Part 1

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Proposition 16.9.

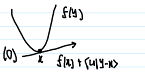
⟦ $f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex ; $x \in \mathcal{H}$ ⟧

$x \in \operatorname{dom} f \Rightarrow \begin{cases} f^{**}(x) = f(x) \\ \partial f^{**}(x) = \partial f(x) \end{cases}$

Proof:

take $u \in \partial f(x)$

$\overset{\text{def}}{\Leftrightarrow} \forall_{y \in \mathcal{H}} \; \langle y - x \mid u \rangle + f(x) \leq f(y)$ (0)



so this first order approximation of f at x acts as a continuous affine minorant of f

/* recall:

Proposition 13.39. *

[$f: \mathcal{H} \to]-\infty, +\infty]$]

- f: has a continuous affine minorant $\Rightarrow f^{**} = \breve{f}$
  $\Leftrightarrow \operatorname{dom} f^* \neq \emptyset$ (from proposition 13.12(ii))
- f: doesnot have a continuous affine minorant $\Rightarrow f^{**} = -\infty$

/* This theorem can have a practical significance, when the first antecedent holds, even if the function f is highly nonconvex, its convex envelope can be found by bi conjugating the original problem, as conjugating may be a relatively easy optimization problem, that is very good practically */

*/

$f^{**} = \breve{f}$

/* recall:

---

*Corollary 9.10.

[$f: \mathcal{H} \to [-\infty, +\infty]$, convex] $\Rightarrow \bar{f} = \breve{f}$

/* $\breve{f} = (g \in \Gamma(\mathcal{H}) \,\|\, g \leq f) \,\|\, \sup_g$ [?] — lower semicontinuous convex envelope

$\bar{f} = (g$: lower semicontinuous $\| g \leq f) \,\|\, \sup_g$ [?] — lower semicontinuous envelope */

* Lemma 1.31.

[$\mathcal{X}$: hausdorff space,
$f: \mathcal{X} \to [-\infty, +\infty]$] $\Rightarrow$

(i) $\bar{f}$: largest lower semicontinuous function majorized by f

(ii) epi $\bar{f}$: closed

(iii) $\operatorname{dom} f \subseteq \operatorname{dom} \bar{f} \subseteq \overline{\operatorname{dom}} f$, (iv) $\forall_{x \in \mathcal{X}} \; \bar{f}(x) = \underline{\lim}_{y \to x} f(y)$

(v) $x \in \mathcal{X} \Rightarrow (f$: lower semicontinuous at $f \Leftrightarrow \bar{f}(x) = f(x))$

(vi) epi $\bar{f} = \overline{\text{epi}}\, f$.

and finally recall that:

Proposition 16.3. [$f \in \mathcal{H} \to]-\infty, +\infty]$, proper, $x \in \operatorname{dom} f$] $\Rightarrow$

(i) $\operatorname{dom} \partial f \subseteq \operatorname{dom} f$

(ii) $\partial f(x) = \bigcap_{y \in \operatorname{dom} f} \{u \in \mathcal{H} \mid \langle y - x \mid u \rangle \leq f(y) - f(x)\}$

(iii) $\partial f(x)$: closed, convex

(iv) $x \in \operatorname{dom} \partial f \Rightarrow f$: lower semicontinuous at x */

$\Updownarrow$

$f = \bar{f}$ /* i.e., f will be its own lower-semicontinuous envelope at x */

at x: $f(x) = \bar{f}(x)$

$f^{**}(x) = \breve{f}(x) = \bar{f}(x) = \underline{\lim}_{y \to x} f(x) = f(x)$ (0)

thus from (0) and (1) we have:

$\forall_{y \in \mathcal{H}} \; \langle y - x \mid u \rangle + f^{**}(x) \leq f^{**}(y)$

$\Leftrightarrow u \in \partial f^{**}(x)$ (2)

$\therefore \partial f(x) \subseteq \partial f^{**}(x)$

Now consider the other side: take $u \in \partial f^{**}(x) \Leftrightarrow \forall_{y \in \mathcal{H}} \; \langle y - x \mid u \rangle + f^{**}(x) \leq f^{**}(y)$ /* recall for any function, f its bi-conjugate is a lower bound for the function.

So, $f^{**} \leq f$ /* Proposition 16.4 */ */

$\Rightarrow \forall_{y \in \mathcal{H}} \; \langle y - x \mid u \rangle + f^{**}(x) \leq f(y)$

now as f: proper, convex, $x \in \operatorname{dom} \partial f$ // By given

$\Updownarrow$

f: lower semicontinuous at x

$\Leftrightarrow f$: proper, convex, lower-semicontinuous at x

$\Leftrightarrow f^{**}(x) = f(x)$ // From Fenchel-Moreau theorem $f \in \Gamma_0(\mathcal{H}) \Rightarrow f^{**} = f$

(3)

$\forall_{y \in \mathcal{H}} \; \langle y - x \mid u \rangle + f(x) \leq f(y)$

$\Leftrightarrow u \in \partial f(x)$

So, we have proven, $\partial f^{**}(x) \subseteq \partial f(x)$ (4)

from (0), (2), (3), (4):

$f^{**}(x) = f(x)$

$\partial f^{**}(x) = \partial f(x)$ ∎

Proposition 16.21. **

[$f \in \Gamma_0(\mathcal{H})$] $\operatorname{int} \operatorname{dom} f = \operatorname{cont} f \subseteq \operatorname{dom} \partial f \subseteq \operatorname{dom} f$

Proof:

/* Recall: as $f \in \Gamma_0(\mathcal{H}) \Rightarrow$

*Corollary 8.30. *

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper ☑, convex ☑,

one of the following holds

- f: bounded above on some neighborhood
- f: lower semicontinuous ☑
- $\mathcal{H}$: finite-dimensional] $\Rightarrow$ $\boxed{\operatorname{cont} f = \operatorname{int} \operatorname{dom} f}$ /* cont f: domain of continuity of a function f */

