Part 0

10:10 AM

Proposition 14.7.

[$L : \mathcal{H}\times\mathcal{H} \to \mathcal{H} : (y,z) \mapsto \frac{1}{2}(y+z)$;

$f, g \in \Gamma_0(\mathcal{H})$;

$F : \mathcal{H}\times\mathcal{H} \to]-\infty,+\infty] : (y,z) \mapsto \frac{1}{2}f(y) + \frac{1}{2}g(z) + \frac{1}{8}\|y-z\|^2$]

(i) $\operatorname{pav}(f,g) = \operatorname{pav}(g,f)$

(ii) $\operatorname{pav}(f,g) = L \triangleright F$

(iii) $\operatorname{dom}\operatorname{pav}(f,g) = \frac{1}{2}\operatorname{dom} f + \frac{1}{2}\operatorname{dom} g$

(iv) $\operatorname{pav}(f,g)$: proper, convex function

Proof:

(i)
$$\operatorname{pav}(f,g) \overset{\text{def}}{=} \begin{bmatrix} \inf_{(y,z)} & \frac{1}{2}\left(f(y) + g(z) + \frac{1}{4}\|y-z\|^2\right) \\ \text{s.t.} & y+z = 2x \\ & (y,z)\in\mathcal{H}\times\mathcal{H} \end{bmatrix} \quad \ldots (1)$$

from definition,

it is clear that $\therefore$ $\operatorname{pav}(f,g) = \operatorname{pav}(g,f)$

(ii) recall that

$$(L \triangleright f)(\tilde{y}) = \inf_{\tilde{x}} f(\tilde{x}) \quad \text{s.t. } L\tilde{x} = \tilde{y}$$

so,
$$(L \triangleright F)(x) = \begin{pmatrix} \inf_{(y,z)} & F(y,z) \\ \text{s.t.} & L(y,z) = x \\ & (y,z)\in\mathcal{H}\times\mathcal{H} \end{pmatrix}$$
/* given, $F(y,z) = \frac{1}{2}f(y) + \frac{1}{2}g(z) + \frac{1}{8}\|y-z\|^2$
$L(y,z) = \frac{1}{2}(y+z)$ */

$$= \begin{pmatrix} \inf_{(y,z)} & \frac{1}{2}f(y) + \frac{1}{2}g(z) + \frac{1}{8}\|y-z\|^2 \\ \text{s.t.} & \frac{1}{2}(y+z) = x \leftrightarrow y+z = 2x \\ & (y,z)\in\mathcal{H}\times\mathcal{H} \end{pmatrix} = \operatorname{pav}(f,g)(x) \quad // \text{ from (1)}$$

first note that:

$$\operatorname{pav}(f,g)(x) = \inf_{(y,z)} \frac{1}{2}f(y) + \frac{1}{2}g(z) + \frac{1}{4}\|y-z\|^2 + \iota_{y+z=2x}(y,z)$$

$$= \frac{1}{2}\left(\inf_{y\in\mathcal{H}} \inf_{z\in\mathcal{H}} f(y) + g(z) + \frac{1}{4}\|y-z\|^2 + \iota_{y+z=2x}(y,z)\right)$$
// recall $\inf_{(x,y)\in\mathcal{X}\times\mathcal{Y}} f(x,y) = \inf_{x\in\mathcal{X}}\inf_{y\in\mathcal{Y}} f(x,y) = \inf_{y\in\mathcal{Y}}\inf_{x\in\mathcal{X}} f(x,y)$ Fact 1.8.3.

$$= \frac{1}{2}\left(\inf_{y\in\mathcal{H}} f(y) + \underbrace{\inf_{z\in\mathcal{H}} g(z) + \frac{1}{4}\|y-z\|^2 + \iota_{z=2x-y}(y,z)}_{\|}\right)$$

