Part 1

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Example 13.9.

$[\![\ \varphi : \mathcal{H} \to]-\infty, +\infty]$, proper;

$\gamma \in \mathbb{R}_{++}$;

$f = \varphi + \frac{1}{2\gamma}\|\cdot\|^2\ ]\!]$

$$f^* = \frac{\gamma}{2}\|\cdot\|^2 - {}^{\gamma}\varphi \circ \gamma \mathrm{Id} = \frac{\gamma}{2}\|\cdot\|^2 - \left(\varphi \,\square\, \frac{1}{2\gamma}\|\cdot\|^2\right)\circ \gamma \mathrm{Id}$$

Proof:

$u \in \mathcal{H}$

$$f^*(u) = \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - f(x)\right)$$

$$= -\inf_{x\in\mathcal{H}} \left(-\langle x|u\rangle + f(x)\right) \quad /* \text{using } \sup(\cdot) = -\inf(-\cdot) */$$

$$= -\inf_{x\in\mathcal{H}} \left(-\langle x|u\rangle + \varphi(x) + \frac{1}{2\gamma}\|x\|^2\right) \quad /* \ \frac{1}{2\gamma}\|x\|^2 - \langle x|u\rangle = \frac{1}{2\gamma}\big(\underbrace{\|x\|^2 - 2\gamma\langle x|u\rangle + \gamma^2\|u\|^2}_{\|x-\gamma u\|^2} - \gamma^2\|u\|^2\big)$$

$$= \frac{1}{2\gamma}\left(\|x-\gamma u\|^2 - \gamma^2\|u\|^2\right) */$$

$$= -\inf_{x\in\mathcal{H}} \Big(\varphi(x) + \frac{1}{2\gamma}\|x-\gamma u\|^2 - \underbrace{\frac{\gamma}{2}\|u\|^2}_{\substack{\text{constanst w.r.t } x \\ \text{so বাইরে আনা যাবে}}}\Big)$$

$$= \frac{\gamma}{2}\|u\|^2 - \underbrace{\inf_{x\in\mathcal{H}} \left(\varphi(x) + \frac{1}{2\gamma}\|x-\gamma u\|^2\right)}_{{}^{\gamma}\varphi(\gamma u)} \quad /* \text{ now: Moreau envelope, } {}^{\gamma}f(x) = \inf_{y\in\mathcal{H}}\left(f(x) + \frac{1}{2\gamma}\|x-y\|^2\right) */$$

$$= \frac{\gamma}{2}\|u\|^2 - {}^{\gamma}\varphi(\gamma u)$$

■

Example 13.8. $[\ \varphi : \mathcal{H} \to]-\infty,+\infty]$; f: perspective function of φ : $\mathbb{R}\times\mathcal{H} \to]-\infty,+\infty] : (\xi, x) \mapsto \begin{cases} \xi\varphi(x/\xi), & \text{if } \xi > 0 \\ +\infty, & \text{else}\end{cases}$

$C = \{(v,u) \in \mathbb{R}\times\mathcal{H} \mid v + \varphi^*(u) \leq 0\}\ ]\!] \Rightarrow f^* = \iota_C$

Proof:

$$f^*(v,u) = \sup_{(\xi,x)\in\mathbb{R}\times\mathcal{H}} \left(-f(\xi,x) + \langle(\xi,x) \mid (v,u)\rangle\right) \quad /* \ \tilde{f}^*(\tilde{u}) = \sup_{\tilde{x}\in\mathcal{H}}\left(-\tilde{f}(\tilde{x}) + \langle\tilde{x}|\tilde{u}\rangle\right) */$$

$$= \sup_{(\xi,x)\in\mathbb{R}\times\mathcal{H}} \left(-\left[\begin{cases}\xi\varphi(x/\xi), & \text{if } \xi>0 \\ +\infty, & \text{else}\end{cases}\right] + \langle\xi|v\rangle + \langle x|u\rangle\right)$$

$$= \sup_{\substack{(\xi,x)\in\mathbb{R}\times\mathcal{H} \\ \xi>0}} \left(-\xi\varphi(x/\xi) + \langle\xi|v\rangle + \langle x|u\rangle\right) \quad /* \text{ now } (x,\xi) \text{ independent, so we can sup over } x \text{ first and then over } \xi */$$

/* otherwise the value is $-\infty$ */

$$= \sup_{\xi\in\mathbb{R}_{++}} \sup_{x\in\mathcal{H}} \Big(-\xi\varphi(x/\xi) + \underbrace{\langle\xi|v\rangle}_{\text{constant w.r.t. } x} + \langle x|u\rangle\Big)$$

$$= \sup_{\xi\in\mathbb{R}_{++}} \Big(\langle\xi|v\rangle + \underbrace{\sup_{x\in\mathcal{H}}\left(\langle x|u\rangle - \xi\varphi(x/\xi)\right)}\Big)$$

$$\sup_{x\in\mathcal{H}} \xi\left(\left\langle \frac{x}{\xi}\Big|u\right\rangle - \varphi(x/\xi)\right) = \xi \sup_{x\in\mathcal{H}}\left(-\varphi\left(\frac{x}{\xi}\right) + \left\langle\frac{x}{\xi}\Big|u\right\rangle\right) = \xi\,\underbrace{\sup_{y=\frac{x}{\xi}\in\mathcal{H}}\left(-\varphi(y) + \langle y|u\rangle\right)}_{\varphi^*(u)} = \xi\varphi^*(u)$$

$$= \sup_{\xi\in\mathbb{R}_{++}}\Big(\underbrace{\langle\xi|v\rangle}_{\xi v \ //\text{as both } \xi, v \text{ are real numbers}} + \xi\varphi^*(u)\Big) = \sup_{\xi\in\mathbb{R}_{++}}\left(\xi v + \xi\varphi^*(u)\right) = \sup_{\xi\in\mathbb{R}_{++}} \xi\left(v + \varphi^*(u)\right) = \left(v+\varphi^*(u)\right)\sup_{\xi\in\mathbb{R}_{++}}\xi = \begin{cases} 0, & \text{if } v+\varphi^*(u) \leq 0 \\ +\infty, & \text{else}\end{cases} = \iota_C \ \{C = \{(v,u)\in\mathbb{R}\times\mathcal{H} \mid v + \varphi^*(u) \leq 0\}\}$$

$$\therefore f^* = \iota_C \ \{C = \{(v,u)\in\mathbb{R}\times\mathcal{H} \mid v+\varphi^*(u) \leq 0\}\}$$

■

***Proposition 13.10.**

$[\ f : \mathcal{H} \to [-\infty, +\infty]\ ]$

(i) $[\ (u,v) \in \mathcal{H}\times\mathbb{R}\ ]$

$(u,v) \in \operatorname{epi} f^* \leftrightarrow \langle\cdot|u\rangle - v \leq f$

(ii) $f^* \equiv +\infty \leftrightarrow f$: posseses no continuous affine minorant

(iii) $[\ \operatorname{dom} f^* \neq \varnothing\ ]$

f: bounded below on every bounded subset of $\mathcal{H}$.

Proof:

(iii) [$\operatorname{dom} f^* \neq \emptyset$]

f: bounded below on every bounded subset of $\mathcal{H}$.