☑

So. $\operatorname{int} \operatorname{dom} f = \operatorname{cont} f$ (1)

/* now

Proposition 16.3. [$f \in \mathcal{H} \to]-\infty, +\infty]$ ☑, proper ☑, $x \in \operatorname{dom} f$ ☑] $\Rightarrow$ → $\operatorname{dom} \partial f \subseteq \operatorname{dom} f$ (2)

(i) $\operatorname{dom} \partial f \subseteq \operatorname{dom} f$ ☑

(ii) $\partial f(x) = \bigcap_{y \in \operatorname{dom} f} \{u \in \mathcal{H} \mid \langle y - x \mid u \rangle \leq f(y) - f(x)\}$

(iii) $\partial f(x)$: closed, convex

(iv) $x \in \operatorname{dom} \partial f \Rightarrow f$: lower semicontinuous at x

Proposition 16.14. [$f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex ; $x \in \operatorname{dom} f$] $\Rightarrow$

(i) [$\operatorname{int} \operatorname{dom} f \neq \emptyset$, $x \in \operatorname{bdry} \operatorname{dom} f$] $\Rightarrow \partial f(x)$: empty or unbounded

(ii) $x \in \operatorname{cont} f \Rightarrow \partial f(x)$: nonempty, weakly compact

(iii) $x \in \operatorname{cont} f \Rightarrow \exists_{\delta \in \mathbb{R}_{++}} \; \partial f(B(x;\delta))$: bounded

(iv) $\operatorname{cont} f \neq \emptyset \Rightarrow \operatorname{int} \operatorname{dom} f \subseteq \operatorname{dom} \partial f$ ☑

*/

(ii) $x \in \operatorname{cont} f \Rightarrow \partial f(x)$: nonempty, weakly compact

(iii) $x \in \operatorname{cont} f \Rightarrow \exists_{\delta \in \mathbb{R}_{++}} \; \partial f(B(x;\delta))$: bounded

(iv) $\operatorname{cont} f \neq \emptyset \Rightarrow \operatorname{int} \operatorname{dom} f \subseteq \operatorname{dom} \partial f$ ☑

*/

assume $\operatorname{int}\operatorname{dom} f = \operatorname{cont} f \neq \emptyset$ /* otherwise, $\operatorname{int}\operatorname{dom} f \subseteq \operatorname{dom}\partial f$ trivially */

(from (1))

↳ then $\operatorname{int}\operatorname{dom} f = \operatorname{cont} f \subseteq \operatorname{dom}\partial f$ (3)

from (2), (3):

$$\operatorname{int}\operatorname{dom} f = \operatorname{cont} f \subseteq \operatorname{dom}\partial f \subseteq \operatorname{dom} f$$

■

Proposition 16.28.

$f \in \Gamma_0(\mathcal{H}) \Rightarrow \operatorname{gra}(f + \iota_{\operatorname{dom}\partial f})$: dense subset of $\operatorname{gra} f$

$$\leftrightarrow \forall_{x \in \operatorname{dom} f} \; \exists_{(x_n)_{n\in\mathbb{N}} \subseteq \operatorname{dom}\partial f} \; (x_n \to x \wedge f(x_n) \to f(x))$$

Proof:

$x \in \operatorname{dom} f$, $\varepsilon \in \mathbb{R}_{++}$

$(p, \pi) = P_{\operatorname{epi} f}(x, f(x) - \varepsilon)$ . . . (i)

/* proposition 9.18.

[$f \in \Gamma_0(\mathcal{H})$, $x \in \operatorname{dom} f$, $\xi \in \,]-\infty, f(x)[$]

$$(p,\pi) = P_{\operatorname{epi} f}(x,\xi) \leftrightarrow \begin{cases} \xi < f(p) = \pi \\ \forall_{y \in \operatorname{dom} f} \; \langle y-p \mid x-p\rangle \leq (f(y)-f(p))(f(p)-\xi) \end{cases}$$

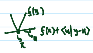
*/

using this: (i) is equivalent to:

$f(x) - \varepsilon < f(p) = \pi \leftrightarrow f(x) - f(p) < \varepsilon$ . . . (0)

$\forall_{y \in \operatorname{dom} f} \; \langle y-p \mid x-p\rangle \leq (f(y)-f(p))(f(p)-f(x)+\xi)$

/* recall: $u \in \partial f(x) \overset{\text{def}}{\leftrightarrow} \forall_{y \in \operatorname{dom} f} \; f(y) \geq f(x) + \langle u \mid y-x\rangle$ */ . . . (ii)



$\leftrightarrow \forall_{y \in \operatorname{dom} f} \; (f(y)-f(p))(f(p)-f(x)+\xi) \geq \left\langle y-p \,\middle|\, \dfrac{x-p}{f(p)-f(x)+\xi}\right\rangle + f(p)$

$\leftrightarrow \forall_{y \in \operatorname{dom} f} \; f(y) \geq f(p) + \left\langle \dfrac{x-p}{f(p)-f(x)+\xi} \,\middle|\, y-p \right\rangle$ . . . (iii)

From (ii) and (iii):

$$\underbrace{\frac{x-p}{f(p)-f(x)+\xi}}_{\text{finite}} \in \partial f(p) \quad // \operatorname{dom} A = \{x : Ax \neq \emptyset\}$$