/* $\left(\inf_{z\in\mathcal{H}:\, z=\underbrace{2x-y}} g(z) + \frac{1}{4}\|y-z\|^2\right) = g(2x-y) + \frac{1}{4}\|y-2x+y\|^2 = g(2x-y) + \frac{1}{4}\|2(y-x)\|^2 = g(2x-y) + \cancel{\frac{1}{4}}\cdot\cancel{4}\,\|y-x\|^2$ */

this is a single vector, so inf will be achieved at $z = 2x-y$

$$= \frac{1}{2}\left(\inf_{y\in\mathcal{H}} \underbrace{f(y) + g(2x-y) + \|y-x\|^2}_{h(y,x)}\right) = \frac{1}{2}\inf_{y\in\mathcal{H}} h(y,x) \quad \ldots (2)$$

define: $h: h(y,x) = f(y) + g(2x-y) + \|x-y\|^2$

$$= \underbrace{\tfrac{1}{2}\cdot 2\|y\|^2}_{\frac{\gamma}{2}\|y\|^2 :\ \gamma=2} + \underbrace{\left(f(y) + g(2x-y) - 2\langle y|x\rangle + \|x\|^2 \right)}_{\in \Gamma_0(\mathcal{H}) \text{ in } y}$$

so, $h \in \Gamma_0(\mathcal{H})$, strongly convex in y // as $\underbrace{f}_{\in\Gamma_0(\mathcal{H})} + \frac{\gamma}{2}\|\cdot\|^2$: strongly convex

... (3)

/* from (2), (3) and using

*Corollary 11.16. ☆☆
[$f \in \Gamma_0(\mathcal{H})$, strongly convex] $\Rightarrow$ f: supercoercive, has exactly one minimizer over $\mathcal{H}$.

we have:

$\operatorname{pav}(f,g)(x) = \frac{1}{2} \inf_{y\in\mathcal{H}} h(y,x)$ has a unique minimizer

so,

$\operatorname{pav}(f,g) = L \triangleright F$

(iii)

from (ii):

$\operatorname{pav}(f,g) = L \triangleright F$

recall:

Proposition 12.34.
[$f: \mathcal{H} \to]-\infty, +\infty]$,
$\mathcal{K}$: real Hilbert space
$L: \mathcal{H} \to \mathcal{K}$]
(i) $\operatorname{dom}(L \triangleright f) = L(\operatorname{dom} f)$
(ii) (f: convex, L: affine) $\Rightarrow$ $L \triangleright f$: convex /*wow!*/

$\therefore \operatorname{dom} \operatorname{pav}(f,g) = \operatorname{dom}(L \triangleright F) = L \operatorname{dom} F$

$$= L \left\{ (y,z) \in \mathcal{H}\times\mathcal{H} \,\middle|\, F(y,z) = \tfrac{1}{2} f(y) + \tfrac{1}{2} g(z) + \tfrac{1}{8}\|y-z\|^2 < +\infty \right\}$$

$$= L \left\{ (y,z) \in \mathcal{H}\times\mathcal{H} \,\middle|\, f(y) < +\infty,\ g(z) < +\infty \right\} \quad // L(y,z) = \tfrac{1}{2}y + \tfrac{1}{2}z$$

$$= \left\{ \tfrac{1}{2}y + \tfrac{1}{2}z \in \mathcal{H}\times\mathcal{H} \,\middle|\, y \in \operatorname{dom} f,\ z \in \operatorname{dom} g \right\}$$

$$= \tfrac{1}{2} \left\{ y+z \in \mathcal{H}\times\mathcal{H} \,\middle|\, y \in \operatorname{dom} f,\ z \in \operatorname{dom} g \right\}$$

$$= \tfrac{1}{2} \left(\{ y \in \mathcal{H} \mid y \in \operatorname{dom} f \} + \{ z \in \mathcal{H} \mid z \in \operatorname{dom} g \} \right)$$

$$= \tfrac{1}{2} \operatorname{dom} f + \tfrac{1}{2} \operatorname{dom} g$$

(iv) F: convex, L: linear $\Rightarrow$ L: affine $\therefore (L \triangleright f) = \operatorname{pav}(f,g)$: convex

now $f, g \in \Gamma_0(\mathcal{H}) \Rightarrow f, g$: proper $\Rightarrow \operatorname{dom} f \neq \emptyset$, $\operatorname{dom} g \neq \emptyset$

in (iii)

$$\left.\begin{array}{l} \operatorname{dom} \operatorname{pav}(f,g) = \tfrac{1}{2}\operatorname{dom} f + \tfrac{1}{2}\operatorname{dom} g \neq \emptyset \\ \text{also, as for } \forall_{\tilde{x}\in\mathcal{H}}\ \operatorname{pav}(f,g)(\tilde{x}) = (L \triangleright f)(\tilde{x}) \Rightarrow \operatorname{pav}(f,g)(\tilde{x}) \neq -\infty \end{array}\right\}$$