Proof:

(i) $(u,v) \in \operatorname{epi} f^*$

$\Leftrightarrow f^*(u) \leqslant v$ /* $f^*(u) = \sup_{x\in\mathcal{H}} \left(\langle u|x\rangle - f(x) \right)$ */

$\Leftrightarrow \sup_{x\in\mathcal{H}} \left(\langle u|x\rangle - f(x) \right) \leqslant v$

$\Leftrightarrow \forall_{x\in\mathcal{H}} \quad \langle u|x\rangle - f(x) \leqslant v$

$\Leftrightarrow \forall_{x\in\mathcal{H}} \quad \langle u|x\rangle - v \leqslant f(x)$

$\Leftrightarrow \langle u|\cdot\rangle - v \leqslant f$ ☺

(ii)

$f^* = +\infty$

$\Leftrightarrow \operatorname{epi} f^* = \{(x,t) \in \mathcal{H}\times\mathbb{R} \mid f(x) \leqslant t\} = \varphi$

$\Leftrightarrow \neg\left(\exists_{(u,v)\in \operatorname{epi} f^*}\right)$

$\Leftrightarrow \neg\left(\exists_{(u,v)\in\mathcal{H}\times\mathbb{R}} \langle u|\cdot\rangle - v \leqslant f\right)$ /* from (i) */

$\Leftrightarrow \forall_{(u,v)\in\mathcal{H}\times\mathbb{R}} \quad \langle u|\cdot\rangle - v > f$

$\Leftrightarrow f$: posseses no continuous affine minorant.

(iii)

$\operatorname{dom} f^* = \{x\in\mathcal{H} \mid f(x) < +\infty\} \neq \emptyset$

$\Leftrightarrow \exists_{u\in\mathcal{H}} \quad f^*(u) < +\infty$

$\Leftrightarrow \exists_{(u,v)\in\mathcal{H}\times\mathbb{R}} \quad f^*(u) = v < +\infty$

$\Leftrightarrow (u,v) \in \operatorname{epi} f^*$

$\Leftrightarrow \langle \cdot|u\rangle - v \leqslant f$ /* from (i) */

Consider a bounded set C in $\mathcal{H}$

take $\beta = \sup_{x\in C} \|x\|$ then $\forall_{x\in C} \; \beta = \sup_{x\in C} \|x\| \geqslant \|x\| \Rightarrow \forall_{x\in C} \; -\beta \leqslant -\|x\|$ . . . (1)

$\forall_{x\in C} \quad f(x) \geqslant \langle x|u\rangle - v$

$\geqslant -\|u\|\,\|x\| - v$

/* Cauchy Schwartz:

$\underbrace{|\langle x|u\rangle|}_{\max\{\langle x|u\rangle, -\langle x|u\rangle\}} \leqslant \|x\|\,\|u\|$

so, $\langle x|u\rangle \leqslant \|x\|\,\|u\|$

$-\langle x|u\rangle \leqslant \|x\|\,\|u\|$

$\Leftrightarrow \langle x|u\rangle \geqslant -\|x\|\,\|u\|$ */

$\Leftrightarrow \forall_{x\in C} \quad f(x) \geqslant -\|x\|\,\|u\| - v$

$\geqslant -\beta\,\|u\| - v$

/* from (1) */

$= \underbrace{-\beta\|u\|}_{\text{finite}} \underbrace{-v}_{\text{finite}}$

$> -\infty$

$\therefore f$: bounded below on every bounded subset of $\mathcal{H}$.

■

Proposition 13.13. (Fenchel-Young inequality)

[[$f: \mathcal{H} \to]-\infty, +\infty]$, proper]]

$\forall_{x\in\mathcal{H}} \; \forall_{u\in\mathcal{H}} \quad f(x) + f^*(u) \geqslant \langle x|u\rangle$

Proof:

$\forall_{x\in\mathcal{H}} \; \forall_{u\in\mathcal{H}}$

f: proper $\Rightarrow -\infty \notin f(\mathcal{H})$, $\operatorname{dom} f = \{\tilde{x}\in\mathcal{H} \mid f(\tilde{x}) < +\infty\} \neq \emptyset$

if $f(x) = +\infty$, the inequality trivially holds

$f(x) < +\infty \Rightarrow f^*(u) = \sup_{\tilde{x}\in\mathcal{H}} \left(\langle \tilde{x}|u\rangle - f(\tilde{x}) \right) \geqslant \langle \tilde{x}|u\rangle - f(\tilde{x}) \quad \forall_{\tilde{x}\in\mathcal{H}}$

$\overset{\tilde{x}:=x}{\Rightarrow} \quad f^*(u) \geqslant \langle x|u\rangle - f(x)$

$\therefore f^*(u) + f(x) \geqslant \langle x|u\rangle$

■

* Proposition 13.18.

[$f: \mathcal{H} \to]-\infty, +\infty]$, f: even]

f^*: even

Proof: $\forall u \in \mathcal{H}$

$f^*(-u) = \sup_{x\in\mathcal{H}} \left(\langle x|-u\rangle - f(x) \right)$

$f(-x)$ /* f: even */

$= \sup_{x\in\mathcal{H}} \left(\langle -x|u\rangle - f(-x)\right)$

$= \sup_{(-x)\in\mathcal{H}} \left(\langle -x|u\rangle - f(-x)\right)$ /* e.g. $\max_{x\in\mathcal{H}} f(x) = \max_{(-x)\in\mathcal{H}} f(-x)$ */

$= \sup_{y\in\mathcal{H}} \left(\langle y|u\rangle - f(y)\right)$

$= f^*(u)$

$\therefore \forall_{u\in\mathcal{H}}\ f^*(-u) = f^*(u)$

$\leftrightarrow f^*$: even.

■

✱ Proposition 13.19.

⟦ $f:\mathcal{H}\to]-\infty,+\infty]$, $f \geq f(0)=0$ ⟧

$f^* \geq f^*(0) = 0$

Proof:

$f^*(u) = \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - f(x)\right)$

$\therefore f^*(0) = \sup_{x\in\mathcal{H}} \left(\underbrace{\langle x|0\rangle}_{=0} - f(x)\right)$

$= \sup_{x\in\mathcal{H}} \left(-f(x)\right)$

$= -\inf_{x\in\mathcal{H}} \left(f(x)\right)$ /* $\inf(\cdot) = -\sup(-\cdot)$ */

$= -\inf f(\mathcal{H})$

now given, $f \geq f(0) = 0$

$\leftrightarrow \forall_{x\in\mathcal{H}}\ f(x) \geq f(0) = 0$

$\leftrightarrow \inf_{x\in\mathcal{H}} f(x) \geq f(0) = 0$, as $f(0)=0$, and $\inf f(\mathcal{H}) \geq 0$, the minimum value will be achieved at $x=0$

$\leftrightarrow \inf_{x\in\mathcal{H}} f(x) = \min_{x\in\mathcal{H}} f(x) = \underbrace{f(0)}_{=0}$ /* $0\in\mathcal{H}$. so equality will hold */

$\therefore \inf f(\mathcal{H}) = 0$

$\therefore f^*(0) = 0 = f(0)$

Again, $\forall_{u\in\mathcal{H}}\ f^*(u) = \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - f(x)\right) \geq \langle x|u\rangle - f(x)\ \forall_{x\in\mathcal{H}}$

$\overset{x:=0}{\Rightarrow} \forall_{u\in\mathcal{H}}\ f^*(u) \geq \underbrace{\langle 0|u\rangle}_{=0} - f(0) = -f^*(0) = 0 = f^*(0)$

$\therefore \forall_{u\in\mathcal{H}}\ f^*(u) \geq f^*(0) = 0$

$\leftrightarrow f^* \geq f^*(0) = 0$

■

Proposition 13.21.