$\Rightarrow p \in \operatorname{dom}\partial f$ as $\partial f(p) \neq \emptyset$ . . . (iv)

in (iii) set $y := x$ we have:

$$f(x) \geq f(p) + \frac{1}{f(p)-f(x)+\xi}\|x-p\|^2$$

$$\leftrightarrow f(x) - f(p) \geq \frac{1}{-(f(x)-f(p)-\xi)}\|x-p\|^2$$

$$\leftrightarrow -\|f(x)-f(p)\|^2 + \xi(f(x)-f(p)) \geq \|x-p\|^2$$

$$\leftrightarrow \|(x-p)\|^2 + \|f(x)-f(p)\|^2 \leq \xi(f(x)-f(p))$$

$$\leftrightarrow \|(x,f(x))-(p,f(p))\|^2 \leq \varepsilon\underbrace{(f(x)-f(p))}_{<\varepsilon \text{ /* from (0) */}} < \varepsilon^2 \quad . . (v)$$

so, essentially we have proven that

$\forall_{x \in \operatorname{dom} f} \; \forall_{\varepsilon \in \mathbb{R}_{++}} \; (p,\pi) = P_{\operatorname{epi} f}(x, f(x)-\varepsilon) \Rightarrow p \in \operatorname{dom}\partial f \wedge \|(x,f(x))-(p,f(p))\|^2 < \varepsilon$

for $\varepsilon = \frac{1}{n}$ we can construct sequence $(\underbrace{x_n}_{\in \operatorname{dom} f})_{n\in\mathbb{N}} : (x_n, \pi_n) = P_{\operatorname{epi} f}(x, f(x) - \frac{1}{n})$ which will

satisfy: $x_n \in \operatorname{dom}\partial f$ and $\forall_n \; \|(x,f(x))-(x_n,f(x_n))\|^2 < \frac{1}{n^2}$ /* by taking $n \to \infty$, $x_n \to x$, $f(x_n) \to f(x)$ */

$\Rightarrow x_n \to x$, $f(x_n) \to f(x)$

■

Proposition 16.13.

[$f : \mathcal{H} \to \,]-\infty, +\infty]$, proper, convex ;

$x \in \mathcal{H}$; $u \in \mathcal{H}$] $\Rightarrow$

$$\left.\begin{aligned} u \in \partial f(x) &\Leftrightarrow (u,-1) \in N_{\operatorname{epi} f}(x, f(x)) \\ &\Leftrightarrow f(x) + f^*(u) = \langle x \mid u\rangle \end{aligned}\right\} \Rightarrow x \in \partial f^*(u)$$

Proof:

f : convex, proper

$\Rightarrow \operatorname{epi} f$: convex, nonempty

now: $(u,-1) \in N_{\operatorname{epi} f}(x, f(x))$ /* recall that: $N_C(x) = \begin{cases} (C-x)^\ominus = \{u \in \mathcal{H} \mid \sup\langle C-x \mid u\rangle \leq 0\}, & \text{if } x \in C \\ \emptyset, & \text{else} \end{cases}$ */

$$\therefore N_{\operatorname{epi} f}(x, f(x)) = \begin{cases} (\operatorname{epi} f - (x, f(x)))^\ominus = \{(\tilde{u},\tilde{v}) \in \mathcal{H}\times\mathbb{R} \mid \sup\langle \operatorname{epi} f - (x,f(x)) \mid (\tilde{u},\tilde{v})\rangle \leq 0\}, & \text{if } (x, f(x)) \in \operatorname{epi} f \\ \emptyset, & \text{else} \end{cases}$$

↳ this cannot be the case as $(u,-1) \in N_{\operatorname{epi} f}(x, f(x))$

so, $(u,-1) \in N_{\operatorname{epi} f}(x, f(x))$

$\Leftrightarrow \sup\langle \operatorname{epi} f - (x, f(x)) \mid (u,-1)\rangle \leq 0$, $(\underbrace{x}_{\in\mathcal{H}}, \underbrace{f(x)}_{\in\mathbb{R}}) \in \operatorname{epi} f$

⇕

$f(x) \leq f(x) \in \mathbb{R}$

⇕

$x \in \operatorname{dom} f$, as $f : \mathcal{H} \to \,]-\infty, \infty]$

$\Leftrightarrow \forall_{(y,\eta) \in \operatorname{epi} f} \; \langle (y,\eta) - (x, f(x)) \mid (u,-1)\rangle \leq 0$, $x \in \operatorname{dom} f$

$f(x) \leq f(x) \in \mathbb{R}$

$\Updownarrow$

$x \in \operatorname{dom} f$, as $f: \mathcal{H} \to]-\infty, \infty]$

$\Leftrightarrow \forall_{(y,\eta) \in \operatorname{epi} f} \; \langle (y,\eta) - (x, f(x)) \mid (u,-1) \rangle \leq 0, \quad x \in \operatorname{dom} f$

$\left\langle \begin{bmatrix} y-x \\ \eta - f(x) \end{bmatrix} \middle| \begin{bmatrix} u \\ -1 \end{bmatrix} \right\rangle = \langle y-x \mid u \rangle + \langle \eta - f(x) \mid -1 \rangle = \langle y-x \mid u \rangle - \eta + f(x)$

(number, number: $(-1)(\eta - f(x)) = -\eta + f(x)$)

$\Leftrightarrow \forall_{(y,\eta) \in \operatorname{epi} f} \; \langle y-x \mid u \rangle + f(x) \leq \eta, \quad x \in \operatorname{dom} f$

$\Updownarrow$

$f(y) \leq \eta \in \mathbb{R}, \; y \in \operatorname{dom} f$

$\Leftrightarrow \forall_{y \in \operatorname{dom} f} \; \langle y-x \mid u \rangle + f(x) \leq f(u)$

/* miniproof: $\forall_{(y,\eta) \in \operatorname{epi} f} \langle y-x \mid u \rangle + f(x) \leq \eta \Leftrightarrow \forall_{y \in \operatorname{dom} f} \langle y-x \mid u \rangle + f(x) \leq f(y)$

($\Rightarrow$): set $\eta = f(y)$ then $(y, f(y)) \in \operatorname{epi} f$ which gives $\forall_{y \in \operatorname{dom} f} \langle y-x \mid u \rangle + f(x) \leq f(y)$

