Part 1

6:42 AM

$\ast$ Proposition 15.1.

$[\, f, g \in \Gamma_0(\mathcal{H}) : \operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset \,]$

$f^* \square g^*$: proper, convex

$f^* \square g^*$: has continuous affine minorant.

$(f+g)^* = (f^* \square g^*)^{**} = (f^* \square g^*)^{\vee}$

Proof:

/*

> **$\ast$ Proposition 13.21. (very important)**
>
> $[\, f, g : \mathcal{H} \to]-\infty, +\infty] \,]$
>
> (i) $(f \square g)^* = f^* + g^*$
>
> (ii) f, g: proper $\Rightarrow (f+g)^* \leq f^* \square g^*$
>
> (iii) $\forall_{\gamma \in \mathbb{R}_{++}} \; ({}^{\gamma} f)^* = f^* + \frac{\gamma}{2} \|\cdot\|^2$
>
> (iv) $L \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \Rightarrow (L \triangleright f)^* = f^* \circ L^*$
>
> (v) $L \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \Rightarrow (f \circ L)^* \leq L^* \triangleright f^*$

Corollary 13.33. (Fenchel-Moreau theorem)

$[\, f \in \Gamma_0(\mathcal{H}) \,]$

- $f^* \in \Gamma_0(\mathcal{H})$
- $f^{**} = f$

*/

$f, g \in \Gamma_0(\mathcal{H}) \Rightarrow f^*, g^* \in \Gamma_0(\mathcal{H})$

$\Rightarrow (f^* \square g^*)^* = f^{**} + g^{**}$ (1)

$= f + g \in \Gamma_0(\mathcal{H})$ /* Γ ness is preserved under addition when number of functions are finite and none contains $-\infty$ (Corollary 9.4), what is left is properness, now $\operatorname{dom} f+g = \operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ given ✓ so, $\operatorname{dom}(f+g) \neq \emptyset$, $-\infty \notin (f+g)(\mathcal{H})$, and $f+g$: proper $\therefore f+g \in \Gamma_0(\mathcal{H})$ */

$\therefore (f^* \square g^*)^*$: proper $\Rightarrow (f^* \square g^*)$: proper

as, $f^*, g^* \in \Gamma_0(\mathcal{H}) \Rightarrow (f^* \square g^*)$: convex

$\therefore (f^* \square g^*)$: proper, convex ✓

now $(f^* \square g^*)^* \in \Gamma_0(\mathcal{H}) \Rightarrow (f^* \square g^*)^* \neq +\infty$
(1)

$\Leftrightarrow f^* \square g^*$: has a continuous affine minorant ✓

as $f^* \square g^*$ has a continuous affine minorant

$\Rightarrow (f^* \square g^*)^{**} = (f^* \square g^*)^{\vee}$ /* $\breve{f}$: lower semicontinuous convex envelope of f */

/* $(f^* \square g^*)^{**} = [(f^* \square g^*)^*]^* = (f+g)^*$

$f^{**} + g^{**}$ From (1)

$= f+g$ /* $\because f, g \in \Gamma_0(\mathcal{H}) \Rightarrow f^*, g^* \in \Gamma_0(\mathcal{H})$ using Fenchel-Moreau theorem */

$\therefore (f^* \square g^*)^{**} = (f+g)^* = (f^* \square g^*)^{\vee}$

■

> **$\ast$ Proposition 13.9.** ★
>
> $[\, f : \mathcal{H} \to [-\infty, +\infty] \,] \Rightarrow$
>
> (i) $f^*(0) = -\inf f(\mathcal{H})$ /* So, inf $f(\mathcal{H})$ [illegible] conjugate function [illegible] 0 [illegible] evaluate [illegible] -1 [illegible] ! */
>
> (ii) $-\infty \in f^*(\mathcal{H}) \Leftrightarrow f \equiv +\infty \Leftrightarrow f^* \equiv -\infty$
>
> (iii) f^*: proper $\Rightarrow f$: proper
>
> (iv) $[\, u \in \mathcal{H} \,]$
>
> $f^*(u) = \sup_{x \in \operatorname{dom} f} (\langle x | u \rangle - f(u)) = \sup_{(x,\xi) \in \operatorname{epi} f} (\langle x | u \rangle - \xi)$
>
> (v) $f^* = \iota^*_{\operatorname{gra} f}(\cdot, -1) = \iota^*_{\operatorname{epi} f}(\cdot, -1)$