⟦ $f,g:\mathcal{H}\to]-\infty,+\infty]$ ⟧

(i) $(f\,\square\,g)^* = f^* + g^*$

(ii) f,g: proper $\Rightarrow (f+g)^* \leq f^*\,\square\,g^*$

(iii) $\forall_{\gamma\in\mathbb{R}_{++}}\ ({}^{\gamma}f)^* = f^* + \frac{\gamma}{2}\|\cdot\|^2$

(iv) $L\in\mathcal{B}(\mathcal{H},\mathcal{K}) \Rightarrow (L \triangleright f)^* = f^* \circ L^*$

(v) $L\in\mathcal{B}(\mathcal{K},\mathcal{H}) \Rightarrow (f\circ L)^* \leq L^* \triangleright f^*$

Proof:

(i) denote $f\,\square\,g = h$ /* $(f\,\square\,g)(z) = \inf_{v\in\mathcal{H}} \left(f(v) + g(z-v)\right)$ */

$\therefore h(x) = \inf_{y\in\mathcal{H}} \left(f(y) + g(x-y)\right)$

$\forall_{u\in\mathcal{H}}$

$h^*(u) = \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - h(x)\right)$

$= \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - \underbrace{\inf_{y\in\mathcal{H}} \left(f(y)+g(x-y)\right)}_{-\sup_{y\in\mathcal{H}}\left(-f(y)-g(x-y)\right)}\right)$ /* $\because \inf(\cdot) = -\sup(-\cdot)$ */

$= \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle + \sup_{y\in\mathcal{H}} \left(-f(y) - g(x-y)\right)\right)$

$= \sup_{x\in\mathcal{H}} \sup_{y\in\mathcal{H}} \left(\langle x|u\rangle - f(y) - g(x-y)\right)$

$$= \sup_{y\in\mathcal{H}} \sup_{x\in\mathcal{H}} \Big(\langle x|u\rangle - f(y) - g(x-y) \Big)$$

/* $\langle y|u\rangle - \langle y|u\rangle + \langle x|u\rangle - f(y) - g(x-y)$

$= \langle y|u\rangle + \langle x-y|u\rangle - f(y) - g(x-y)$ /* $\langle a+b|c\rangle = \langle a|c\rangle + \langle b|c\rangle$ */

$= \big(\langle y|u\rangle - f(y)\big) + \big(\langle x-y|u\rangle - g(x-y)\big)$ */

$$= \sup_{y\in\mathcal{H}} \sup_{x\in\mathcal{H}} \Big(\big(\langle y|u\rangle - f(y)\big) + \big(\langle x-y|u\rangle - g(x-y)\big) \Big)$$

$$= \sup_{y\in\mathcal{H}} \Big[\big(\langle y|u\rangle - f(y)\big) + \sup_{x\in\mathcal{H}} \big(\langle x-y|u\rangle - g(x-y)\big) \Big]$$

$= \sup_{x-y\in\mathcal{H}} \big(\langle x-y|u\rangle - g(x-y)\big)$ /* as y is a constant w.r.t x we can optimize w.r.t $x-y \in \mathcal{H}$ and the sup$(\cdot)$ value would be the same */

$= g^*(u)$

$$= \sup_{y\in\mathcal{H}} \big(\langle y|u\rangle - f(y) + g^*(u) \big)$$

$$= g^*(u) + \underbrace{\sup_{y\in\mathcal{H}} \big(\langle y|u\rangle - f(y) \big)}_{f^*(u)}$$

$$= f^*(u) + g^*(u)$$

$\therefore \forall_{u\in\mathcal{H}} \quad (f \,\square\, g)^*(u) = f^*(u) + g^*(u)$

$\Leftrightarrow (f \,\square\, g)^* = f^* + g^*$ ☺

(ii)

/* Recall

* Proposition 13.14.

$[\, f, g : \mathcal{H} \to [-\infty, +\infty] \,]$

(i) $f^{**} \leq f$

(ii) $f \leq g \Rightarrow (f^* \geq g^*,\ f^{**} \leq g^{**})$

(iii) $f^{***} = f^*$

(iv) $(\breve{f})^* = f^*$ /* $\breve{f} = \sup\{ g \in \Gamma(\mathcal{H}) : g \leq f \}$: lower semicontinuous convex envelope of f */

*/

$f \geq f^{**}$

$g \geq g^{**}$

$f + g \geq f^{**} + g^{**}$

$\Rightarrow (f+g)^* \leq (f^{**} + g^{**})^*$

/* in (i) we have shown $f^* + g^* = (f \,\square\, g)^*$

so, $f^{**} + g^{**} = (f^*)^* + (g^*)^* = (f^* \,\square\, g^*)^*$ */

$= \big((f^* \,\square\, g^*)^* \big)^*$

$= (f^* \,\square\, g^*)^{**}$

$\leq f^* \,\square\, g^*$

$\therefore (f+g)^* \leq f^* \,\square\, g^*$ ☺

(iii)

recall that, ${}^{\gamma}f = f \,\square\, \frac{1}{2\gamma}\|\cdot\|^2$

so, $({}^{\gamma}f)^* = \big(f \,\square\, \frac{1}{2\gamma}\|\cdot\|^2 \big)^*$

$= f^* + \frac{1}{\gamma}\big(\frac{1}{2}\|\cdot\|^2\big)^*$ /* from (i) $(f \,\square\, g)^* = f^* + g^*$ */

$= f^* + \frac{1}{2\gamma}\|\cdot\|^2$ /* $\frac{1}{2}\|\cdot\|^2$ is the only self-conjugate function */

(iv)

$L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ /* recall that : $(L \triangleright f)(y) = \inf_{x\in\mathcal{H} : Lx = y} f(x)$: infimal postcomposition of f by L

$L \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \Leftrightarrow L$: continuous linear operator */

$\forall_{v\in\mathcal{K}}$

$(L \triangleright f)^*(v)$ /* say $L \triangleright f = h$ */

$= h^*(v)$

$(L \triangleright f)^*(v)$ /* say $L \triangleright f = h$ */

$= h^*(v)$

$= \sup\limits_{y \in \mathcal{K}} \left(\langle y | v \rangle - \underbrace{h(y)}_{\parallel} \right)$

/* $(L \triangleright f)(y) = \inf\limits_{x \in \mathcal{H}: Lx = y} f(x)$ */

$= \sup\limits_{y \in \mathcal{K}} \left(\langle y | v \rangle - \underbrace{\inf\limits_{x \in \mathcal{H}: Lx = y} f(x)}_{-\sup\limits_{x \in \mathcal{H}: Lx=y} -f(x)} \right)$ /* $\because \inf(\cdot) = -\sup(-\cdot)$ */

$= \sup\limits_{y \in \mathcal{K}} \left(\langle y | v \rangle + \sup\limits_{x \in \mathcal{H}: Lx = y} (-f(x)) \right)$

$= \sup\limits_{y \in \mathcal{K}} \sup\limits_{x \in \mathcal{H}: Lx = y} \left[\langle y | v \rangle - f(x) \right]$

/*

* Supremum and infimum of functions : Results :

- $[f, g : \mathcal{H} \to \mathbb{R}; \forall_{x,y \in \mathcal{H}} |f(x) - f(y)| \leq |g(x) - g(y)|]$ $\sup f(\mathcal{H}) - \inf f(\mathcal{H}) \leq \sup g(\mathcal{H}) - \inf g(\mathcal{H})$
- $[\text{term} \in \{\sup, \inf, \overline{\lim}, \underline{\lim}, \underbrace{\lim}_{\text{if exists}}\}; f : \mathcal{H} \times \mathcal{K} \to \mathbb{R}]$ $\text{term}_{x \in \mathcal{X}} \sup\limits_{y \in \mathcal{Y}} f(x,y) \geq \sup\limits_{y \in \mathcal{Y}} \text{term}_{x \in \mathcal{X}} f(x,y)$