($\Leftarrow$): have: $\forall_{y \in \operatorname{dom} f} \langle y-x \mid u \rangle + f(x) \leq f(y)$, $\forall_{(y,\eta)} (y,\eta) \in \operatorname{epi} f \Leftrightarrow f(y) \leq \eta \to \forall_{(y,\eta) \in \operatorname{epi} f} \langle y-x \mid u \rangle + f(x) \leq f(y) \leq \eta$

$\therefore \forall_{(y,\eta) \in \operatorname{epi} f} \langle y-x \mid u \rangle + f(x) \leq \eta$ */

$\Leftrightarrow \quad u \in \partial f(x)$

/* Proposition 16.9.
[$f: \mathcal{H} \to]-\infty, +\infty]$, proper ;
$x \in \mathcal{H}$; $u \in \mathcal{H}$]
$u \in \partial f(x) \Leftrightarrow f(x) + f^*(u) = \langle x \mid u \rangle \Rightarrow x \in \partial f^*(u)$ */

the rest of the proof follows from this ∎

Proposition 16.9.

[[$f: \mathcal{H} \to]-\infty, +\infty]$, proper ; $x \in \mathcal{H}$; $u \in \mathcal{H}$]]

$u \in \partial f(x) \Leftrightarrow f(x) + f^*(u) = \langle x \mid u \rangle \Rightarrow x \in \partial f^*(u)$

Proof:

$u \in \partial f(x) \overset{\text{def}}{\Leftrightarrow} \forall_{y \in \operatorname{dom} f} \; \underbrace{\langle y - x \mid u \rangle}_{\langle y|u \rangle - \langle x|u \rangle} + f(x) \leq f(y)$

$\Leftrightarrow \forall_{y \in \operatorname{dom} f} \; \langle y \mid u \rangle - f(y) \leq \langle x \mid u \rangle - f(x)$

$\Leftrightarrow \underbrace{\sup_{y \in \operatorname{dom} f} \langle y \mid u \rangle - f(y)}_{f^*(u)} \leq \langle x \mid u \rangle - f(x)$

$\Leftrightarrow f^*(u) \leq \langle x \mid u \rangle - f(x)$

$\Leftrightarrow f^*(u) + f(x) \leq \langle x \mid u \rangle \quad \ldots (1)$

But from Fenchel-Moreau inequality:

*Proposition 13.13. (Fenchel-Young inequality)
[$f: \mathcal{H} \to]-\infty, +\infty]$, proper] /* wow! see how general the function is, infact any sensible function would satisfy this! */
$\forall_{x \in \mathcal{H}} \forall_{u \in \mathcal{H}} \; f(x) + f^*(u) \geq \langle x \mid u \rangle$

$\Leftrightarrow f(x) + f^*(u) = \langle x \mid u \rangle$ /* ∵ Fenchel Moreau is an independent piece of result */

(2)

*Proposition 13.14. ★ ★ ★
[$f, g: \mathcal{H} \to [-\infty, +\infty]$] /* these properties are actually quite useful */

(i) $f^{**} \leq f$ /* Biconjugate is always a lower bound */

(ii) $f \leq g \Rightarrow (f^* \geq g^*, \; f^{**} \leq g^{**})$ /* Conjugation flips the relation between two functions */

(iii) $f^{***} = f^*$ /* Conjugating thrice and once is the same */

(iv) $(\breve{f})^* = f^*$ /* $\breve{f} = \sup\{g \in \Gamma(\mathcal{H}) : g \leq f\}$: lower semicontinuous convex envelope of f */

setting $f := f^*$

$\forall_{x \in \mathcal{H}} \forall_{u \in \mathcal{H}} \; \langle x \mid u \rangle \leq f^*(x) + f^{**}(u)$

interchanging x, u

$\Leftrightarrow$

$\forall_{u \in \mathcal{H}} \forall_{x \in \mathcal{H}} \; \langle u \mid x \rangle \leq f^*(u) + f^{**}(x) \leq f^*(u) + f(x) = \langle u \mid x \rangle$ /* using (2) which is equivalent to $u \in \partial f(x)$ */

↓ collapse

$\therefore f^*(u) + f^{**}(x) = \langle u \mid x \rangle$ /* use (2) again */

$\Leftrightarrow x \in \partial f^*(u)$

∎

Part 2

10:23 PM

Proposition 16.34.

⟦ $f \in \Gamma_0(\mathcal{H})$; $x, p \in \mathcal{H}$ ⟧

$p = \mathrm{Prox}_f x \Leftrightarrow x - p \in \partial f(p)$

i.e., $\mathrm{Prox}_f = (\mathrm{Id} + \partial f)^{-1}$

Proof: /* */

Proposition 12.26. (Defining properties of Prox operator) ★★ /* This shows the similarities between projection and proximal operator */

[$f \in \Gamma_0(\mathcal{H})$; $x, p \in \mathcal{H}$]

$p = \mathrm{Prox}_f x \Leftrightarrow \forall_{y \in \mathcal{H}} \; \langle y - p \mid x - p \rangle + f(p) \leq f(y)$

Definition of subdifferential: $\forall_{y \in \mathcal{H}} \; \langle y - p \mid \overbrace{x - p}^{\tilde{u}} \rangle + f(p) \leq f(y) \Leftrightarrow (x - p) \in \partial f(p)$:)

Definition 16.1. [$f : \mathcal{H} \to]-\infty, +\infty]$, proper] (Definition of subdifferential)

∂f : subdifferential of $f \overset{\text{def}}{\Leftrightarrow} \partial f : \mathcal{H} \to 2^{\mathcal{H}} : x \mapsto \{u \in \mathcal{H} \mid \forall_{y \in \mathcal{H}} \; \langle y - x \mid u \rangle + f(x) \leq f(y)\}$

now $x - p \in \partial f(p) \Leftrightarrow x \in \partial f(p) + p = (\mathrm{Id} + \partial f)(p)$

$\therefore \; (\mathrm{Id} + \partial f)^{-1}(x) \ni p \Leftrightarrow p = \mathrm{Prox}_f x$ (1)

so, $(\mathrm{Id} + \cdot)^{-1} = \mathrm{Prox}_f x$ (2)   But $\mathrm{Prox}_f(\cdot)$: is a singleton operator

so, $(\mathrm{Id} + \partial f)^{-1}$ " " " "

thus (1) becomes: $p = (\mathrm{Id} + \partial f)^{-1}(x)$

■

Proposition 16.32.