$\therefore \operatorname{dom} \operatorname{pav}(f,g) \neq \emptyset$, $\operatorname{pav}(f,g)(\mathcal{H}) \not\ni -\infty$

$\Updownarrow$ def

$\mathrm{pav}(f,g)$: proper

So, $\mathrm{pav}(f,g)$: proper, convex function. ∎

* Proposition 11.11. (Coercivity of a function in terms of lower level set)
[$f: \mathcal{H} \to [-\infty, +\infty]$] f : coercive $\leftrightarrow$ $(\mathrm{lev}_{\leq \xi} f)_{\xi \in \mathbb{R}}$: bounded

Part 1

1:51 PM

* Theorem 14.3.

[$f \in \Gamma_0(\mathcal{H})$, $\gamma \in \mathbb{R}_{++}$]

(i) $\frac{1}{2\gamma}\|\cdot\|^2 = \left(f \,\square\, \frac{1}{2\gamma}\|\cdot\|^2\right) + \left(f^* \,\square\, \frac{\gamma}{2}\|\cdot\|^2\right) \circ \frac{1}{\gamma}\mathrm{Id} = {}^{\gamma}f + {}^{(\frac{1}{\gamma})}(f^*) \circ \frac{1}{\gamma}\mathrm{Id}$

(ii) $\mathrm{Id} = \mathrm{Prox}_{\gamma f} + \gamma\, \mathrm{Prox}_{\frac{1}{\gamma} f^*} \circ \frac{1}{\gamma}\mathrm{Id}$

(iii) $\forall_{x \in \mathcal{H}}$

$f(\mathrm{Prox}_{\gamma f} x) + f^*\left(\mathrm{Prox}_{\frac{1}{\gamma} f^*}\left(\frac{x}{\gamma}\right)\right) = \langle \mathrm{Prox}_{\gamma f} x \mid \mathrm{Prox}_{\frac{1}{\gamma} f^*} \frac{x}{\gamma} \rangle$

Proof: take $\tilde{\gamma} = \frac{1}{\gamma} \in \mathbb{R}_{++}$, take $\hat{f} = f^* \in \Gamma_0(\mathcal{H})$ /* $\because f : \Gamma_0(\mathcal{H}) \Rightarrow f^* \in \Gamma_0(\mathcal{H})$

using Fenchel-Moreau theorem */

/* Recall:

* Proposition 12.15. /* Properties of power norm infimal convolution of a convex function */

[$\hat{f} \in \Gamma_0(\mathcal{H})$

$\tilde{\gamma} \in \mathbb{R}_{++}$

$p \in]1, +\infty[$

$\hat{f} \,\square\, \frac{1}{\tilde{\gamma} p}\|\cdot\|^p : \mathcal{H} \to]-\infty, +\infty] : x \mapsto \inf_{y \in \mathcal{H}} \left(\hat{f}(y) + \frac{1}{\tilde{\gamma} p}\|x - y\|^p\right)$]

- $\hat{f} \,\square\, \frac{1}{\tilde{\gamma} p}\|\cdot\|^p$: convex, real-valued, continuous, exact, infimum uniquely attained.

*

Example 13.4

[$\cancel{\varphi}\ \hat{f} : \mathcal{H} \to]-\infty, +\infty]$, proper;

$\tilde{\gamma} \in \mathbb{R}_{++}$;

$\tilde{f} = \cancel{\varphi}\ \hat{f} + \frac{1}{2\tilde{\gamma}}\|\cdot\|^2$]

$\tilde{f}^* = \frac{\tilde{\gamma}}{2}\|\cdot\|^2 - {}^{\tilde{\gamma}}\cancel{\varphi}\hat{f} \circ \tilde{\gamma}\mathrm{Id} = \frac{\tilde{\gamma}}{2}\|\cdot\|^2 - \underbrace{\left(\cancel{\varphi}\ \hat{f} \,\square\, \frac{1}{2\tilde{\gamma}}\|\cdot\|^2\right)}_{{}^{\gamma}\varphi} \circ \tilde{\gamma}\mathrm{Id}$

* Proposition 14.1.