Proposition 12.11. (infimal convolution between convex functions preserves convexity)

$[\, f, g : \mathcal{H} \to]-\infty, +\infty]$, convex $]$

$f \square g$: convex

$\ast$ Proposition 13.10. ★

$[\, f : \mathcal{H} \to [-\infty, +\infty] \,]$

(i) $[(u, v) \in \mathcal{H} \times \mathbb{R}]$ $(u, v) \in \operatorname{epi} f^* \leftrightarrow \langle \cdot | u \rangle - v \leq f$ /* Any continuous affine minorant of a function has the (slope, shift) in the epigraph of the conjugate function! */

(ii) $f^* \equiv +\infty \leftrightarrow f$: posseses no continuous affine minorant

(iii) $[\operatorname{dom} f^* \neq \emptyset]$ f: bounded below on every bounded subset of $\mathcal{H}$.

Proof: See 2/15/2016 2:44 PM.

Proposition 13.39.

$[\, f : \mathcal{H} \to]-\infty, +\infty] \,]$

- f: has a continuous affine minorant $\Rightarrow f^{**} = \breve{f}$
  $\Updownarrow$
  $\operatorname{dom} f^* \neq \emptyset$ (from Proposition 13.10(ii))
- f: doesnot have a continuous affine minorant $\Rightarrow f^{**} = -\infty$

> **$\ast$ Theorem 15.3. (Attouch-Brézis theorem)**
>
> $[\, f, g \in \Gamma_0(\mathcal{H})$
>
> $0 \in \operatorname{sri}(\operatorname{dom} f - \operatorname{dom} g) \leftrightarrow \operatorname{cone}(\operatorname{dom} f - \operatorname{dom} g) = \overline{\operatorname{span}}\,(\operatorname{dom} f - \operatorname{dom} g) \,]$
>
> $(f+g)^* = f^* \square g^* \in \Gamma_0(\mathcal{H})$

Proof:

$f, g \in \Gamma_0(\mathcal{H})$

$\Rightarrow f+g \in \Gamma_0(\mathcal{H})$ /* Γ_0: closed under addition */

$\Rightarrow (f+g)^* \in \Gamma_0(\mathcal{H})$ /* Fenchel-Moreau Corollary: $f \in \Gamma_0(\mathcal{H}) \Rightarrow \begin{cases} f^* \in \Gamma_0(\mathcal{H}) \\ f^{**} = f \end{cases}$ */

$\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$ because

$0 \in \mathrm{sri}(\underbrace{\mathrm{dom} f - \mathrm{dom} g}_{[illegible]}) = \{ x \in [illegible] \mid \mathrm{cone}([illegible]) = \overline{\mathrm{span}}([illegible]) \}$

$\Leftrightarrow 0 \in \underbrace{[illegible]}_{\mathrm{dom} f - \mathrm{dom} g} \Leftrightarrow 0 \in \mathrm{dom} f - \mathrm{dom} g \Leftrightarrow \exists \underset{f([illegible]) < +\infty}{[illegible] \in \mathrm{dom} f} \; \exists \underset{g([illegible]) < +\infty}{[illegible] \in \mathrm{dom} g} \quad 0 = [illegible] - [illegible] \Leftrightarrow \underbrace{[illegible]}_{\in \mathrm{dom} f} = \overbrace{[illegible]}^{\in \mathrm{dom} g} \Leftrightarrow \exists_{[illegible] = [illegible]} [illegible] \in \mathrm{dom} f \cap \mathrm{dom} g.$

$\therefore \mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$

set, $\varphi : x \mapsto f(x+z)$

$\psi : y \mapsto g(y+z)$

Note that, $0 \in \mathrm{dom}\varphi - \mathrm{dom}\psi$, $\mathrm{dom}\varphi - \mathrm{dom}\psi = \mathrm{dom} f - \mathrm{dom} g$

$0 \in \mathrm{dom} f - \mathrm{dom} g \Leftrightarrow \exists_z \; z \in \mathrm{dom} f \cap \mathrm{dom} g$