[$\mathcal{K}$: real Hilbert space ; $f \in \Gamma_0(\mathcal{H})$; $g \in \Gamma_0(\mathcal{K})$; $L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$: $L(\mathrm{dom} f) \cap (\mathrm{dom}\, g) \neq \emptyset$; $(f + g \circ L)^* = f^* \square (L^* \triangleright g^*)$]]

$\partial(f + g \circ L) = \partial f + L^* \circ \partial g \circ L$

Proof:

*

Proposition 16.5. [$\mathcal{K}$: real Hilbert space ; $f : \mathcal{H} \to]-\infty, +\infty]$; $g : \mathcal{K} \to]-\infty, +\infty]$, proper ; $L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; $\lambda \in \mathbb{R}_{++}$]

(i) $\partial(\lambda f) = \lambda \partial f$

(ii) $\mathrm{dom}\, g \cap L(\mathrm{dom} f) \neq \emptyset \Rightarrow \partial f + L^* \circ (\partial g) \circ L \subseteq \partial(f + g \circ L)$ ✓

so, we have: $\partial f + L^* \circ (\partial g) \circ L \subseteq \partial(f + g \circ L)$ (-1)

and we need to prove. $\partial(f + g \circ L) \subseteq \partial f + L^* \circ (\partial g) \circ L$

take $(x, u) \in \mathrm{gra}\, \partial(f + g \circ L)$ (0) $\Leftrightarrow (f + g \circ L)(x) + (f + g \circ L)^*(u) = \langle x \mid u \rangle$ (1)

Proposition 16.9.
[$f : \mathcal{H} \to]-\infty, +\infty]$, proper ; $x \in \mathcal{H}$; $u \in \mathcal{H}$]
$u \in \partial f(x) \Leftrightarrow f(x) + f^*(u) = \langle x \mid u \rangle \Rightarrow x \in \partial f^*(u)$

also, given: $(f + g \circ L)^* = f^* \square (L^* \triangleright g^*)$

$\therefore \; (f + g \circ L)^*(u) = \left(f^* \square (L^* \triangleright g^*)\right)(u)$

$= \min_{\tilde{y} \in \mathcal{H}} \left((L^* \triangleright g^*)(\tilde{y}) + f^*(u - \tilde{y}) \right)$ /* say the optimizer is y^* */

$= (L^* \triangleright g^*)(y^*) + f^*(u - y^*)$

/* $\left(\begin{array}{l} \min \; g^*(\tilde{x}) \\ \text{s.t.} \; L^* \tilde{x} = y^* \end{array} \right)$ say the minimizer is v, then $y^* = L^* v$ */

$= g^*(v) + f^*(u - y^*) = g^*(v) + f^*(u - L^* v)$ ($L^* v = y^*$)

so, we have: $(f + g \circ L)^*(u) = g^*(v) + f^*(u - L^* v)$ (2)

$\underbrace{(f + g \circ L)(x)}_{f(x) + g(Lx)} + g^*(v) + f^*(u - L^* v) = \underbrace{\langle x \mid u \rangle}_{\langle x \mid u - L^* v + L^* v \rangle}$

$= \langle x \mid u - L^* v \rangle + \langle x \mid L^* v \rangle$

$\Leftrightarrow \underbrace{\left(f(x) + f^*(u - L^* v) - \langle x \mid u - L^* v \rangle\right)}_{\geq 0} + \underbrace{\left(g(Lx) + g^*(v) - \langle x \mid L^* v \rangle\right)}_{\geq 0} = 0$ /* now using Fenchel-Young inequality

* Proposition 13.13. (Fenchel-Young inequality)

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper] /* wow! See how general the function is, infact any sensible function would satisfy this! */ */

$\Leftrightarrow \underbrace{\left(f(x)+f^{*}(u-L^{*}v)-\langle x|u-L^{*}v\rangle\right)}_{\geq 0}+\underbrace{\left(g(Lx)+g^{*}(v)-\langle x|L^{*}v\rangle\right)}_{\geq 0}=0$ /* now using Fenchel-Young inequality

> *Proposition 13.13. (Fenchel-Young inequality)
> [$f:\mathcal{H}\to]-\infty,+\infty]$, proper] /* wow! see how general the function is, infact any sensible function would satisfy this! */
> $\forall_{x\in\mathcal{H}}\ \forall_{u\in\mathcal{H}}\quad f(x)+f^{*}(u)\geq\langle x|u\rangle$

*/

so both of these terms must be seperately zero: /* if sum of two nonegative numbers is zero, then they seperately must be zero */

$f(x)+f^{*}(u-L^{*}v)=\langle x|u-L^{*}v\rangle$ and

$g(Lx)+g^{*}(v)=\langle x|L^{*}v\rangle\overset{\text{def}}{=}\langle v|Lx\rangle$

using

> Proposition 16.9.
> [$f:\mathcal{H}\to]-\infty,+\infty]$, proper ; $x\in\mathcal{H}$; $u\in\mathcal{H}$]
> $u\in\partial f(x)\Leftrightarrow f(x)+f^{*}(u)=\langle x|u\rangle\Rightarrow x\in\partial f^{*}(u)$

$u-L^{*}v\in\partial f(x)$

$v\in\partial g(Lx)$

$\Rightarrow L^{*}v\in L^{*}\partial g(Lx)$

$\Rightarrow u\in L^{*}v+\partial f(x)\subseteq L^{*}g(Lx)+\partial f(x)=\left(L^{*}\circ g\circ L+\partial f\right)(x)$

$\updownarrow$

$(x,u)\in \operatorname{gra}\left(\partial f+L^{*}\circ(\partial g)\circ L\right)$ (3)

so, from (0) and (3) we have:

$\partial(f+g\circ L)\subseteq \operatorname{gra}\left(\partial f+L^{*}\circ(\partial g)\circ L\right)$ (4)

so, from (-1), (4) we have:

$\partial(f+g\circ L)=\partial f+L^{*}\circ(\partial g)\circ L$

∎

Proposition 16.46.