$\text{term}_{x \in \mathcal{X}} \inf\limits_{y \in \mathcal{Y}} f(x,y) \leq \inf\limits_{y \in \mathcal{Y}} \text{term}_{x \in \mathcal{X}} f(x,y)$; $\sup\limits_{(x,y) \in \mathcal{X} \times \mathcal{Y}} f(x,y) = \sup\limits_{x \in \mathcal{X}} \sup\limits_{y \in \mathcal{Y}} f(x,y) = \sup\limits_{y \in \mathcal{Y}} \sup\limits_{x \in \mathcal{X}} f(x,y)$

$\inf\limits_{(x,y) \in \mathcal{X} \times \mathcal{Y}} f(x,y) = \inf\limits_{x \in \mathcal{X}} \inf\limits_{y \in \mathcal{Y}} f(x,y) = \inf\limits_{y \in \mathcal{Y}} \inf\limits_{x \in \mathcal{X}} f(x,y)$

*/

$= \left[\begin{array}{ll} \sup\limits_{(x,y) \in \mathcal{H} \times \mathcal{K}} & \langle y | v \rangle - f(x) \\ \text{s.t} & Lx = y \end{array} \right]$

$= \sup\limits_{(x,y) \in \mathcal{H} \times \mathcal{K}} \langle y | v \rangle - f(x) + \iota_{Lx=y}(x,y)$

$= \sup\limits_{x \in \mathcal{H}} \sup\limits_{y \in \mathcal{K}} \left(\langle y | v \rangle - f(x) + \iota_{Lx=y}(x,y) \right)$

$= \sup\limits_{x \in \mathcal{H}} -f(x) + \underbrace{\sup\limits_{y \in \mathcal{K}} \left(\langle y | v \rangle + \iota_{Lx=y}(x,y) \right)}_{\langle Lx | v \rangle}$ /* as : $y = Lx$ for a fixed x is a single vector, so we are supremizing over a single vector

$\therefore \sup\limits_{y \in \mathcal{K}} \left(\langle y | v \rangle + \iota_{Lx=y}(x,y) \right) = \langle Lx | v \rangle$ */

$= \sup\limits_{x \in \mathcal{H}} \langle Lx | v \rangle - f(x)$ /* recall that, a linear bounded (continuous) operator $L \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has its adjoint defined as:

$\forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} \langle Lx | y \rangle = \langle x | L^* y \rangle$ */

$= \sup\limits_{x \in \mathcal{H}} \langle x | L^* v \rangle - f(x)$

$= f^*(L^* v)$ /* $f^*(u) = \sup\limits_{x \in \mathcal{H}} \langle x | u \rangle - f(x)$ */

$\forall_{v \in \mathcal{K}} \quad (L \triangleright f)^*(v) = f^*(L^* v) = f^* L^* v = (f^* \circ L^*) v$

$\leftrightarrow \quad (L \triangleright f)^* = (f^* \circ L^*)$ /* Caution : f^* : conjugate of f. L^* : adjoint of L */

☺

(v)

these stars stand for conjugate

these stars stand for adjoint

$(L^* \triangleright f^*)^* = (f^*)^* \circ (L^*)^*$ /* from (iv) $(L \triangleright f)^* = f^* \circ L^*$ */

$= f^{**} \circ \underbrace{L^{**}}_{L}$ /* for a continuous linear operator L, $L^{**} = L$: Fact 2.18-(i) */

$= f^{**} \circ L$ /* for any $g : \mathcal{H} \to [-\infty, +\infty]$, $g^{**} \leq g \leftrightarrow \forall_{x \in \mathcal{H}} g^{**}(x) \leq g(x)$

now, $\forall_{x \in \mathcal{H}} \quad f^{**}(x) \leq f(x)$

set $x := Ly$

then $\forall_{y \in \mathcal{K}} \quad f^{**}(Ly) \leq f(Ly) \leftrightarrow f^{**} \circ L \leq f \circ L$ */

$\leq f \circ L$

so,

$(L^* \triangleright f^*)^* \leq f \circ L$ /* from Proposition 13.14 (ii) $f \leq g \Rightarrow (f^* \geq g^*, f^{**} \leq g^{**})$

$\Rightarrow (L^* \triangleright f^*)^{**} \geq (f \circ L)^*$ /* from Proposition 13.14 (i) : $h^{**} \leq h$

$\therefore (L^* \triangleright f^*)^{**} \leq L^* \triangleright f^*$ */

$\therefore L^* \triangleright f^* \geq (f \circ L)^*$

$\blacksquare$

* Proposition 13.25.

$[\![(f_i)_{i \in I}$: family of proper functions, $: \mathcal{H} \to]-\infty, +\infty]]\!]$

(i) $\left(\inf\limits_{i \in I} f_i \right)^* = \left(\sup\limits_{i \in I} f_i^* \right)$

(ii) $\left(\sup_{i\in I} f_i\right)^* \leq \inf_{i\in I} f_i^*$

Proof:

(i) $\forall_{u\in\mathcal{H}}$ /* take $\left(\inf_{i\in I} f_i\right) = h$ */

$$h^*(u) = \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - h(x)\right)$$

/* interchangable as x, i independant */

$$= \sup_{x\in\mathcal{H}} \Big(\langle x|u\rangle - \underbrace{\left(\inf_{i\in I} f_i(x)\right)}_{-\sup_{i\in I} - f_i(x)}\Big) = \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle + \sup_{i\in I} - f_i(x)\right) = \sup_{x\in\mathcal{H}} \sup_{i\in I} \left(\langle x|u\rangle - f_i(x)\right)$$

$$= \sup_{i\in I} \underbrace{\sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - f_i(x)\right)}_{f_i^*(u)} = \sup f_i^*(u)$$

$\therefore$ $\left(\inf_{i\in I} f_i\right)^*(u) = \left(\sup_{i\in I} f_i^*\right)(u) \leftrightarrow \left(\inf_{i\in I} f_i\right)^* = \left(\sup_{i\in I} f_i^*\right)$ ☺

(ii) take $g = \sup_{i\in I} f_i$

by definition, $g \geq f_i \ \forall_{i\in I}$

$\Rightarrow$ $g^* \leq f_i^* \ \forall_{i\in I}$ /* By proposition 13.14 (ii): $g \geq u \Rightarrow g^* \leq u^*$ */

$\leftrightarrow$ $\underbrace{g^*}_{\left(\sup_{i\in I} f_i\right)^*} \leq \inf_{i\in I} f_i^*$

$\therefore$ $\left(\sup_{i\in I} f_i\right)^* \leq \inf_{i\in I} f_i^*$ ■

Part 2

1:45 PM

Proposition 13.28.

[$\mathcal{K}$: real Hilbert space;

$F : \mathcal{H}\times\mathcal{K} \to]-\infty,+\infty]$, proper

$f : \mathcal{H} \to [-\infty,+\infty] : x \mapsto \inf F(x,\mathcal{K}) = \inf_{y\in\mathcal{K}} F(x,y)$]

$f^* = F^*(\cdot,0)$

Proof:

Fix $u \in \mathcal{H}$:

$$f^*(u) = \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - \overbrace{f(x)}^{\inf_{y\in\mathcal{K}} F(x,y)}\right)$$

$$= \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle - \underbrace{\inf_{y\in\mathcal{K}} F(x,y)}_{-\sup_{y\in\mathcal{K}} -F(x,y)}\right)$$

constant w.r.t y

$$= \sup_{x\in\mathcal{H}} \left(\langle x|u\rangle + \sup_{y\in\mathcal{K}} -F(x,y)\right)$$

$$= \sup_{x\in\mathcal{H}} \sup_{y\in\mathcal{K}} \left(\langle x|u\rangle - F(x,y)\right)$$ /* recall $\sup_{x\in\mathcal{X}} \sup_{y\in\mathcal{Y}} f(x,y) = \sup_{y\in\mathcal{Y}} \sup_{x\in\mathcal{X}} f(x,y) = \sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} f(x,y)$ */

$$= \sup_{(x,y)\in\mathcal{H}\times\mathcal{K}} \left(\underbrace{\langle x|u\rangle}_{\langle (x,y)|(u,0)\rangle = \langle x|u\rangle + \langle y|0\rangle = \langle x|u\rangle} - F(x,y)\right) = \sup_{(x,y)\in\mathcal{H}\times\mathcal{K}} \left(\langle (x,y)|(u,0)\rangle - F(x,y)\right) = F^*(u,0) \quad \therefore \quad \blacksquare$$

