Part 1

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*Proposition 12.6.

$[\![f,g,h : \mathcal{H} \to]-\infty,+\infty]]\!] \Rightarrow$

(i) f,g : have continuous affine minorant with slope $u \in \mathcal{H}$

$\Rightarrow f \square g$: has " " " " " $u \in \mathcal{H}$, $-\infty \notin (f\square g)(\mathcal{H})$

(ii) $\operatorname{dom} f\square g = \operatorname{dom} f + \operatorname{dom} g$

(iii) $f \square g = g \square f$

Proof:

(i) f : has continuous affine minorant with slope $u \in \mathcal{H} \overset{def}{\Leftrightarrow} \exists_{\eta \in \mathbb{R}}\ f - \langle \cdot | u\rangle \geq \eta > -\infty \Leftrightarrow \exists_{\eta \in \mathbb{R}} \forall_{\tilde{x} \in \mathcal{H}}\ f(\tilde{x}) - \langle \tilde{x} | u \rangle \geq \eta$

g : " " " " " " $u \in \mathcal{H} \overset{def}{\Leftrightarrow} \exists_{\mu \in \mathbb{R}}\ g - \langle \cdot | u \rangle \geq \mu > -\infty \Leftrightarrow \exists_{\mu \in \mathbb{R}} \forall_{\tilde{y} \in \mathcal{H}}\ g(\tilde{y}) - \langle \tilde{y} | u \rangle \geq \mu$

$\tilde{x} := y$

$$\left.\begin{array}{l} f(y) - \langle y | u \rangle \geq \eta \\ \tilde{y} := x - y \\ g(x-y) - \langle x-y | u \rangle \geq \mu \end{array}\right\} \stackrel{(+)}{\Rightarrow} f(y) + g(x-y) \underbrace{- \langle y | u \rangle - \langle x-y | u \rangle}_{\substack{-\langle y|u\rangle - \langle x|u\rangle + \langle y|u\rangle \\ = -\langle x|u\rangle}} \geq \eta + \mu > -\infty$$

$$f(y) + g(x-y) - \langle x | u \rangle \geq \eta + \mu > -\infty$$

now y : arbitrary. x : arbitrary

$\therefore \forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}}\ f(y) + g(x-y) - \langle x | u \rangle \geq \eta + \mu > -\infty$

$$\Leftrightarrow \forall_{x \in \mathcal{H}}\ \underbrace{\inf_{y \in \mathcal{H}} \big(f(y) + g(x-y) \big)}_{(f\square g)(x)} \underbrace{- \langle x | u \rangle}_{\text{constant w.r.t } y} \geq \eta + \mu > -\infty$$

$-\infty$

$\therefore \forall_{x \in \mathcal{H}}\ (f\square g)(x) - \langle x | u \rangle \geq \eta + \mu > -\infty$

$\Leftrightarrow\ (f \square g) - \langle \cdot | u \rangle \geq \eta + \mu > -\infty$

so, $f\square g$: has a continuous affine minorant.

$\Rightarrow \forall_{x \in \mathcal{H}}\ (f\square g)(x) \geq (\eta + \mu) + \langle x | u \rangle > -\infty$

$\Rightarrow -\infty \notin (f\square g)(\mathcal{H})$ ☺

(ii), (iii)

$f\square g : \mathcal{H} \to [-\infty, +\infty] : x \mapsto \inf_{y \in \mathcal{H}} \big(f(y) + g(x-y) \big)$

$$(f\square g)(x) = \inf_{u \in \mathcal{H}} f(u) + g(x-u)$$

$$= \inf_{u}\ f(u) + g(\underbrace{x-u}) \quad \text{s.t. } u \in \mathcal{H}$$

V : new decision vector $= x - u$

$$= \left(\begin{array}{ll} \inf_{u,v} & f(u) + g(v) \\ \text{s.t.} & u + v = x \\ & u, v \in \mathcal{H} \end{array} \right) \quad \text{(eq 1)}$$

note that this inf problem stays same if we switch f and g, so,

$(f\square g)(x) = (g \square f)(x)\quad \forall_{x \in \mathcal{H}}$

$\Leftrightarrow \quad f\Box g = g\Box f \qquad$ (iii) proved

(ii): want to prove: $\mathrm{dom}(f\Box g) = \underbrace{\mathrm{dom} f + \mathrm{dom} g}$ // $(\mathrm{dom} f + \mathrm{dom} g)(x) = (\mathrm{dom} f)(x) + (\mathrm{dom} g)(x)$