[$\mathcal{K}$: real Hilbert space; $F\in\Gamma_0(\mathcal{H}\times\mathcal{K})$; $f:\mathcal{H}\to[-\infty,+\infty]:x\mapsto\inf F(x,\mathcal{K})$;

f: proper ; $f(x)=F(x,y)$; $u\in\mathcal{H}$] $u\in\partial f(x)\Leftrightarrow(u,0)\in\partial F(x,y)$

$\updownarrow$

$y=\underset{\tilde{y}\in\mathcal{K}}{\operatorname{argmin}}\,F(x,\tilde{y})$

Proof:

> *Proposition 13.28: **
> [$\mathcal{K}$: real Hilbert space
> $F:\mathcal{H}\times\mathcal{K}\to]-\infty,+\infty]$, proper
> $f:\mathcal{H}\to[-\infty,+\infty]:x\mapsto\inf F(x,\mathcal{K})$]
> $f^{*}=F^{*}(\cdot,0)$

we have: $f^{*}(u)=F^{*}(u,0)$

> Proposition 16.9. (Very important theorem, subdifferential membership can be tested using it */
> [$f:\mathcal{H}\to]-\infty,+\infty]$, proper ; $x\in\mathcal{H}$; $u\in\mathcal{H}$]
> $u\in\partial f(x)\Leftrightarrow f(x)+f^{*}(u)=\langle x|u\rangle\Rightarrow x\in\partial f^{*}(u)$

$u\in\partial f(x)\Leftrightarrow \underbrace{f(x)}_{F(x,y)}+\underbrace{f^{*}(u)}_{F^{*}(u,0)}=\langle x|u\rangle$

$\Leftrightarrow F^{*}(u,0)=\langle x|u\rangle-F(x,y)=\langle (x,y)|(u,0)\rangle-F(x,y)$

$\Leftrightarrow F(x,y)+F^{*}(u,0)=\langle (x,y)|(u,0)\rangle$

$\Leftrightarrow (u,0)\in\partial F(x,y)$ ∎

Proposition 16.47.

$[\![f \in \Gamma_0(\mathcal{H})$; $\mathcal{K}$: real Hilbert space ; $L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$; $y \in \operatorname{dom}(L \triangleright f)$; $x \in \mathcal{H}$; $Lx = y \,]\!] \Rightarrow$

(i) $(L \triangleright f)(y) = f(x) \Rightarrow \partial(L \triangleright f)(y) = (L^*)^{-1}(\partial f(x))$

(ii) $(L^*)^{-1}(\partial f(x)) \neq \emptyset \Rightarrow (L \triangleright f)(y) = f(x)$

Proof: $v \in \mathcal{K}$

suppose,

$f(x) + (L \triangleright f)^*(v) = \langle y \mid v \rangle$ (-1) then this is equivalent to saying:

$\leftrightarrow f(x) + (f^* \circ L^*)(v) = \langle Lx \mid v \rangle$ /* ∵ $Lx = y$: given, and

* Proposition 13.21 (very important)
$[\![f, g : \mathcal{H} \to]-\infty, +\infty] \,]\!]$
(i) $(f \,\square\, g)^* = f^* + g^*$
(ii) f, g : proper $\Rightarrow (f+g)^* \leq f^* \,\square\, g^*$
(iii) $\forall_{\gamma \in \mathbb{R}_{++}} \; ({}^{\gamma}f)^* = f^* + \frac{\gamma}{2}\|\cdot\|^2$
(iv) $L \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \Rightarrow (L \triangleright f)^* = f^* \circ L^*$
(v) $L \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \Rightarrow (f \circ L)^* \leq L^* \triangleright f^*$

*/

$\leftrightarrow f(x) + f^*(L^* v) = \langle Lx \mid v \rangle = \langle x \mid L^* v \rangle$ ... (0)

/* using the definition of adjoint operator */

/* using

Proposition 16.9 (Very important theorem, subdifferential membership can be tested using ...)
$[\![f : \mathcal{H} \to]-\infty, +\infty]$, proper ; $x \in \mathcal{H}$; $u \in \mathcal{H}]\!]$
$u \in \partial f(x) \Leftrightarrow f(x) + f^*(u) = \langle x \mid u \rangle \Leftrightarrow x \in \partial f^*(u)$

*/

$\leftrightarrow L^* v \in \partial f(x)$ (1)

↳ we are going to use this later

* Proposition 13.15 (Fenchel-Young inequality)
$[\![f : \mathcal{H} \to]-\infty, +\infty]$, proper $]\!]$ /* wow! see how general the function is, infact any sensible function would satisfy this! */
$\forall_{x \in \mathcal{H}} \forall_{u \in \mathcal{H}} \; f(x) + f^*(u) \geq \langle x \mid u \rangle$

(i)

$v \in \partial(L \triangleright f)(y) \leftrightarrow (L \triangleright f)(y) + (L \triangleright f)^*(v) = \langle y \mid v \rangle$ /* given $Lx = y$, and $(L \triangleright f)(y) = f(x)$ */

where $(L \triangleright f)^*(v) = (f^* \circ L^*)v$

$\leftrightarrow f(x) + f^*(L^* v) = \langle Lx \mid v \rangle$ /* this is (0), so use (1) */

