beta}{2}\|x-y\|^2$ $\Leftrightarrow$

(iv) f : Frechet differentiable on $\mathcal{H}$ $\wedge$

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $f^*(\nabla f(y)) \ge f^*(\nabla f(x)) + \langle x \mid \nabla f(y) - \nabla f(x)\rangle + \frac{1}{2\beta}\|\nabla f(x) - \nabla f(y)\|^2$ $\Leftrightarrow$

(v) f : Frechet differentiable on $\mathcal{H}$ $\wedge$ ∇f : $\frac{1}{\beta}$ cocoercive $\Leftrightarrow$

(vi) $\beta q - f = \frac{\beta}{2}\|\cdot\|^2 - f$: convex. /* this is quite amazing, because in generally this is subtraction of two convex functions */ $\Leftrightarrow$

(vii) $f^* - \frac{1}{\beta} q$: convex /* same as saying f^* : $\frac{1}{\beta}$ strongly convex */ $\Leftrightarrow$

(viii) $h \in \Gamma_0(\mathcal{H})$ $\wedge$ $f = {}^{(1/\beta)}(h^*) = \beta q - {}^{\beta}h \circ \beta \mathrm{Id}$ $\Leftrightarrow$

(ix) $h \in \Gamma_0(\mathcal{H})$ $\wedge$ $\nabla f = \mathrm{Prox}_{\beta h} \circ \beta \mathrm{Id} = \beta(\mathrm{Id} - \mathrm{Prox}_{h^*/\beta})$

$h = \sum_{i\in I} \alpha_i (f_i^* \square q)^* - q \in \Gamma_0(\mathcal{H})$

$\nabla f = \underbrace{(\mathrm{Prox}_h \circ \mathrm{Id})}_{} = (\mathrm{Id} - \mathrm{Prox}_{h^*})$

$\Rightarrow$ $\boxed{\mathrm{Prox}_h = \sum \alpha_i \mathrm{Prox}_{f_i}}$

■