$\forall_x$ // $x \in \mathrm{dom} f + \mathrm{dom} g \Leftrightarrow \exists_{u,v:\, x=u+v}\ u\in\mathrm{dom} f,\ v\in\mathrm{dom} g$

$$x\in\mathrm{dom}(f\Box g) \Leftrightarrow (f\Box g)(x) = \begin{pmatrix}\inf\limits_{u,v\in\mathcal{H}} & f(u)+g(v)\\ \text{s.t} & u+v=x\end{pmatrix} < +\infty \quad \text{// from (eq 1)}$$

$\Rightarrow \forall_{u,v\in\mathcal{H}:\, u+v=x}\quad f(u)+g(v)<+\infty$ // as $f,g,h:\mathcal{H}\to\,]-\infty,+\infty]$, this implies: $f(u)<+\infty,\ f(v)<+\infty$

$\Rightarrow \forall_{u,v\in\mathcal{H}:\, u+v=x}\quad f(u)<+\infty,\ g(v)<+\infty$

$\Leftrightarrow \forall_{u,v\in\mathcal{H}:\, u+v=x}\quad u\in\mathrm{dom} f,\ v\in\mathrm{dom} g$

$\Rightarrow \exists_{u,v\in\mathcal{H}:\, u+v=x}\quad u\in\mathrm{dom} f,\ v\in\mathrm{dom} g \ \Leftrightarrow\ x\in\mathrm{dom} f+\mathrm{dom} g \qquad \therefore\ \mathrm{dom}(f\Box g)\subseteq \mathrm{dom} f+\mathrm{dom} g \quad (1)$

now: $\forall_x$

$x\in\mathrm{dom} f+\mathrm{dom} g \Leftrightarrow \exists_{\tilde u,\tilde v:\, x=\tilde u+\tilde v}\ \tilde u\in\mathrm{dom} f,\ \tilde v\in\mathrm{dom} g$

$f(\tilde u)<+\infty \quad g(\tilde v)<+\infty$ // as $f,g,h:\mathcal{H}\to\,]-\infty,+\infty]$, this implies: $f(u)+g(v)<+\infty$

$\Rightarrow \exists_{\tilde u,\tilde v:\, x=\tilde u+\tilde v}\quad f(\tilde u)+g(\tilde v)<+\infty$

now by definition: $\begin{pmatrix}\inf\limits_{u,v\in\mathcal{H}:\, u+v=x} & f(u)+g(v)\end{pmatrix} = (f\Box g)(x) < f(\tilde u)+g(\tilde v)$ //from eq 1 $\Rightarrow (f\Box g)(x)<+\infty \Leftrightarrow x\in\mathrm{dom}(f\Box g)$

$\therefore\ \mathrm{dom} f+\mathrm{dom} g\subseteq\mathrm{dom}(f\Box g) \quad (2)$

Thus from (1), (2): $\mathrm{dom}(f\Box g) = \mathrm{dom} f+\mathrm{dom} g$. :)

(iv) f,g,h: continuous affine minorants with slope u

$\Rightarrow f\Box g,\ g\Box h:\ -\infty\notin\mathrm{dom} f\Box g,\ -\infty\notin\mathrm{dom}\, g\Box h$ /* from (i) */

so. $f\Box(g\Box h),\ (f\Box g)\Box h$: well defined

now $\quad (g\Box h)(y) = \inf\limits_{v,w} \quad g(v)+h(w)$

$\quad$ s.t. $\quad v+w=y,\ v,w\in\mathcal{H}$

now, $\quad f\Box(g\Box h)x = \inf\limits_{u,y\in\mathcal{H}} \quad f(u)+(g\Box h)(y)$

$\quad$ s.t $\quad u+y=x$

constant w.r.t v,w so we can take it inside

$$= \inf_{u,y\in\mathcal{H}}\Big(\big(f(u)+\mathbb{1}_{u+y=x}(u,y)\big)+\inf_{v,w}\big(g(v)+h(w)+\mathbb{1}_{v+w=y}(v,w)\big)\Big)$$

$$= \inf_{u,y\in\mathcal{H}}\ \inf_{v,w}\Big[f(u)+g(v)+h(w)+\underbrace{\mathbb{1}_{u+y=x}(u,y)+\mathbb{1}_{v+w=y}(v,w)}_{\mathbb{1}_{u+v+w=x}(u,v+w)}\Big]$$