[$\hat{f} \in \Gamma_0(\mathcal{H})$; $\gamma \in \mathbb{R}_{++}$]

$\left(\hat{f} + \frac{\tilde{\gamma}}{2}\|\cdot\|^2\right)^* = \hat{f}^* \,\square\, \frac{1}{2\tilde{\gamma}}\|\cdot\|^2 = {}^{\gamma}(\hat{f}^*)$

$\underbrace{\tilde{f}^*}_{=} = \left(\hat{f} + \frac{1}{2\tilde{\gamma}}\|\cdot\|^2\right)^* = \frac{\tilde{\gamma}}{2}\|\cdot\|^2 - {}^{\tilde{\gamma}}\hat{f} \circ \tilde{\gamma}\mathrm{Id} = \frac{\tilde{\gamma}}{2}\|\cdot\|^2 - \left(\hat{f} \,\square\, \frac{1}{2\tilde{\gamma}}\|\cdot\|^2\right) \circ \tilde{\gamma}\mathrm{Id}$

$\left(\underbrace{\hat{f}}_{\in \Gamma_0(\mathcal{H})} \,\square\, \frac{1}{2\underbrace{\tilde{\gamma}}_{\in \mathbb{R}_{++}}}\|\cdot\|^2\right)$

$\Downarrow$

${}^{\tilde{\gamma}}\hat{f} = \hat{f} \,\square\, \frac{1}{2\tilde{\gamma}}\|\cdot\|^2$: convex, continuous, real valued, exact

$\Rightarrow {}^{\tilde{\gamma}}\hat{f} = \hat{f} \,\square\, \frac{\hat{f}}{2\tilde{\gamma}}\|\cdot\|^2$

$\Rightarrow \underbrace{\left(\hat{f} + \frac{1}{2\tilde{\gamma}}\|\cdot\|^2\right)^*}_{\hat{f}^* \,\square\, \frac{1}{2\tilde{\gamma}}} = \frac{\tilde{\gamma}}{2}\|\cdot\|^2 - \left(\hat{f} \,\square\, \frac{1}{2\tilde{\gamma}}\|\cdot\|^2\right) \circ \tilde{\gamma}\mathrm{Id}$

$\Leftrightarrow \frac{\tilde{\gamma}}{2}\|\cdot\|^2 = \left(\hat{f}^* \,\square\, \frac{1}{2\tilde{\gamma}}\|\cdot\|^2\right)^* + \left(\hat{f} \,\square\, \frac{1}{2\tilde{\gamma}}\|\cdot\|^2\right) \circ \tilde{\gamma}\mathrm{Id}$ /* set $\gamma = \frac{1}{\tilde{\gamma}}$, $\tilde{f} = f^*$, and recall from Fenchel-Moreau that,

$f \in \Gamma_0(\mathcal{H}) \Rightarrow f^{**} = f$ */

$\Leftrightarrow \frac{1}{2\gamma}\|\cdot\|^2 = \left(\underbrace{f^{**}}_{f} \,\square\, \frac{\gamma}{2}\|\cdot\|^2\right)^* + \left(f^* \,\square\, \frac{\gamma}{2}\|\cdot\|^2\right) \circ \frac{1}{\gamma}\mathrm{Id}$

$= \underbrace{\left(f \,\square\, \frac{\gamma}{2}\|\cdot\|^2\right)}_{{}^{\gamma}f}{}^* + \underbrace{\left(f^* \,\square\, \frac{\gamma}{2}\|\cdot\|^2\right)}_{{}^{\gamma}f^*} \circ \frac{1}{\gamma}\mathrm{Id}$ ☺

(ii) this part of the proof is incomplete. I need to comeback to it later.