(0 vector)

Proposition 13.31.

transposition operator

$F^{\top} : (u,x) \mapsto F(x,u)$

[[$F \in \Gamma(\mathcal{H}\times\mathcal{H})$: autoconjugate]] /* [$F : \mathcal{H}\times\mathcal{H} \to [-\infty,+\infty]$] F: autoconjugate $\overset{\text{def}}{\Leftrightarrow}$ $F^* = \overbrace{F^{\top}}$ */

$\therefore$ $\boxed{F^*(u,x) = F^{\top}(u,x) = F(x,u)}$

$F \geq \langle\cdot|\cdot\rangle$

$F^* \geq \langle\cdot|\cdot\rangle$

Proof:

* Proposition 13.13. (Fenchel-Young inequality)

[$f : \mathcal{H} \to]-\infty,+\infty]$, proper] /* wow! see how general the function is, infact any sensible function would satisfy this! */

$\forall_{x\in\mathcal{H}} \forall_{u\in\mathcal{H}} \quad f(x) + f^*(u) \geq \langle x|u\rangle$

$\forall_{(x,u)\in\mathcal{H}\times\mathcal{H}} \forall_{(y,v)\in\mathcal{H}\times\mathcal{H}}$

$F(x,u) + F^*(y,v) \geq \langle (x,u)|(y,v)\rangle$

set $(y,v) := (u,x)$

then $F(x,u) + \underbrace{F^*(u,x)}_{F^{\top}(u,x) = F(x,u)} \geq \langle (x,u)|(u,x)\rangle = \langle x|u\rangle + \langle u|x\rangle = 2\langle x|u\rangle$

$\Leftrightarrow$ $2F(x,u) \geq 2\langle x|u\rangle$ $\therefore F \geq \langle\cdot|\cdot\rangle$

as $F^*(u,x) = F(x,u)$ we also have $F^*(u,x) \geq \langle x|u\rangle = \langle u|x\rangle$

$\therefore F^* \geq \langle\cdot|\cdot\rangle$ $\blacksquare$

* Theorem 13.32. (Fenchel-Moreau Theorem)

[$f : \mathcal{H} \to]-\infty,+\infty]$, proper]]

- f: lower semicontinuous, convex $\Leftrightarrow$ $f = f^{**}$
- f: lower semicontinuous, convex $\Rightarrow$ f^*: proper

Proof: $(\Leftarrow)$

given:

$f = f^{**}$

/* recall that. $\forall_{f:\mathcal{H}\to[-\infty,+\infty]}$ $f^* \in \Gamma(\mathcal{H})$ */ (Proposition 13.32; set of all lower semicontinuous and convex functions, $:\mathcal{H}\to[-\infty,+\infty]$)

$\Rightarrow$ $\underbrace{(f^*)^*}_{\text{irrespective of what } f^* \text{ is}} = f$: lower semicontinuous and convex

$(\Rightarrow)$

/* Proof sketch:

Part 1: We prove that:

/* Proof sketch :

Part 1: we prove that:

$$[\ f\in\Gamma_0(\mathcal{H}),\ \overbrace{(x,\xi)}^{\in \operatorname{dom} f}\in\mathcal{H}\times\mathbb{R},\ \xi\in\,]-\infty,f(x)[\,,\ (p,\pi)=P_{\operatorname{epi}}(x,\xi)\ ]\quad f^{**}(x)>\xi$$

Proof sketch for Part 1: (i) use

* Proposition 9.18. * (Charecterization of projection of the epigraph of a Γ_0 function)

$[\ f\in\Gamma_0(\mathcal{H}),\ x\in\operatorname{dom} f,\ \xi\in\,]-\infty,f(x)[\,,\ (p,\pi)\in\mathcal{H}\times\mathbb{R}\ ]$

$$(p,\pi)=P_{\operatorname{epi} f}(x,\xi)\Leftrightarrow\begin{cases}\cdot\ \boxed{\xi<f(p)=\pi}\ \checkmark\\ \cdot\ \forall_{y\in\operatorname{dom} f}\ \langle y-p\mid x-p\rangle\leq(f(y)-f(p))(f(p)-\xi)\end{cases}$$

then (ii) prove :

$$\pi>\xi\Rightarrow f^{**}(x)>\xi\quad */$$

... (goal 1.(i))

Part 2: using part 1 : we show that ($\operatorname{dom} f\neq\emptyset$ as $f\in\Gamma_0(\mathcal{H})$ for this case)

$$\forall_{x\in\operatorname{dom} f}\ f(x)=f^{**}(x)\quad\ldots\text{(goal 2)}$$

Part 3: we prove that :

$$\forall_{x\notin\operatorname{dom} f}\ f(x)=f^{**}(x)=+\infty\quad */$$

thus part 2 and part 3 proves that :

$$\forall_{x\in\mathcal{H}}\ f(x)=f^{**}(x)$$

Part 4: $f=f^{*}\Rightarrow f^{*}$: proper

*/

/* proving goal 1.(i) */

given: f: lower semicontinuous, convex, $\underbrace{\text{proper}}_{\text{universal given}}$ $\Leftrightarrow f\in\Gamma_0(\mathcal{H})$

/* recall:

Proposition 9.17.

$[\ f\in\Gamma_0(\mathcal{H}),\ (x,\xi)\in\mathcal{H}\times\mathbb{R},\ (p,\pi)\in\mathcal{H}\times\mathbb{R}\ ]$

$$(p,\pi)=P_{\operatorname{epi} f}(x,\xi)\Leftrightarrow\begin{cases}\max\{\xi,f(p)\}\leq\pi\\ \forall_{y\in\operatorname{dom} f}\ \langle y-p\mid x-p\rangle+(f(y)-\pi)(\xi-\pi)\leq 0\end{cases}$$

*/

take any $x\in\mathcal{H}$, take $\xi\in\,]-\infty,f(x)[$ /* we are allowed to pick any $\xi\in\mathbb{R}$ */ otherwise $f(x)\leq\xi\Leftrightarrow(x,\xi)\in\operatorname{epi} f$ and $P_{\operatorname{epi} f}(x,\xi)=(x,\xi)$ a noninteresting case

$$(p,\pi)=P_{\operatorname{epi} f}(x,\xi)\Leftrightarrow\begin{cases}\max\{\xi,f(p)\}\leq\pi\Rightarrow\xi\leq\pi\quad /*\ \begin{smallmatrix}-\ \pi\\ -\ f(p)\\ -\ \xi\end{smallmatrix}\ */\\ \forall_{y\in\operatorname{dom} f}\ \langle y-p\mid x-p\rangle+(f(y)-\pi)(\xi-\pi)\leq 0\end{cases}\quad\text{[Eq:13.20]}$$