$\leftrightarrow L^* v \in \partial f(x) \leftrightarrow \exists_{\eta \in \partial f(x)} \; L^* v = \eta$

$\leftrightarrow \exists_{\eta \in \partial f(x)} \; v = (L^*)^{-1} \eta$ /* Provided that L^{*-1} exists */

$\leftrightarrow$ $v = (L^*)^{-1} \eta \in (L^*)^{-1} \partial f(x)$

$\therefore \partial(L \triangleright f)(y) = (L^*)^{-1}(\partial f(x))$

(ii) take $v \in (L^*)^{-1}(\partial f(x)) \leftrightarrow L^* v \in \partial f(x) \leftrightarrow f(x) + (L \triangleright f)^*(v) = \langle y \mid v \rangle$ /* In a manner similar to this */ (from (1)) ... (4)

/* using

* Proposition 13.15 (Fenchel-Young inequality)
$[\![f : \mathcal{H} \to]-\infty, +\infty]$, proper $]\!]$ /* wow! see how general the function is, infact any sensible function would satisfy this! */
$\forall_{x \in \mathcal{H}} \forall_{u \in \mathcal{H}} \; f(x) + f^*(u) \geq \langle x \mid u \rangle$

*/

$\langle y \mid v \rangle \leq (L \triangleright f)(y) + (L \triangleright f)^*(v)$

$\quad\; \| $

/* $\begin{pmatrix} \min_{\tilde{x}} & f(\tilde{x}) \\ \text{s.t.} & L\tilde{x} = y \end{pmatrix} \leq f(x)$, as $Lx = y$, x is a feasible point for the optimization problem */

$\leq f(x) + (L \triangleright f)^*(v) = \langle y \mid v \rangle$ /* from (4) */

So, the inequalities collapse:

$\therefore \langle y \mid v \rangle = (L \triangleright f)(y) + (L \triangleright f)^*(v) = f(x) + (L \triangleright f)^*(v) = \langle y \mid v \rangle$

$\therefore (L \triangleright f)(y) = f(x)$ ■

/* info: [$F: \mathcal{H}\times\mathcal{H} \to [-\infty,+\infty]$] F: autoconjugate $\overset{def}{\Leftrightarrow} F^* = F^T$
$F^T(u,x) = F(x,u)$ $(\cdot)^T$: transposition operator */

Proposition 16.52. (Subdifferential of autoconjugate function)

[$F \in \Gamma_0(\mathcal{H}\times\mathcal{H})$, autoconjugate ; $(x,u) \in \mathcal{H}\times\mathcal{H}$]

$$F(x,u) = \langle x|u\rangle \leftrightarrow F^*(u,x) = \langle x|u\rangle \leftrightarrow (u,x) \in \partial F(x,u)$$

$$\leftrightarrow (x,u) = \mathrm{Prox}_F(x+u, x+u)$$

Proof: (i) ⇔ (ii):

(i) ⇔ $F(x,u) = \langle x|u\rangle$ /* By definition: $F^T(u,x) = F(x,u)$ */

$\leftrightarrow F^T(u,x) = F(x,u) = \langle x|u\rangle$ /* F: autoconjugate $\overset{def}{\Leftrightarrow}$ $F^T(u,x) = F^*(u,x)$ */

$\leftrightarrow F^*(u,x) = \langle x|u\rangle = F^T(u,x)$

⇔ (ii)

(i) ⇔ (iii)

$F(x,u) = \langle x|u\rangle$

$$\leftrightarrow 2F(x,u) = 2\langle x|u\rangle \leftrightarrow F(x,u) + \underbrace{F(x,u)}_{\substack{\| \\ F^T(u,x) = F^*(u,x)}} = \langle x|u\rangle + \underbrace{\langle x|u\rangle}_{\langle u|x\rangle}$$

/* F: autoconjugate */

$\leftrightarrow F(x,u) + F^*(u,x) = \langle (x,u) | (u,x)\rangle$ /* (Testing subdifferential membership via conjugate function)
Proposition 16.9.
[$f: \mathcal{H} \to]-\infty,+\infty]$, proper ; $x\in\mathcal{H}$; $u\in\mathcal{H}$]
$u\in\partial f(x) \Leftrightarrow f(x)+f^*(u) = \langle x|u\rangle \Leftrightarrow x \in \partial f^*(u)$ */

$\leftrightarrow (u,x) \in \partial F(x,u)$ ⇔ (iii)

(iii) ⇔ (iv)

(iii): $(u,x) \in \partial F(x,u)$ /* recall: $f \in \Gamma_0(\mathcal{H}) \Rightarrow \mathrm{Prox}_f = (Id + \partial f)^{-1}$ */

$\leftrightarrow (u,x) + (x,u) \in \partial F(x,u) + (x,u) = (\partial F + Id)(x,u)$

$$\leftrightarrow \underbrace{(\partial F + Id)^{-1}}_{\mathrm{Prox}_F}\big(\underbrace{(u,x)+(x,u)}_{\begin{bmatrix} u \\ x\end{bmatrix} + \begin{bmatrix} x \\ u\end{bmatrix} = \begin{bmatrix} u+x \\ x+u\end{bmatrix} = \begin{bmatrix} x+u \\ x+u\end{bmatrix} = (x+u, x+u)}\big) \ni (x,u)$$

$\leftrightarrow \mathrm{Prox}_F(x+u, x+u) \ni (x,u)$ /* But: $\mathrm{Prox}_{(\cdot)}(\cdot)$: is a single valued operator */

$\leftrightarrow \mathrm{Prox}_F(x+u, x+u) = (x,u)$ ⇔ (iv)

∎

Proposition 16.53.