* Proposition 12.29.
[$f \in \Gamma_0(\mathcal{H})$
$\gamma \in \mathbb{R}_{++}$]

${}^{\gamma}f : \mathcal{H} \to \mathbb{R}$, Fréchet differentiable

$\nabla({}^{\gamma}f) = \frac{1}{\gamma}(Id - \mathrm{Prox}_{\gamma f})$; γ^{-1} Lipschitz continuous.

so, $\nabla^{\gamma} f = \frac{1}{\gamma}(Id - \mathrm{Prox}_{\gamma f}) \Leftrightarrow \nabla^{\gamma} f(\cdot) = \frac{1}{\gamma}(Id - \mathrm{Prox}_{\gamma f})(\cdot) = \frac{1}{\gamma}(\cdot - \mathrm{Prox}_{\gamma f}\cdot)$

$\nabla^{\gamma} f^* = \frac{1}{\gamma}(Id - \mathrm{Prox}_{\gamma f^*})$

now $\left({}^{\gamma}f^* \circ \frac{1}{\gamma} Id\right)(x) = {}^{\gamma}f^*\left(\frac{1}{\gamma}x\right)$

recall, $(\nabla \cdot) = (D \cdot)^T$

chain rule says: $D(R \circ T)(x) = DR(Tx) \circ DT(x)$

so, $D(\underbrace{{}^{\gamma}f^* \circ \tfrac{1}{\gamma} Id}_{\mho})(x) = D\left[{}^{\gamma}f^*\left(\frac{1}{\gamma} Id\, x\right)\right] \circ D \frac{1}{\gamma} Id(x)$

$= D\left[{}^{\gamma}f^*\left(\frac{x}{\gamma}\right)\right] \circ \frac{1}{\gamma} \underbrace{Dx}_{Id}$

$= D\left[{}^{\gamma}f^*\left(\frac{x}{\gamma}\right)\right] \circ \frac{1}{\gamma} Id$

* $DT(x) \overset{\text{def}}{\Leftrightarrow} \lim_{0 \neq y \to 0} \dfrac{\| T(x+y) - Tx - DT(x)y \|}{\|y\|} = 0$

here $T = Id$

$\lim_{0 \neq y \to 0} \dfrac{\| x+y-x-(\overset{Id}{\widetilde{Dx}})y \|}{\|y\|} = 0$ Id satisfies the equation

$\therefore Dx = Id$ */

$\therefore D\mho(x)\, y = \left(D\, {}^{\gamma}f^*\left(\frac{x}{\gamma}\right) \circ \frac{1}{\gamma} Id\right)(y)$

$= \underbrace{D\, {}^{\gamma}f^*\left(\frac{x}{\gamma}\right)}_{\text{linear operator}} \left(\frac{y}{\gamma}\right) = \frac{1}{\gamma} D\, {}^{\gamma}f^*\left(\frac{x}{\gamma}\right)(y)$

now the relation between ∇ and D says

$\forall_{y \in \mathcal{H}} \quad D\mho(x)y = \langle y \mid \nabla \mho(x) \rangle$

$\Leftrightarrow \frac{1}{\gamma} \underbrace{D\, {}^{\gamma}f^*\left(\frac{x}{\gamma}\right)}(y) = \langle y \mid \nabla \mho(x) \rangle$

from (i)

$\frac{1}{2\gamma} \|\cdot\|^2 = {}^{\gamma}f + {}^{\gamma}f^* \circ \frac{1}{\gamma} Id$

taking derivative on both sides.

$Id = \mathrm{Prox}_{\gamma f} + \gamma\, \mathrm{Prox}_{\frac{1}{\gamma} f^*} \circ \frac{1}{\gamma} Id$.

(iii) incomplete.

☹

Theorem 14.17.
(Moreau-Rockafellar theorem)

[[$f \in \Gamma_0(\mathcal{H})$, $u \in \mathcal{H}$]]

$f - \langle \cdot \mid u \rangle$: coercive $\Leftrightarrow u \in \mathrm{int}\,\mathrm{dom} f^*$

Proof:

Required info:

* Proposition 14.16. (Alternative charecterization of coercive functions */
[$f \in \Gamma_0(\mathcal{H})$] /* info: f : coercive $\Leftrightarrow \lim_{\|x\| \to +\infty} f(x) = +\infty$ */

(i) f : coercive $\Leftrightarrow$

(ii) $(\mathrm{lev}_{\leq \xi} f)_{\xi \in \mathbb{R}}$: bounded $\Leftrightarrow$

(iii) $\lim_{\|x\| \to +\infty} \dfrac{f(x)}{\|x\|} > 0$ $\Leftrightarrow$

(iv) $\exists_{(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}}$ $f \geq \alpha \|\cdot\| + \beta$ $\Leftrightarrow$