/* $P\wedge(P\Rightarrow Q)\Leftrightarrow(P\wedge Q)$ */ /* now we can have two cases $\pi>\xi$ and $\pi=\xi$

case $\pi>\xi$, then

$$\forall_{y\in\operatorname{dom} f}\ \langle y-p\mid x-p\rangle\leq\underbrace{(\pi-\xi)}_{>0}(f(y)-\pi)$$

$$\Leftrightarrow\left\langle y-p\ \middle|\ \frac{x-p}{\pi-\xi}\right\rangle\leq(f(y)-\pi)$$

/* set $v:=\dfrac{x-p}{\pi-\xi}$ */

$$\Leftrightarrow\langle y-p\mid v\rangle=\langle y\mid v\rangle-\langle p\mid v\rangle\leq f(y)-\pi$$

$$\Leftrightarrow\langle y\mid v\rangle-f(y)\leq\langle p\mid v\rangle-\pi$$

/* $v=\dfrac{x-p}{\pi-\xi}\Leftrightarrow x-p=v(\pi-\xi)$ $\Leftrightarrow p=x-v(\pi-\xi)$ */

$$=\langle x-v(\pi-\xi)\mid v\rangle-\pi$$

$$=\langle x\mid v\rangle-(\pi-\xi)\underbrace{\langle v\mid v\rangle}_{\|v\|^2}-\pi$$

$$=\langle x\mid v\rangle-\underbrace{\underbrace{(\pi-\xi)\|v\|^2}_{\geq 0}}_{\leq 0\ :\ \text{so removing it will yield larger stuff}}-\pi$$

$$\leq\langle x\mid v\rangle-\pi$$

$\underbrace{\qquad}_{\leq 0}^{\geq 0}$: so removing it will yield larger stuff

$\leq \langle x|v\rangle - \pi$

So, $\forall_{y \in \text{dom} f}$ $\langle y|v\rangle - f(y) \leq \langle x|v\rangle - \pi$

$\leftrightarrow$ $\underbrace{\sup_{y \in \text{dom} f} \langle y|v\rangle - f(y)}$ $\leq \langle x|v\rangle - \pi$

$f^*(v) = \sup_{y \in \mathcal{H}} \langle y|v\rangle - f(y)$ /* Because if $y \notin \text{dom} f$ $f(y) = +\infty$

$\langle y|v\rangle - f(y) = -\infty \leq \langle x|v\rangle - \pi$

*/

$\therefore$ $f^*(v) \leq \langle x|v\rangle - \pi$

$\Rightarrow$ $\pi \leq \langle x|v\rangle - f^*(v) \overset{\text{obviously}}{\leq} \sup_{w \in \mathcal{H}} \langle x|w\rangle - f^*(w) = f^{**}(x)$ /* now $\pi > \xi \Rightarrow f^{**} \geq \pi > \xi$ */

$v = \frac{x-p}{\pi - \xi}$

so we have shown that: $\boxed{\pi > \xi \Rightarrow f^{**}(x) > \xi}$ $\forall_{x \in \text{dom} f}$ /* goal 1.(i) proved */

consider the case $x \in \text{dom} f$

/* recall that,

* Proposition 9.18. * (characterization of projection of the epigraph of a Γ_0 function)

[$f \in \Gamma_0(\mathcal{H})$, $x \in \text{dom} f$, $\xi \in]-\infty, f(x)[$, $(p,\pi) \in \mathcal{H} \times \mathbb{R}$]

$(p,\pi) = P_{\text{epi} f}(x,\xi) \Leftrightarrow \begin{cases} \xi \leq f(p) = \pi \\ \forall_{y \in \text{dom} f} \langle y-p|x-p\rangle \leq (f(y)-f(p))(f(p)-\xi) \end{cases}$

goal 1(i)

*/

Combining both we have proved Part 1:

$\boxed{[f \in \Gamma_0(\mathcal{H}),\ \overbrace{(x,\xi)}^{\in \text{dom} f} \in \mathcal{H} \times \mathbb{R},\ \xi \in]-\infty, f(x)[,\ (p,\pi) = P_{\text{epi}}(x,\xi)]\ f^{**}(x) > \xi}$

/* Proof of part 2 starts here */ want to prove: $\forall_{x \in \text{dom} f}$ $f^{**}(x) = f(x)$, assume $\text{dom} f \neq \emptyset$

/* recall

* Proposition 13.14. *

[$f, g : \mathcal{H} \to [-\infty, +\infty]$] /* these properties are actually quite useful */

(i) $f^{**} \leq f$

(ii) $f \leq g \Rightarrow (f^* \geq g^*,\ f^{**} \leq g^{**})$

(iii) $f^{***} = f^*$

(iv) $(\breve{f})^* = f^*$ /* $\breve{f} = \sup\{g \in \Gamma(\mathcal{H}) : g \leq f\}$: lower semicontinuous convex envelope of f */ */

$\boxed{f^{**}(x) \leq f(x)}$

$\xi < f^{**}(x) \leq f(x)$ /* $\xi \in]-\infty, f(x)[$ */

$\Rightarrow$ $\underbrace{\sup_{\xi \in]-\infty, f(x)[} \xi}_{f(x)} \leq f^{**}(x) \leq f(x)$ /* e.g., $\forall_{x : x^2 < 2}$ $x < \sqrt{2}$

$\leftrightarrow$ $\sup_{x : x^2 < 2} x \leq \sqrt{2}$ */

$\leftrightarrow$ $f(x) \leq f^{**}(x) \leq f(x)$

$\boxed{\forall_{x \in \text{dom} f}\ f(x) = f^{**}(x)}$ (Part 2 proved) [Part_2_proved]

/* Part 3 proof started */

our goal is to prove that: $\forall_{x \notin \text{dom} f}$ $f(x) = f^{**}(x)$

Proposition 9.17. so, $x \notin \text{dom} f$ 일 때 formula 를 유지함

[$f \in \Gamma_0(\mathcal{H})$, $(x,\xi) \in \mathcal{H} \times \mathbb{R}$, $(p,\pi) \in \mathcal{H} \times \mathbb{R}$]

$(p,\pi) = P_{\text{epi} f}(x,\xi) \Leftrightarrow \begin{cases} \max\{\xi, f(p)\} \leq \pi \\ \forall_{y \in \text{dom} f} \langle y-p|x-p\rangle + (f(y)-\pi)(\xi-\pi) \leq 0 \end{cases}$ [Eq13.20]

$\Rightarrow$ $\xi \leq \pi$ so, there are two cases $\boxed{\xi < \pi}$ case 1, $\boxed{\xi = \pi}$ case 2

Consider case 1: $\pi > \xi$

$\pi > \xi \Rightarrow f^{**}(x) > \xi$ $\forall_{x \in \mathcal{H}}$ $\forall_{\xi \in]-\infty, f(x)[}$

if $x \notin \text{dom} f \Rightarrow f(x) = +\infty$ $\left(f(x) \overset{\shortparallel}{=} +\infty\right)$

$\pi > \xi \Rightarrow f^*(x) > \xi \quad \forall_{x\in\mathcal{H}} \forall_{\xi\in]-\infty, f(x)[}$

if $x \notin \mathrm{dom} f \Rightarrow f(x) = +\infty$

$\forall_{x\notin \mathrm{dom} f} \quad f^*(x) > \sup_{\xi\in]-\infty,+\infty[} \xi = +\infty \Rightarrow f^*(x) = +\infty$

$\forall_{x\notin \mathrm{dom} f} \quad f(x) = f^*(x) = +\infty$ for $\xi < \pi$ (case 1)

Now consider case 2: $\pi = \xi$

Now let us consider $\pi = \xi$

now, $\underbrace{(p,\pi)}_{\in \mathrm{epi} f} = P_{\mathrm{epi} f}\underbrace{(x,\xi)}_{\xi\in]-\infty, f(x)[}$