$$= \inf_{u,v+w\in\mathcal{H}}\ \inf_{v,w}\Big[f(u)+g(v)+h(w)+\mathbb{1}_{u+v+w=x}(u,v+w)\Big]$$

$$= \begin{pmatrix}\inf\limits_{u,v,w} & f(u)+g(v)+h(w)\\ \text{s.t.} & u+v+w=x\\ & u,v,w\in\mathcal{H}\end{pmatrix}$$

s.t. $\quad u+v+w=x$

$\quad\quad u,v,w\in\mathcal{H}$

$= \left(\left(f\,\square\, g\right)\square\, h\right)(x)$ // as: if we started with $\left(\left(f\square g\right)\square h\right)(x)$ we would end up with this $(\cdot)$ term.

$\blacksquare$

*Corollary 12.19.

[C: nonempty subset of $\mathcal{H}$

$h: C\to\mathbb{R}$, β-Lipschitz continuous for some $\beta\in\mathbb{R}_+$

$f:\mathcal{H}\to\,]-\infty,+\infty]: x\mapsto \begin{cases} h(x), & \text{if } x\in C \\ +\infty, & \text{otherwise}\end{cases}$

]

$f\,\square\,\beta\|\cdot\|$: β-Lipschitz continuous extension of h

Proof:

$g = f\,\square\,\beta\|\cdot\|$

$\forall_{x\in C}\quad g(x) = \inf_{y\in\mathcal{H}}\left(f(y)+\beta\|x-y\|\right)$

$= \inf_{y\in\mathcal{H}}\left(h(y)[y\in C] + \infty[y\notin C] + \beta\|x-y\|\right)$

$= \inf_{y\in C}\left(h(y)+\beta\|x-y\|\right)$ /* h: β-Lipschitz continuous on C

$\Rightarrow \|h(x)-h(y)\|\leq\beta\|x-y\| \Rightarrow h(x)-h(y)\leq\|h(x)-h(y)\|\leq\beta\|x-y\|$ */

$\geq \inf_{y\in C}\left(h(y)+\|h(x)-h(y)\|\right)$

$\geq \inf_{y\in C}\left(h(y)+h(x)-h(y)\right)$ /* $\forall_{a\in\mathbb{R}}\ \|a\|\geq a \Leftrightarrow b+\|a\|\geq b+a$ */

$= \inf_{y\in C} \underbrace{h(x)}_{\text{const ant w.r.t } y} = \overset{h}{h(x)} > -\infty$ /* As, $\forall_{x\in C}\ h(x)\in\mathbb{R}$ $(\because h: C\to\mathbb{R})$

$\Rightarrow \forall_{x\in C}\ h(x)>-\infty$ */

/* So we have, $\forall_{x\in C}\ g(x)\geq h(x)$ */

$\therefore \forall_{x\in C}\ g(x)>-\infty$

Recall that,

*Proposition 12.17.

[$f:\mathcal{H}\to\,]-\infty,+\infty]$, proper, $\beta\in\mathbb{R}_+$

g: β-Pasch-Hausdorff envelope of f]

Exactly one of the following holds:

(i) f: posseses a β-Lipschitz continuous minorant,

g: largest β-Lipschitz continuous minorant of f.

(ii) f: posses no β-Lipschitz continuous minorant, $g\equiv-\infty$ $\to$ So (ii) doesnot hold

as (ii) doesnot hold, (i) must hold

so, f: posses a β-Lipschitz continuous minorant,

g: largest β-Lipschitz continuous minorant of f. $\therefore \forall_{x\in\mathcal{H}}\ f(x)\geq g(x)$

$\Rightarrow \forall_{x\in C}\ g(x)\leq f(x)$

$\forall_{x\in C}\ g(x)=f(x)=h(x)$ /* as on C, $f=h$ */

as g: β-Lipschitz continuous in general /* $\forall_{x,y\in\mathcal{H}}\ \|g(x)-g(y)\|\leq\beta\|x-y\|$ */

$g = f\,\square\left(\beta\|\cdot\|\right)$: β-Lipschitz continuous extension of h.

$\blacksquare$

*Proposition 12.17.