$\Leftrightarrow f(z) < +\infty, \; g(z) < +\infty$

$\Leftrightarrow f(0+z) = \varphi(0) < +\infty, \; g(0+z) = \psi(0) < +\infty$

$\Leftrightarrow 0 \in \mathrm{dom}\varphi, 0 \in \mathrm{dom}\psi$

$\Leftrightarrow 0 \in \mathrm{dom}\varphi \cap \mathrm{dom}\psi$

$\mathrm{dom} f - \mathrm{dom} g = \{ x-y \mid x \in \mathrm{dom} f, \; y \in \mathrm{dom} g \}$

$= \{ x-y \mid f(x) < +\infty, \; g(y) < +\infty \}$

$= \{ x-y \mid f(x-z+z) = \varphi(x-z) < +\infty$

(incomplete...)

Part 2

8:44 PM

* Proposition 15.12.

$[\ f, g: \text{proper}, : \mathcal{H} \to]-\infty, +\infty]$

$\mu = \inf (f+g)(\mathcal{H})$,

$\mu^* = \inf (f^* + g^{*\vee})(\mathcal{H})\ ]$

(i) $\mu \geq -\mu^*$ (essentially the weak duality condition)

(ii) $\Delta(f, g) \in [0, +\infty]$

(iii) $\mu = -\mu^* \Leftrightarrow \Delta(f,g) = 0$

Proof:

(i) $\forall_{x \in \mathcal{H}} \forall_{u \in \mathcal{H}} \quad f(x) + g(x) + f^*(u) + g^*(-u)$

$= \underbrace{f(x) + f^*(u)}_{\geq \langle x|u \rangle} + \underbrace{g(x) + g^*(-u)}_{\geq \langle x|-u \rangle = -\langle x|u \rangle}$

$\geq \langle x|u \rangle - \langle x|u \rangle = 0$

// Fenchel-Young inequality: $[\ f: \mathcal{H} \to]-\infty, +\infty],\ \text{proper}]$ $\forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} \quad f(x) + f^*(u) \geq \langle x|u \rangle$

$\Leftrightarrow \forall_{x \in \mathcal{H}} \forall_{u \in \mathcal{H}} \quad f(x) + g(x) \geq -f^*(u) - g^*(-u)$

$\Leftrightarrow \forall_{u \in \mathcal{H}} \inf_{x \in \mathcal{H}} f(x) + g(x) \geq -f^*(u) - g^*(-u)$

$\Leftrightarrow \underbrace{\inf_{x \in \mathcal{H}} f(x) + g(x)}_{\mu} \geq \underbrace{\sup_{u \in \mathcal{H}} -f^*(u) - g^*(-u)}_{= -\inf_{u \in \mathcal{H}} f^*(u) + g^*(-u)} = -\mu^*$

$\therefore \mu \geq -\mu^*$

(ii), (iii) follows from definition. ■

Proposition 15.13. (strong duality)

$[\ f, g \in \Gamma_0(\mathcal{H}) : 0 \in \text{sri}(\text{dom} f - \text{dom} g)\ ]$

$\inf(f+g)(\mathcal{H}) = -\min(f^* + g^{*\vee})(\mathcal{H})$

Proof:

$f: \mathcal{H} \to]-\infty, +\infty] \Rightarrow f^*(0) = -\inf f(\mathcal{H})$ // Proposition 13.9.(i)

$f, g \in \Gamma_0(\mathcal{H}) : 0 \in \text{sri}(\text{dom} f - \text{dom} g) \Rightarrow (f+g)^* = f^* \boxdot g^* \in \Gamma_0(\mathcal{H})$ // Atlouch–Brézis theorem

$\inf(f+g)(\mathcal{H}) = -(f+g)^*(0)$

$= -(f^* \boxdot g^*)(0)$

$= -\min_{x \in \mathcal{H}} (f^*(x) + g^*(0-x))$ // $(f \boxdot g)(y) = \min_{x \in \mathcal{H}} (f(x) + g(x-y))$

$= -\min_{x \in \mathcal{H}} (f^*(x) + g^*(-x))$, where $g^*(-x) = g^{*\vee}(x)$

$= -\min_{x \in \mathcal{H}} (f^*(x) + g^{*\vee}(x))$

$\therefore \inf(f+g)(\mathcal{H}) = -\min (f^* + g^{*\vee})(\mathcal{H})$ ■

Corollary 15.14.