/* By definition */ so. $\xi < f(x) \Leftrightarrow (x,\xi) \notin \mathrm{epi} f$

so, $\underbrace{(p,\pi)}_{\in \mathrm{epi} f} \neq \underbrace{(x,\xi)}_{\notin \mathrm{epi} f}$ and $\pi = \xi$

$\Rightarrow p \neq x \Leftrightarrow \|x-p\| > 0$

take $w \in \mathrm{dom} f^* \Leftrightarrow f^*(w) < +\infty$,

$u = \|x-p\|$

[Eq13.20] says:

$\xi < \pi, \quad \forall_{y\in \mathrm{dom} f} \quad \langle y-p | x-p\rangle + (f(y)-\pi)(\xi-\pi) \leq 0$

now $\xi = \pi$ so $\forall_{y\in\mathrm{dom} f} \quad \underbrace{\langle y-p | \underbrace{x-p}_{u}\rangle}_{} \leq 0$

$\langle y|u\rangle - \langle p|u\rangle \leq 0$

$\Leftrightarrow \langle y|u\rangle \leq \langle p|u\rangle$

$\therefore \forall_{y\in\mathrm{dom} f} \quad \langle y|u\rangle \leq \langle p|u\rangle$

$w \in \mathrm{dom} f^* \Rightarrow f^*(w) = \sup_{y\in\mathcal{H}} \langle y|w\rangle - f(y) < +\infty$

$\Rightarrow \forall_{y\in\mathcal{H}} \quad \langle y|w\rangle - f(y) \leq f^*(w) < +\infty$

$\Leftrightarrow \forall_{y\in\mathrm{dom} f} \quad \langle y|w\rangle - f(y) \leq f^*(w)$ /* As outside dom f, $\langle y|w\rangle - \overbrace{f(y)}^{+\infty} = -\infty < +\infty$, so we can confine y to dom f */

So, we have $\boxed{\forall_{y\in\mathrm{dom} f} \quad \langle y|w\rangle - f(y) \leq f^*(w), \quad \langle y|u\rangle \leq \langle p|u\rangle}$

take $\lambda \in \mathbb{R}_{++}$ then $\lambda\langle y|u\rangle \leq \lambda\langle p|u\rangle$

$\Leftrightarrow \langle y|\lambda u\rangle \leq \langle p|\lambda u\rangle$

$\langle y|w\rangle - f(y) \leq f^*(w) \quad \oplus$

$\forall_{y\in\mathrm{dom} f} \quad \langle y|w+\lambda u\rangle - f(y) \leq f^*(w) + \langle p|\lambda u\rangle$

$\Leftrightarrow \underbrace{\sup_{y\in\mathrm{dom} f} \langle y|w+\lambda u\rangle - f(y)}_{f^*(w+\lambda u)} \leq f^*(w) + \langle p|\lambda u\rangle$

$\therefore f^*(w+\lambda u) \leq f^*(w) + \langle p|\lambda u\rangle$ /* $x-p=u \Leftrightarrow p = x-u$ */

$\lambda\langle p|u\rangle = \lambda\langle x-u|u\rangle$

$= \lambda\langle x|u\rangle - \lambda\langle u|u\rangle$

$= \langle x|\lambda u\rangle - \lambda\|u\|^2$

$= \langle x|w\rangle + \langle x|\lambda u\rangle - \langle x|w\rangle - \lambda\|u\|^2$

$= \langle x|w+\lambda u\rangle - \langle x|w\rangle - \lambda\|u\|^2$

$\Leftrightarrow f^*(w+\lambda u) \leq f^*(w) + \langle x|w+\lambda u\rangle - \langle x|w\rangle - \lambda\|u\|^2$

$\Leftrightarrow \langle x|w+\lambda u\rangle - f^*(w+\lambda u) \geq \langle x|w\rangle + \lambda\|u\|^2 - f^*(w)$

now. $f^{**}(x) = \sup_{\tilde{y}\in\mathcal{H}} \langle x|\tilde{y}\rangle - f^*(\tilde{y}) \geq \langle x|\tilde{y}\rangle - f^*(\tilde{y}) \quad \forall_{\tilde{y}\in\mathcal{H}}$

set $\tilde{y} := w + \lambda u \in \mathcal{H}$, then:

$\forall_{\lambda\in\mathbb{R}_{++}} \quad f^{**}(x) \geq \langle x|w+\lambda u\rangle - f^*(w+\lambda u) \geq \langle x|w\rangle + \lambda\|u\|^2 - f^*(w)$

$f^{**}(x) \geq \langle x|w\rangle + \lambda\|u\|^2 - f^*(w)$

$\Leftrightarrow f^{**}(x) \geq \sup_{\lambda\in\mathbb{R}_{++}} \underbrace{\langle x|w\rangle}_{} + \lambda\|u\|^2 - \underbrace{f^*(w)}_{} = +\infty$

$f^{**}(x) \geq \langle x|u\rangle + \lambda\|u\|^2 - f^*(w)$

$\Leftrightarrow \quad f^{**}(x) \geq \sup_{\lambda \in \mathbb{R}_{++}} \underbrace{\langle x|w\rangle}_{\text{finite}} + \lambda\|u\|^2 - \underbrace{\underbrace{f^*(w)}_{<+\infty}}_{>-\infty} = +\infty$

$\therefore f^{**}(x) = +\infty$

$\forall_{\substack{x \notin \operatorname{dom} f \\ \Updownarrow \\ f(x)=+\infty}} \quad f(x) = f^{**}(x) = +\infty \quad$ for $y = \pi$ (case 2)

so, for both cases we have: $\forall_{x \notin \operatorname{dom} f} \; f(x) = f^{**}(x) = +\infty$ . . . (part 3 proved) [Part_3_proved]

so, from (part 2) and (part 3) we have: $\forall_{x \in \mathcal{H}} \; f(x) = f^{**}(x)$:)

[Part_2_proved] [Part_3_proved] $\quad \Updownarrow \; f(\mathcal{H}) = f^{**}(\mathcal{H})$

(part 4 proof start)

Now let us prove that: when $\left(\forall_{x \in \mathcal{H}} f(x) = f^{**}(x)\right) \Rightarrow f^*$: proper /* to prove this we use Proposition 13.9. g^*: proper $\Rightarrow g$: proper */

also, $f \in \Gamma_0(\mathcal{H}) \Rightarrow -\infty \notin f(\mathcal{H}) = f^{**}(\mathcal{H})$ ✓

also, $f(\operatorname{dom} f) = f^{**}(\operatorname{dom} f)$, $\underset{\substack{\Downarrow \\ \operatorname{dom} f \neq \emptyset}}{f \in \Gamma_0(\mathcal{H})} \Rightarrow \operatorname{dom} f^{**} \neq \emptyset$ ✓

$\Big\} \left(-\infty \notin f^{**}(\mathcal{H}), \operatorname{dom} f^{**} \neq \emptyset\right) \Leftrightarrow f^{**}$ proper $\Rightarrow f^*$: proper.

∎

*Proposition 13.39.