[$L: \mathcal{H}\times\mathcal{H} \to \mathcal{H}\times\mathcal{H} : (x,u) \mapsto (u,x)$; $F \in \Gamma_0(\mathcal{H}\times\mathcal{H})$]

$$\mathrm{Prox}_{F^{*T}} = (Id - L\,\mathrm{Prox}_F\,L)$$

Proof:

$L\begin{bmatrix} x \\ u\end{bmatrix} = \begin{bmatrix} u \\ x\end{bmatrix}$, now $\langle L(x,u) | (\tilde{x},\tilde{u})\rangle = \left\langle \begin{bmatrix} u \\ x\end{bmatrix} \middle| \begin{bmatrix} \tilde{x} \\ \tilde{u}\end{bmatrix}\right\rangle$

$= \langle u|\tilde{x}\rangle + \langle x|\tilde{u}\rangle$

$\langle (x,u) | L(\tilde{x},\tilde{u})\rangle = \left\langle \begin{bmatrix} x \\ u\end{bmatrix} \middle| \begin{bmatrix} \tilde{u} \\ \tilde{x}\end{bmatrix}\right\rangle = \langle x|\tilde{u}\rangle + \langle u|\tilde{x}\rangle$

$= \langle u|\tilde{x}\rangle + \langle x|\tilde{u}\rangle$

so, $\langle L(x,u) | (\tilde{x},\tilde{u})\rangle = \langle (x,u) | L(\tilde{x},\tilde{u})\rangle$ /* recall: $\langle Lx|y\rangle = \langle x|L^*y\rangle$ */

$\therefore L = L^*$

also, $L = L^{-1}$ as $L\left(L\begin{bmatrix} x \\ u\end{bmatrix}\right) = L\begin{bmatrix} u \\ x\end{bmatrix} = \begin{bmatrix} x \\ u\end{bmatrix}$

so, $L^* = L = L^{-1}$ $\therefore Id = LL^{-1} = LL$ (0.1) also, $L = L^{-1}$ as $L\left(L\begin{bmatrix} x \\ u \end{bmatrix}\right) = L\begin{bmatrix} u \\ x \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix}$

so, $L^* = L = L^{-1}$ (0) $\quad \therefore \mathrm{Id} = LL^{-1} = LL$ (0.1)

now:

$F^* \in \Gamma_0(\mathcal{H}\times\mathcal{H})$

Theorem 16-47: [$\mathcal{K}$: real Hilbert space; $f \in \Gamma_0(\mathcal{H})$; $g \in \Gamma_0(\mathcal{K})$; $L \in \mathcal{B}(\mathcal{H},\mathcal{K})$; one of the following holds:

(i) $0 \in \mathrm{sri}(\mathrm{dom}\, g - L(\mathrm{dom}\, f))$ = $\mathrm{sri}(\mathcal{H}) = \mathcal{H}$ ✓ ($0 = \mathcal{H}$)

(ii) $\mathcal{K}$: finite-dimensional; g: polyhedral; $\mathrm{dom}\, g \cap \mathrm{ri}\, L(\mathrm{dom}\, f) \neq \emptyset$

(iii) $\mathcal{H},\mathcal{K}$: finite-dimensional; f, g: polyhedral; $\mathrm{dom}\, g \cap L(\mathrm{dom}\, f) \neq \emptyset$] $\Rightarrow \partial(f + g\circ L) = \partial f + L^*\circ(\partial g)\circ L$

so, $\partial(F^*\circ L) = \underbrace{L^*}_{=L}\circ(\partial F^*)\circ L = L\circ(\partial F^*)\circ L$ ... (1)

now, $F^{*T}(x,u) = (F^*)^T(x,u) = F^*(u,x) = F^*\left(L\begin{bmatrix} x \\ u \end{bmatrix}\right)$

$= (F^*\circ L)(x,u)$

so, $F^{*T} = F^*\circ L$

$\Rightarrow \partial F^{*T} = \partial(F^*\circ L) = L\circ(\partial F^*)\circ L$

so, $(\mathrm{Id} + \partial F^{*T}) = \underbrace{L\circ L}_{\text{// from (0.1)}} + \underbrace{L\circ(\partial F^*)\circ L}_{\text{// from (0)}} = L\circ \mathrm{Id}\circ L + L\circ(\partial F^*)\circ L = L(\mathrm{Id} + \partial F^*)L$

now: recall $\mathrm{Prox}_f(\cdot) = (\mathrm{Id} + \partial f)^{-1}$

so, $\mathrm{Prox}_{F^{*T}} = (\mathrm{Id} + \partial F^{*T})^{-1} = (L\circ(\mathrm{Id} + \partial F^*)\circ L)^{-1}$ /* all of L, $(\mathrm{Id} + \partial F^*)$ are matrices, where $(AB)^{-1} = B^{-1}A^{-1}$ */

$= \underbrace{L^{-1}}_{L}\circ\underbrace{(\mathrm{Id} + \partial F^*)^{-1}}_{\mathrm{Prox}_{F^*}}\circ\underbrace{L^{-1}}_{L}$ // from (0)

$= L\circ \mathrm{Prox}_{F^*}\circ L$

/* now using Moreau's decomposition:
[$f \in \Gamma_0(\mathcal{H})$] $\mathrm{Prox}_f + \mathrm{Prox}_{f^*} = \mathrm{Id}$ */

$= L\circ(\mathrm{Id} - \mathrm{Prox}_F)\circ L = L\circ(\underbrace{\mathrm{Id}\circ L}_{L} - \mathrm{Prox}_F\circ L) = \underbrace{(L\circ L)}_{\mathrm{Id}\ \text{// from (0.1)}} - L\circ \mathrm{Prox}_F\circ L$

$= \mathrm{Id} - L\,\mathrm{Prox}_F\, L$

$\therefore \mathrm{Prox}_{F^{*T}} = \mathrm{Id} - L\,\mathrm{Prox}_F\, L$ ∎