(v) f^*: bounded above on a neighborhood of 0 $\Leftrightarrow$

(vi) $0 \in \mathrm{int}\,\mathrm{dom} f^*$

* Proposition 13.20. (Easy conjugate formulas)

(vi) $0 \in \operatorname{int} \operatorname{dom} f^*$

* Proposition 13.20. (Easy conjugate formulas)

$[\![\, f : \mathcal{H} \to]-\infty, +\infty] \,]\!]$

(i) $\forall_{\alpha \in \mathbb{R}_{++}} \quad (\alpha f)^* = \alpha f^*(\cdot/\alpha)$

(ii) $\forall_{\alpha \in \mathbb{R}_{++}} \quad (\alpha f(\cdot/\alpha))^* = \alpha f^*$

(iii) $\forall_{y,v \in \mathcal{H}} \forall_{\alpha \in \mathbb{R}} \quad (\tau_y f + \langle \cdot | v \rangle + \alpha)^* = \tau_v f^* + \langle y | \cdot \rangle - \langle y | v \rangle - \alpha$ /* here $\tau_z f$: translation of f by z, $x \mapsto f(x-z)$ */

(iv) $L : \in \mathcal{B}(\mathcal{H})$, bijective $\Rightarrow (f \circ L)^* = f^* \circ L^{*-1}$

(v) $f^{\vee *} = f^{* \vee}$ /* $f^\vee$: reversal of a function $f \overset{\text{def}}{\Leftrightarrow} f^\vee : x \mapsto f(-x)$ */

(vi) (V: closed linear subspace of $\mathcal{H}$, $\operatorname{dom} f \subseteq V$) $\Rightarrow (f|_V)^* \circ P_V = f^* = f^* \circ P_V$

$f - \langle \cdot | u \rangle$: coercive

$\Leftrightarrow 0 \in \operatorname{int} \operatorname{dom} (f - \langle \cdot | u \rangle)^*$

/* now lets us show, $(f - \langle \cdot | u \rangle)^* = \tau_{-u} f^*$

$(\tau_y f + \langle \cdot | v \rangle + \alpha)^* = \tau_v f^* + \langle y | \cdot \rangle - \langle y | v \rangle - \alpha \quad \forall_{y,v,\alpha}$

set $y := 0$ then $\tau_y f(\cdot) = f(\cdot - \ ) = f(\cdot)$

$v := -u$, $\alpha := 0$

then

$(f - \langle \cdot | u \rangle)^* = \tau_{-u}$ */

$\Leftrightarrow 0 \in \operatorname{int} \operatorname{dom} (\underbrace{\tau_{-u} f^*}) = \operatorname{int} \operatorname{dom} f^*(\cdot + u)$

$\tau_{-u} f^* = f^*(\cdot + u)$

$\Leftrightarrow u \in \operatorname{int} \operatorname{dom} f^*$

/* As, $f \in \Gamma_0(\mathcal{H}) \Rightarrow f^* \in \Gamma_0(\mathcal{H}) \Rightarrow \begin{cases} -\infty \notin f^*(\mathcal{H}). \\ \operatorname{dom} f^* = \{x \in \mathcal{H} \mid f^*(x) < +\infty\} \end{cases}$

$\Rightarrow \operatorname{dom} f^* = \{x \in \mathcal{H} \mid -\infty < f^*(x) < +\infty\}$

$\exists_{M \in \mathbb{R}} \forall_{x \in \operatorname{dom} f} \quad f^*(x) \leq M$

$\therefore \operatorname{dom} f^* = \{x \in \mathcal{H} \mid f^*(x) \leq M\}$

$\Rightarrow \operatorname{int} \operatorname{dom} f^* = \{x \in \mathcal{H} \mid f^*(x) < M\}$

so. $0 \in \operatorname{int} \operatorname{dom} f^*(\cdot + u) = \{x \in \mathcal{H} \mid f^*(x+u) < M\}$

$\Leftrightarrow f^*(0+u) < M$

$\Leftrightarrow f^*(u) < M \Leftrightarrow u \in \operatorname{int} \operatorname{dom} f^*$ */

Proposition. (14.19) (Conjugate of the difference)

$[\![\; g : \mathcal{H} \to]-\infty, +\infty]$; $h \in \Gamma_0(\mathcal{H})$;