$[\ f \in \Gamma_0(\mathcal{H})$; K: closed convex cone in $\mathcal{H}$, $0 \in \text{sri}(K - \text{dom} f)\ ]$ $\inf f(K) = -\min f^*(K^{\oplus})$

Proof:

/* Proposition 15.13:

$[\ f, g \in \Gamma_0(\mathcal{H}) : 0 \in \text{sri}(\text{dom} f - \text{dom} g)\ ]$

$\inf(f+g)(\mathcal{H}) = -\min(f^* + g^{*\vee})(\mathcal{H})$ */ ... (i)

K: closed convex cone

$\Rightarrow g^{\cdot} = \iota_K$

$\Rightarrow g^* = \iota_{K^{\ominus}}$ ... (iv) /* K: nonempty cone in $\mathcal{H} \Rightarrow f^* = \iota_{K^{\ominus}}$ (Example 13.3 (ii)) */

$K^{\ominus}$: polar cone of K

$= \{u \in \mathcal{H} \mid \sup \langle C|u \rangle \leq 0\}$

$= \{u \in \mathcal{H} \mid \forall_{x \in C} \langle x|u \rangle \leq 0\}$ */ ... (v)

$\therefore \inf(f+g)(\mathcal{H})$

$= \inf(f + \iota_K)(\mathcal{H})$

$= \inf_{x \in K} f(x)$ /* comes from desination */

$= \inf f(K)$ .. (ii)

now. $\min(f^* + g^{*\vee})(\mathcal{H}) = \min(f^* + \iota_{K^{\ominus}}^{\vee})(\mathcal{H})$ /* from (iv) */

$= \min_{x \in \mathcal{H}} f^*(x) + \iota_{K^{\ominus}}^{\vee}(x)$ /* $f^{\vee}(\cdot) = f(-\cdot)$ */

$= \min_{x \in \mathcal{H}} f^*(x) + \iota_{K^{\ominus}}(-x)$

$= \min_{x \in \mathcal{H} : -x \in K^{\ominus}} f^*(x)$

$-x \in K^{\ominus}$
$\Leftrightarrow \forall_{y \in K} (\langle -x|y \rangle \leq 0 \Leftrightarrow -\langle x|y \rangle \leq 0 \Leftrightarrow \langle x|y \rangle \geq 0)$
$\Leftrightarrow \forall_{y \in K} \langle x|y \rangle \geq 0$
$\Leftrightarrow \inf_{y \in K} \langle x|y \rangle \geq 0 \Leftrightarrow \inf \langle K|x \rangle \geq 0$
$\Leftrightarrow x \in K^{\oplus} = \{\tilde{x} \mid \inf \langle K|\tilde{x} \rangle \geq 0\} \subseteq \mathcal{H}$

$= \min_{x \in K^{\oplus}} f^*(x)$

$= \min f^*(K^{\oplus})$ ... (vi)

from (i), (ii), (vi) $\Rightarrow$

$\inf f(K) = -\min f^*(K^{\oplus})$

■

* Conditions for zero duality gap for Fenchel–Rockafellar duality:

Proposition 15.22

$[\ f \in \Gamma_0(\mathcal{H}),\ g \in \Gamma_0(\mathcal{K}),\ L \in \mathcal{B}(\mathcal{H}, \mathcal{K})\ ]$

$0 \in \text{core}(\text{dom}\, g - L(\text{dom} f)) \Rightarrow$

$\inf(f + g \circ L)(\mathcal{H}) = -\min(f^* \circ L^* + g^{*\vee})(\mathcal{K})$

Proof:

this seperable sum is valid as we are talking about functions, so $f: \mathcal{H} \to]-\infty, +\infty]$, $g: \mathcal{K} \to]-\infty, +\infty]$

Denote $\varphi = f \oplus g$: $\forall x \in \mathcal{H}, y \in \mathcal{K}$ $\quad \varphi(x, y) = f(x) + g(y)$ (-i)

$V = \text{gra}\, L = \{(x, Lx) \mid x \in \mathcal{H}\}$, with $x \in \mathcal{H}$, $Lx \in \mathcal{K}$ (0)

take. any $(x, y) \in \mathcal{H} \times \mathcal{K}$

$\Rightarrow y - Lx \in \mathcal{K}$ ... (i)

given: $0 \in \text{core}(\text{dom}\, g - L(\text{dom} f))$ // $\text{core}\, C = \{x \in C \mid \text{cone}(C - x) = \mathcal{H}\}$