[$f:\mathcal{H}\to\,]-\infty,+\infty]$, proper,

$\beta\in\mathbb{R}_+$

g: β-Pasch-Hausdorff envelope of f /* $g = f\,\square\,\overset{\in\mathbb{R}_+}{\beta}\|\cdot\|$ */

]

Exactly one of the following holds:

(i) f: possesses a β-Lipschitz continuous minorant, in this case

g: largest β-Lipschitz continuous minorant of f

(ii) f: posses no β-Lipschitz continuous minorant, in this case $g \equiv -\infty$

Proof: it is true that no matter what

Exactly one of (i) or (ii) is apparent, as f either has a β-Lipschitz continuous minorant or it does not.

g: β-Pasch-Hausdorff envelope of f

$\Leftrightarrow$

$$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \quad g(x) = \inf_{z\in\mathcal{H}} \big(f(z)+\beta\|x-z\|\big) \leqslant \inf_{z\in\mathcal{H}} \big(f(z)+\overbrace{\beta\|x-y\|}^{\text{constant w.r.t } z}+\beta\|y-z\|\big) = \beta\|x-y\| + \underbrace{\inf_{z\in\mathcal{H}}\big(f(z)+\beta\|y-z\|\big)}_{g(y)} = \beta\|x-y\| + \overbrace{g(y)}^{\text{finite}}$$

$f(z) + \beta\|x-y+y-z\|$

$\leqslant f(z) + \beta\|x-y\| + \beta\|y-z\|$ /* Using triangle inequality */

as power norm infimal convolution for a general function has full domain, i.e.,

(Power norm infimal convolution of a general function and its properties)

* Proposition 12.9.

[$f:\mathcal{H}\to]-\infty,+\infty]$, proper

$p\in[1,+\infty[$

$\forall_{\gamma\in\mathbb{R}_{++}} \; g_\gamma = f \,\square\, \left(\frac{1}{\gamma p}\|\cdot\|^p\right)$: power norm infimal convolution]

$$\forall_{\gamma\in\mathbb{R}_{++}} \forall_{x\in\mathcal{H}} \begin{cases} \operatorname{dom} g_\gamma = \mathcal{H} \\ \mu\in]\gamma,+\infty[\Rightarrow \inf f(\mathcal{H}) \leqslant g_\mu(x) \leqslant g_\gamma(x) \leqslant f(x) \\ \inf g_\gamma(\mathcal{H}) = \inf f(\mathcal{H}) \\ g_\mu(x) \downarrow \inf f(\mathcal{H}) \text{ as } \mu\uparrow+\infty \\ g_\gamma : \text{bounded on every ball in } \mathcal{H} \end{cases}$$

As β-Pasch-Hausdorff envelope satisfies all the properties of power norm infimal convolution,

$\operatorname{dom} g = \mathcal{H} \Leftrightarrow \forall_{x\in\mathcal{H}} \; g(x) < +\infty$

$$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \quad g(x) = \overbrace{\beta\|x-y\|}^{\text{finite}} + \overbrace{g(y)}^{<+\infty} < +\infty \quad \ldots \quad (12.15)$$

it is true that no matter what

Exactly one of (i) or (ii) is apparent, as f either has a β-Lipschitz continuous minorant or it does not.

(i) Now consider the first possibility:

f: possesses a β-Lipschitz continuous minorant

f: posses a β-Lipschitz continuous minorant $h:\mathcal{H}\to\mathbb{R}$ $\Leftrightarrow$

- h satisfies

  $\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \; |h(x)-h(y)| \leqslant \beta\|x-y\| \Rightarrow h(x) \leqslant h(y)+\beta\|x-y\|$

- h: minorant of f

  $\Leftrightarrow \forall_{\tilde{x}\in\mathcal{H}} \; h(\tilde{x}) \leqslant f(\tilde{x})$

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \quad h(x) \leqslant h(y) + \beta\|x-y\| \leqslant f(y) + \beta\|x-y\|$

$$\Leftrightarrow \forall_{x\in\mathcal{H}} \quad h(x) \leqslant \inf_{y\in\mathcal{H}} \big(f(y)+\beta\|x-y\|\big)$$

$$= \underbrace{\big(f \,\square\, (\beta\|\cdot\|)\big)}_{g}(x)$$

$$= g(x)$$

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $h(x) \leq h(y) + \beta\|x-y\| \leq f(y) + \beta\|x-y\|$

$\Leftrightarrow \forall_{x\in\mathcal{H}}$ $h(x) \leq \inf_{y\in\mathcal{H}} \left(f(y) + \beta\|x-y\|\right)$