[$f : \mathcal{H} \to]-\infty, +\infty]$]

• f: has a continuous affine minorant $\Rightarrow \breve{f} = f^{**}$
$\Updownarrow$
$\operatorname{dom} f^* \neq \emptyset$

• f: doesnot have a continuous affine minorant $\Rightarrow f^{**} \equiv -\infty$

Proof:

Case 1: $f \equiv +\infty$ $\quad$ In this case f itself is affine, so we will show that f**=+∞

$f \equiv +\infty$: convex $\Rightarrow \breve{f} = f$ /* trivially */
$\hookrightarrow \operatorname{dom} f = \emptyset$ $\quad$ /* Sup over an empty set is -∞ */

then $f^*(u) = \underbrace{\sup_{x \in \operatorname{dom} f}}_{=\emptyset} \left(\langle x|u\rangle - f(x)\right) = -\infty \Leftrightarrow f^* \equiv -\infty$

/* Proposition 13.9. (iv) */

$\therefore f^{**}(v) = \sup_{u \in \mathcal{H}} \langle u|v\rangle - f^*(u) = +\infty \Leftrightarrow f^{**} \equiv +\infty$

$\Big\} \Rightarrow f^{**} = \breve{f}$:)

Case 2: $f \not\equiv +\infty$ $\quad$ /* Now in this case we show that, if f has a continuous affine minorant then, f**=f^
and if does not then f**=-∞

/* recall Proposition 13.10. (ii)

$f^* \equiv +\infty \Leftrightarrow f$: posseses no continuous affine minorant */

• f: posseses no continuous affine minorant $\Rightarrow f^* \equiv +\infty$

$\Leftrightarrow f^{**} \equiv -\infty$ /* using Proposition 13.9 (ii)

$-\infty \in f^*(\mathcal{H}) \Leftrightarrow f \equiv +\infty \Leftrightarrow f^* \equiv -\infty$ */

• f: posseses a continuous affine minorant

$\Leftrightarrow \exists \; a : \mathcal{H} \to \mathbb{R}$, affine function $\quad a \leq f$

/* continuous affine function
e.g. $c^T x + d$ cannot shoot to $\pm\infty$ */

naturally $a = \breve{a}$ /* as a: affine continuous */

eg. $c^T x + d$ cannot shoot to $\pm\infty$ */

naturally $a = \breve{a}$ /* as a: affine continuous */
$\Downarrow$ convex $\Downarrow$ lower semicontinuous

so. $a = \breve{a}$: lower semicontinuous convex function
$\leftrightarrow$ $a = \breve{a} \in \Gamma(\mathcal{H})$

by definition, $\breve{f} = \sup\{g \in \Gamma(\mathcal{H}) \mid g \leq f\} \geq a = \breve{a}$

$\therefore$ $a = \breve{a} \leq \breve{f} \leq f$

now. $\breve{a}$: proper as $\pm\infty \notin \breve{a}(\mathcal{H})$

and $f : \mathcal{H} \to]-\infty, +\infty]$, $f \not\equiv +\infty$

$\leftrightarrow f : \mathcal{H} \to]-\infty, +\infty[= \mathbb{R} \to f$: proper

so we have $\breve{f}$: proper as it is sandwiched between two proper functions

so, $\breve{f}$: proper, and by definition $\breve{f} \in \Gamma(\mathcal{H})$

$\to \breve{f} \in \Gamma_0(\mathcal{H})$

/* recall Proposition 13.14 (iv) $(\breve{f})^* = f^*$
Corollary 13.32. $f \in \Gamma_0(\mathcal{H}) \Rightarrow f^* \in \Gamma_0(\mathcal{H})$, $f^{**} = f$ */

so, $\breve{f} \in \Gamma_0(\mathcal{H}) \Rightarrow (\breve{f})^* = f^* \in \Gamma_0(\mathcal{H})$, $\breve{f}^{**} = f^{**} = \breve{f} = \breve{f}$

$\therefore f^{**} = (\breve{f})^{**} = \breve{f}$

∎

Proposition 13.41.

[Let $(f_i)_{i \in I} : \subseteq \Gamma_0(\mathcal{H})$, $\sup_{i \in I} f_i \not\equiv +\infty$] $(\sup_{i \in I} f_i)^* = (\inf_{i \in I} f_i^*)^{\vee}$

Proof :

/* use Fenchel-Moreau theorem

* Theorem 13.32 (Fenchel-Moreau Theorem) *

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper] /* f: proper $\Leftrightarrow -\infty \notin f(\mathcal{H})$, $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \neq \emptyset$ */

- f : lower semicontinuous, convex $\Leftrightarrow f = f^{**}$
- f : lower semicontinuous, convex $\Rightarrow f^*$: proper

* Proposition 13.25.

[$(f_i)_{i \in I}$: family of proper functions, $: \mathcal{H} \to]-\infty, +\infty]$]

(i) $(\inf_{i \in I} f_i)^* = (\sup_{i \in I} f_i^*)$

(ii) $(\sup_{i \in I} f_i)^* \leq \inf_{i \in I} f_i^*$

*/

$\sup_{i \in I} f_i = \sup_{i \in I} f_i^{**}$ /* $f_i \in \Gamma_0(\mathcal{H}) \Rightarrow f_i^{**} = f_i$ */

$= \sup_{i \in I} (f_i^*)^*$

$= (\inf_{i \in I} f_i^*)^* \Rightarrow \boxed{(\sup_{i \in I} f_i)^* = (\inf_{i \in I} f_i^*)^{**}}$

first note that :

given: $\sup_{i \in I} f_i \not\equiv +\infty$

$\Rightarrow \forall_{i \in I}$ $f_i(\mathcal{H}) \in \mathbb{R}$ // as f_i : proper, $-\infty \notin f_i(\mathcal{H})$

$\Rightarrow \forall_{i \in I}$ $f_i^*(u) = \sup_{x \in \mathcal{H}} (\underbrace{\langle x \mid u \rangle - f(x)}_{\text{finite}}) = \text{finite}$ $\forall_{u \in \mathcal{H}}$

$\Rightarrow$ $\inf_{i \in I} f_i^* = \text{finite}$

using a similar logic

/* using

* Theorem 9.19.

[$f \in \Gamma_0(\mathcal{H})$] f : possesses a continuous affine minorant. /* the proof is constructive */

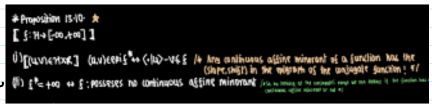
*/

$\Rightarrow \inf_{i \in I} f_i^* = \text{finite}$

using a similar logic

$\Rightarrow \left(\inf_{i \in I} f_i^* \right)^* = \text{finite}$

$\Rightarrow \left(\inf_{i \in I} f_i^* \right)^* \not\equiv +\infty$ /* now



* Proposition 13.10. ★

[$f: \mathcal{H} \to [-\infty, +\infty]$]

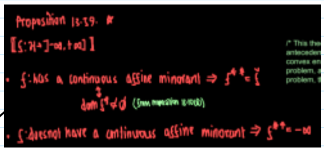
(i) [$(u,v) \in \mathcal{H} \times \mathbb{R}$] $(u,v) \in \operatorname{epi} f^* \Leftrightarrow \langle \cdot | u \rangle - v \leq f$ /* Any continuous affine minorant of a function has the (slope, shift) in the epigraph of the conjugate function; */

(ii) $f^* \equiv +\infty \Leftrightarrow f$: possesses no continuous affine minorant

*/

$\Leftrightarrow \left(\inf_{i \in I} f_i^* \right)$: has a continuous affine minorant.

/* use



Proposition 13.39. ★

[$f: \mathcal{H} \to]-\infty, +\infty]$]

- f: has a continuous affine minorant $\Rightarrow f^{**} = \breve{f}$; $\operatorname{dom} f^* \neq \emptyset$ (from proposition 13.10(ii))
- f: doesnot have a continuous affine minorant $\Rightarrow f^{**} \equiv -\infty$

*/

$$\underbrace{\left(\inf_{i \in I} f_i^* \right)^{**}}_{\left(\sup_{i \in I} f_i \right)^*} = \left(\inf_{i \in I} f_i^* \right)^{\smile}$$

$$\therefore \left(\sup_{i \in I} f_i \right)^* = \left(\inf_{i \in I} f_i^* \right)^{\smile} \quad \blacksquare$$