// $0 \in \text{core}\, C \Leftrightarrow \text{cone}(C) = \mathcal{H}$

$\Leftrightarrow \mathbb{R}_{++} C = \mathcal{H}$ // from Proposition 6.2. $(\forall\, C: \text{cone}, C \subseteq \mathcal{H})$ $\text{cone}\, C = \mathbb{R}_{++} C$

$\Leftrightarrow \mathbb{R}_{++}(\text{dom}\, g - L(\text{dom} f)) = \mathcal{K}$ ... (ii)

$\text{dom}\, g - L(\text{dom} f) = \{\tilde{b} - L\tilde{a} \mid \tilde{b} \in \text{dom}\, g, \tilde{a} \in \text{dom} f\}$, with $\text{dom}\, g \subseteq \mathcal{K}$, $\text{dom} f \subseteq \mathcal{H}$

From (i), (ii):

$\exists_{\gamma \in \mathbb{R}_{++},\ b \in \text{dom}\, g,\ a \in \text{dom} f} \quad \gamma(b - La) = y - Lx$ // set $z = a - \frac{x}{\gamma} \Rightarrow x = \gamma(a - z)$

$\Leftrightarrow y = Lx + \gamma b - \gamma La$

$= L\gamma(a - z) + \gamma b - \gamma La$

$= \gamma L(a - z) + \gamma b - \gamma La$

$= \gamma(La - Lz + b - La)$

$= \gamma(b - Lz)$

$\therefore \forall_{(x,y) \in \mathcal{H} \times \mathcal{K}} \exists_{\gamma \in \mathbb{R}_{++}, b \in \text{dom}\, g, a \in \text{dom} f} \left| \begin{array}{l} x = \gamma(a - z) \\ [illegible] \end{array} \right. \Leftrightarrow (x, y) = \gamma(a - z, b - Lz) = \gamma \begin{bmatrix} a - z \\ b - Lz \end{bmatrix} = \gamma\left(\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} z \\ Lz \end{bmatrix}\right)$ // so, $\varphi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = f(a) + g(b) < +\infty \Leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix} \in \text{dom}\, \varphi$

$a \in \text{dom} f \Rightarrow f(a) < +\infty$

$= \gamma(b - Lz)$

$\therefore \forall_{(x,y)\in\mathcal{H}\times\mathcal{K}} \ \exists_{\gamma\in\mathbb{R}_{++},\, b\in\operatorname{dom} g,\, a\in\operatorname{dom} f}$ $\left| \begin{array}{l} x = \gamma(a-z) \\ y = \gamma(b-Lz) \end{array}\right.$ $\Leftrightarrow (x,y) = \gamma(a-z, b-Lz) = \gamma\begin{bmatrix} a-z \\ b-Lz\end{bmatrix} = \gamma\left(\begin{bmatrix} a \\ b\end{bmatrix} - \underbrace{\begin{bmatrix} z \\ Lz\end{bmatrix}}_{\in V}\right)$

$a \in \operatorname{dom} f \Rightarrow f(a) < +\infty$, $b \in \operatorname{dom} g \Rightarrow g(b) < +\infty$ // so, $\Phi\left(\begin{bmatrix} a \\ b\end{bmatrix}\right) = f(a) + g(b) < +\infty \Leftrightarrow \begin{bmatrix} a \\ b\end{bmatrix} \in \operatorname{dom}\Phi$

$\in \underbrace{\gamma}_{\in\mathbb{R}_{++}}(\operatorname{dom}\Phi - V) \in \operatorname{cone}(\operatorname{dom}\Phi - V)$

$\therefore$ we have shown that, $\forall_{(x,y)\in\mathcal{H}\times\mathcal{K}}$ $(x,y) \in \operatorname{cone}(\operatorname{dom}\Phi - V)$

$\Leftrightarrow \mathcal{H}\times\mathcal{K} \subseteq \operatorname{cone}(\operatorname{dom}\Phi - V)$

Naturally, $\operatorname{cone}(\operatorname{dom}\Phi - V) \subseteq \mathcal{H}\times\mathcal{K}$