$f : \mathcal{H} \to [-\infty, +\infty] : x \mapsto \begin{cases} g(x) - h(x), & x \in \operatorname{dom} g = \{\tilde{x} \in \mathcal{H} \mid g(\tilde{x}) < +\infty\} \\ +\infty, & x \notin \operatorname{dom} g \end{cases}$

$]\!] \Rightarrow$

$$\forall_{u \in \mathcal{H}} \quad f^*(u) = \sup_{v \in \operatorname{dom} h^*} \left(g^*(u+v) - h^*(v) \right)$$

Proof:

$\forall u \in \mathcal{H}$

/* recall

★★★

Corollary 13.33. (An immediate consequence of Fenchel-Moreau theorem)

$[\![\, f \in \Gamma_0(\mathcal{H}) \,]\!]$

- $f^* \in \Gamma_0(\mathcal{H})$
- $f^{**} = f$

*/

$$f^*(u) = \sup_{x \in \mathcal{H}} \left(\langle x | u \rangle - f(x) \right)$$

$$= \sup_{x \in \mathcal{H}} \left(\langle x | u \rangle - (g(x) - h(x)) \; [x \in \operatorname{dom} g] \right)$$

$x \in \mathcal{H}$

$$= \sup_{x \in \mathcal{H}} \left(\langle x | u \rangle - \left(g(x) - h(x) \right) \left[x \in \operatorname{dom} g \right] \right)$$

$$= \sup_{x \in \operatorname{dom} g} \left(\langle x | u \rangle - g(x) + h(x) \right) \quad /^{*} \text{But } h \in \Gamma_0(\mathcal{H}), \text{ so } h^{**} = h \ ^{*}/$$

$$= \sup_{x \in \operatorname{dom} g} \left(\langle x | u \rangle - g(x) + \underbrace{h^{**}(x)}_{= \sup_{v \in \operatorname{dom} h^{*}} \left(-h^{*}(v) + \langle v | x \rangle \right)} \right)$$

$/^{*}$ By def: $f^{*}(u) = \sup_{x \in \mathcal{H}} \left(-f(x) + \langle x | u \rangle \right) = \sup_{x \in \operatorname{dom} f} \left(-f(x) + \langle x | u \rangle \right)$ $^{*}/$

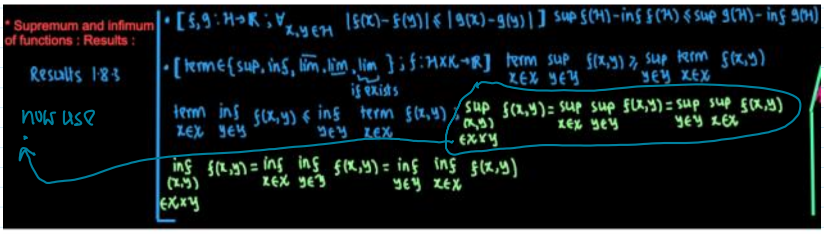
$$= \sup_{x \in \operatorname{dom} g} \left(\underbrace{\langle x | u \rangle - g(x)}_{} + \sup_{v \in \operatorname{dom} h^{*}} \left(\langle v | x \rangle - h^{*}(v) \right) \right)$$

constant w.r.t v, so can be taken inside

interchangeable

$$= \sup_{x \in \operatorname{dom} g} \ \sup_{v \in \operatorname{dom} h^{*}} \left(\langle x | u \rangle - g(x) + \underbrace{\langle v | x \rangle}_{= \langle x | v \rangle} - h^{*}(v) \right)$$

$/^{*}$



$^{*}/$

$$= \sup_{v \in \operatorname{dom} h^{*}} \ \sup_{x \in \operatorname{dom} g} \left(\langle x | u + v \rangle - g(x) - h^{*}(v) \right)$$

$$= \sup_{v \in \operatorname{dom} h^{*}} \left(-h^{*}(v) + \underbrace{\sup_{x \in \operatorname{dom} g} \left(-g(x) + \langle x | u + v \rangle \right)}_{g^{*}(u+v)} \right) \quad ^{*}/$$

$$= \sup_{v \in \operatorname{dom} h^{*}} \left(g^{*}(u+v) - h^{*}(v) \right)$$

■