$\Rightarrow \operatorname{cone}((\operatorname{dom}\Phi) - V) = \mathcal{H}\times\mathcal{K} \quad \ldots (iii)$

so, $\bar{0} = (0,0) \in \operatorname{cone}((\operatorname{dom}\Phi) - V)$

$\Leftrightarrow 0 \in \operatorname{cone}(V - \operatorname{dom}\Phi) \subseteq \operatorname{sri}(V - \operatorname{dom}\Phi)$ // $\forall_{C:\text{ convex set}}$ $\operatorname{int} C \subseteq \operatorname{core} C \subseteq \operatorname{sri} C \subseteq \operatorname{ri} C \subseteq \operatorname{qri} C \subseteq C$

recall corollary 15.14.

[$f \in \Gamma_0(\mathcal{H})$, K: closed convex cone $\subseteq \mathcal{H}$

$0 \in \operatorname{sri}(K - \operatorname{dom} f)$] $\inf f(K) = -\min f^*(K^\ominus)$

in our case set $K := V$: closed convex cone

$\Phi = \underbrace{f}_{\in\Gamma_0(\mathcal{H})} \oplus \underbrace{g}_{\in\Gamma_0(\mathcal{K})} \in \Gamma_0(\mathcal{H}\times\mathcal{K})$

$\Rightarrow \inf \Phi(V) = -\min \Phi^*(V^\ominus)$ // As V: linear subspace $\Rightarrow V^\ominus = V^\perp$ Proposition — orthogonal complement of V

$= -\min \Phi^*(V^\perp) \quad \ldots (iv)$

Now, $V^\perp = \{(x,y)\in\mathcal{H}\times\mathcal{K} \mid \forall_{(\tilde{x},\tilde{y})\in V} \ \langle (x,y) \mid (\tilde{x},\tilde{y})\rangle = 0\}$ =

$V = \operatorname{gra} L$, $L \in \mathcal{B}(\mathcal{H},\mathcal{K})$; $(\tilde{x},\tilde{y}) \in V \Leftrightarrow \tilde{y} = L\tilde{x}$

$\Leftrightarrow \forall_{(\tilde{x},\tilde{y})\in\mathcal{H}\times\mathcal{K}:\ \tilde{y}=L\tilde{x}}$ $\left\langle \begin{bmatrix} x \\ y\end{bmatrix} \middle| \begin{bmatrix} \tilde{x} \\ \tilde{y}\end{bmatrix}\right\rangle$

$\Leftrightarrow \forall_{\tilde{x}\in\mathcal{H}}$ $\left\langle \begin{bmatrix} x \\ y\end{bmatrix} \middle| \begin{bmatrix} \tilde{x} \\ L\tilde{x}\end{bmatrix}\right\rangle = \langle x \mid \tilde{x}\rangle +$

$\Leftrightarrow \forall_{\tilde{x}\in\mathcal{H}}$ $\langle \tilde{x} \mid x + L^* y\rangle = 0$

$\Leftrightarrow x + L^* y = 0$ // As this is the

$\Leftrightarrow x = -L^* y$

also. $\Phi = f \oplus g \Rightarrow \Phi^* = f^* \oplus g^*$ // Proposition

$\ldots (vi)$

now: $\Phi(V) = (f+g)(V)$, $V = \{(x, Lx) \mid x\in\mathcal{H}\}$

$= \left\{(f+g)\begin{bmatrix} x \\ Lx\end{bmatrix} : x\in\mathcal{H}\right\}$

$= \{f(x) + g(Lx) \mid x\in\mathcal{H}\}$

$\Rightarrow \inf \Phi(V) = \inf_{x\in\mathcal{H}} f(x) + g(Lx)$

$= \inf (f + g\circ L)(\mathcal{H})$

$\therefore \inf(f + g\circ L)(\mathcal{H}) = \inf \Phi(V)$ // use (iv) now

$= -\min \Phi^*(V^\perp)$ // use (v) now

$= -\min \Phi^*(\{(x,y)\in\mathcal{H}\times\mathcal{K} \mid x = -L^*$

$= -\min \Phi^*(\{(-L^* y, y) : y\in\mathcal{K}\})$

$= -\min_{y\in\mathcal{K}} \underbrace{\Phi^*(-L^* y, y)}$

$= (f^* \oplus g^*)(-L^* y, y) =$

$= -\min_{y\in\mathcal{K}} \left((f^*\circ L^*)(-y) + g^*(y)\right)$

// now set $y := -z \in \mathcal{K}$

$= -\min_{-z\in\mathcal{K}} \left((f^*\circ L^*)(z) + \underbrace{g^*(-z)}_{= g^{*\vee}(z)}\right)$

$= -\min_{-z\in\mathcal{K}} \left((f^*\circ L^*) + g^{*\vee}\right)(z)$

$= -\min \left((f^*\circ L^*) + g^{*\vee}\right)(\mathcal{K})$

$\therefore \inf(f + g\circ L)(\mathcal{H}) = -\min(f^*\circ L^* + g^{*\vee})(\mathcal{K})$

∎

Fact 15.25.

[$(f\in\Gamma_0(\mathcal{H}),\ g\in\Gamma_0(\mathcal{K}))$: polyhedral

$L \in \mathcal{B}(\mathcal{H},\mathcal{K})$

$\mathcal{K}$: finite dimensional and one of the following holds:

(i) $\operatorname{dom} g \cap \operatorname{ri} L(\operatorname{dom} f) \neq \emptyset$

(ii) $\mathcal{H}$ finite-dimensional. f: polyhedral, $\operatorname{dom} g \cap L(\operatorname{dom} f) \neq \emptyset$]

$\Rightarrow \inf(f + g\circ L)(\mathcal{H}) = -\min((f^*\circ L^*) + g^{*\vee})(\mathcal{K})$

<u>Proof</u>:

$\mu = \inf(f + g\circ L)(\mathcal{H})$

Assume $\mu > -\infty$ then, $\ldots (0)$

$\mu = \inf(f + g\circ L)(\mathcal{H})$

$= \inf_{x\in\mathcal{H}} f(x) + g(Lx)$ // recall $\operatorname{gra} L = \{(\tilde{x}, L\tilde{x}) : \tilde{x}\in\mathcal{H}\}$

6·22

$\{(x,y) \in \mathcal{H} \times \mathcal{K} \mid x = -L^* y\}$ . . . (V)

$= 0$

$\langle y \mid L\tilde{x} \rangle = \langle x \mid \tilde{x} \rangle + \langle L^* y \mid \tilde{x} \rangle = \langle x + L^* y \mid \tilde{x} \rangle = 0$

$\langle L^* y \mid \tilde{x} \rangle$

only possibility */

13·27

$y\})$

$)$

this is linear too

$f^*(-L^* y) + g^*(y) = (f^* \circ L^*)(-y) + g^*(y) = ((f^* \circ L^*)^\vee + g^*)(y)$

$= L^*(-y)$

$(f^* \circ L^*)^\vee (y)$ // By definition of the reversal function

$= \{(x,y) : y = Lx\}$

$= \inf_{\substack{x\in\mathcal{H} \\ y\in K : y = Lx \\ (x,y)\in \text{gra } L}} f(x) + g(y)$

$= \inf_{\substack{(x,y)\in \text{gra } L \\ \Leftrightarrow (y,x)\in \text{gra } L^{-1} \\ \Leftrightarrow L^{-1}y \ni x}} f(x) + g(y) = \inf_{\substack{y\in K \\ x\in\mathcal{H}}} g(y) + f(x) : x \in L^{-1}y$

$= \inf_{y\in K} \inf_{x\in\mathcal{H}} \left(g(y) + f(x) : x\in L^{-1}y \right)$

($y\in K$ ↓ independent variable; $x\in\mathcal{H}$ → dependent variable)

$= \inf_{y\in K} \Big(g(y) + \underbrace{\inf_{\substack{x\in\mathcal{H} \\ x\in L^{-1}y}} f(x)}_{(L\triangleright f)(y)} \Big)$ // recall the definition of infimal post composition:

// $(L\triangleright f) : y \mapsto \inf_{\substack{x\in\mathcal{H} \\ Lx = y}} f(x)$, where $Lx = y \Leftrightarrow (x,y)\in L \Leftrightarrow (y,x)\in L^{-1} \Leftrightarrow L^{-1}y \ni x$

$= \inf_{y\in K} \left(g(y) + (L\triangleright f)(y) \right) \quad \ldots (i)$

/* recall Proposition 12.34.

$\left[\begin{array}{l} f : \mathcal{H} \to]-\infty, +\infty] \\ L : \mathcal{H} \to K \end{array} \right] \Rightarrow \text{dom}(L\triangleright f) = L(\text{dom } f)$ */

(K: real Hilbert space)

as $f \in \Gamma_0(\mathcal{H}) \Rightarrow f : \mathcal{H} \to]-\infty, +\infty] \Rightarrow \text{dom}(L\triangleright f) = L(\text{dom } f) \quad \ldots (ii)$

/* recall Proposition 13.21.

$[\,(f,g) : \mathcal{H} \to]-\infty,+\infty]\,] \Rightarrow (L\triangleright f)^* = f^* \circ L^*$ // what a compact and beautiful statement

*/

similarly, $(L\triangleright f)^* = f^* \circ L^* \quad \ldots (iii)$

Now

- assume antecedent (i): $\text{dom } g \cap \text{ri } L(\text{dom } f) \neq \emptyset$ holds

  let $z \in \text{dom } f$: $Lz \in \text{dom } g \cap \text{ri } L \text{ dom } f$

  $= \text{dom } g \cap \text{ri dom}(L\triangleright f)$ // as this is nonempty, there // will be one such z

  $\ldots (iv)$

- Assume antecedent (ii): $\mathcal{H}$: finite-dimensional, f: polyhedral, $\text{dom } g \cap L(\text{dom } f) \neq \emptyset$

  let $z \in \text{dom } f$: $Lz \in \text{dom } g \cap L(\text{dom } f) = \text{dom } g \cap \text{dom}(L\triangleright f)$

  $\ldots (v)$

From (iv) and (v): $\mu = \inf_{y\in K} \left(g(y) + (L\triangleright f)(y) \right)$

$\leq g(Lz) + (L\triangleright f)(Lz)$ // $\because Lz \in K$

/* $(L\triangleright f)(y) = \inf_{\substack{x\in\mathcal{H} \\ Lx = y}} f(x) \leq f(x) \ \forall x : Lx = y$

$\therefore (L\triangleright f)(Lz) \leq f(\tilde{x}) : \forall \tilde{x} : L\tilde{x} = Lz$

(clearly $\tilde{x} = z$ satisfies $L\tilde{x} = Lz$)

$\leq f(z)$ */

$\leq \underbrace{g(Lz)}_{<+\infty} + \underbrace{f(z)}_{<+\infty}$ // as $z \in \text{dom } f$, $Lz \in \text{dom } g$ from (iv) and (v)

$< +\infty \quad \ldots (vi)$

From (0) and (vi) we have

$-\infty < \mu < +\infty$

$\Leftrightarrow \mu \in \mathbb{R} \quad \ldots (vii)$

Assume $(L\triangleright f)(Lz) = \inf_{\substack{\tilde{x}\in\mathcal{H} \\ L\tilde{x} = Lz}} f(\tilde{x}) = -\infty$

$\Rightarrow \exists\, (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H} : Lx_n = Lz$, $f(x_n) \to -\infty$

now, $\mu = \inf (f + g\circ L)(\mathcal{H})$

$\leq f(x) + g(Lx) \quad \forall x \in \mathcal{H}$

$\leq f(x_n) + g(Lx_n)$

$\to -\infty \quad \ldots (viii)$

⇓ contradicts with (vii)

$\therefore$ the assumption $(L\triangleright f)(Lz) \in \mathbb{R}$

Antecedent (i) holds $\Rightarrow (L\triangleright f)$: convex, proper // f: convex, L: affine $\Rightarrow (L\triangleright f)$: convex

if Antecedent (ii) " $\Rightarrow (L\triangleright f)$: polyhedral, proper // By Rockafellar's convex analysis book

using Rockafellar's convex analysis book: Theorem 31.1 yields:

$\inf_{y\in K} \left(g(y) + (L\triangleright f)(y) \right) = -\min_{v\in K} \left(g^{+v}(v) + (L\triangleright f)^*(v) \right)$

$= -\min_{v\in K} \left(g^{+v}(v) + f^*(L^*v) \right)$

↳ using this with (i) we arrive at the conclusion.

∎