$= \underbrace{\left(f \,\Box\, (\beta\|\cdot\|)\right)}_{g}(x)$

$= g(x)$

$\therefore \forall_{x\in\mathcal{H}}$ $h(x) \leq g(x) < +\infty$

but, $h: \mathcal{H} \to \underbrace{\mathbb{R}}_{\not\ni \pm\infty}$

so, $\forall_{x\in\mathcal{H}}$ $-\infty < h(x) \leq g(x) < +\infty$

so, $g: \mathcal{H} \to \mathbb{R}$,

also, $\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $g(x) - g(y) \overset{\text{from (12.15)}}{=} \beta\|x-y\|$

similarly, swapping x and y:

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $g(y) - g(x) = \beta\|x-y\|$

$\Rightarrow \forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $|g(x) - g(y)| = \beta\|x-y\| \Rightarrow g$: β-Lipschitz-continuous

so, we have: $h \leq g \leq f$, g: β-Lipschitz continuous minorant ☺

/* any β-Lipschitz continuous minorant */

/* also, Proposition 12.9(ii) says $g \leq f$ */ so, g: largest β-Lipschitz continuous minorant of f

(ii)

f: possesses no β-Lipschitz continuous minorant but per absurdum g be real valued at some point in $\mathcal{H}$, say the point is $\bar{y}$

now (12.15) says

$\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $g(x) - g(y) \leq \beta\|x-y\|$

$\forall_{y\in\mathcal{H}} \forall_{x\in\mathcal{H}}$ $g(y) - g(x) \leq \beta\|x-y\|$

$\Rightarrow \forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $|g(x) - g(y)| \leq \beta\|x-y\|$

so, g: β-Lipschitz continuous

set $y := \bar{y}$

$\Rightarrow \forall_{x\in\mathcal{H}}$ $\underbrace{|g(x) - \underbrace{g(\bar{y})}_{\text{finite}}|}_{} \leq \beta \underbrace{\|x-\bar{y}\|}_{\text{finite}}$

$\geq |g(x)| - |g(\bar{y})|$ /* using reverse triangle inequality

$\|a-b\| + \|b\| \geq \|a-b+b\| = \|a\|$

$\Leftrightarrow \|a-b\| \geq \|a\| - \|b\|$ */

$\Rightarrow \forall_{x\in\mathcal{H}}$

$|g(x)| - |\underbrace{g(\bar{y})}_{\text{finite}}| \leq |g(x) - g(\bar{y})| \leq \beta\|x-\bar{y}\|$

$\therefore \forall_{x\in\mathcal{H}}$ $|g(x)| \leq \underbrace{\beta\|x-\bar{y}\| + |g(\bar{y})|}_{\text{finite}}$

so, g: everywhere real valued

$\therefore g$: real valued everywhere and β-Lipschitz continuous.

now suppose $\beta = 0$. then by definition

$g = f \,\Box\, \underbrace{\beta}_{0}\|\cdot\|$

$\therefore \forall_{x\in\mathcal{H}}$ $g(x) = \inf_{y\in\mathcal{H}} \left(f(y) + \overset{0}{\widetilde{\beta}}\|x-y\|\right) = \inf_{y\in\mathcal{H}} f(y) = \inf f(\mathcal{H})$

$\Leftrightarrow \forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}}$ $g(x) \leq f(y)$

set $y := x \Rightarrow$

$\Leftrightarrow \forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \quad g(x) \leq f(y)$

set $y := x \Rightarrow$

$\forall_{x\in\mathcal{H}} \quad g(x) \leq f(x)$

$\leftrightarrow \quad \underbrace{g}_{\substack{\text{real valued everywhere} \\ \beta\text{- Lipschitz continuos}}} \leq f \leftrightarrow g$: β-Lipschitz continuous minorant of $f \Rightarrow$ contradiction

now consider $\beta>0$, then

$$g = f \,\square\, \overset{>0}{\beta} \|\cdot\|$$

$$= f \,\square\, \frac{1}{(1/\beta)\,1} \|\cdot\|^1$$: this is power norm infimal convolution with 1 norm, so we can apply Proposition 12.9 (see above)

$$= g_{1/\beta}$$ property 2 which says

$$\forall_{x\in\mathcal{H}} \quad \underbrace{g_{1/\beta}}_{g}(x) \leq f(x)$$

$$\leftrightarrow \quad g \leq f$$

$\therefore g$: β-Lipschitz continuous minorant of $f \Rightarrow$ contradiction

$\therefore g$ cannot be real valued anywhere, but $\operatorname{dom} g = \mathcal{H}$ (Proposition 12.9 (i))

$\forall_{x\in\mathcal{H}} \quad g(x) < +\infty$

$\therefore \forall_{x\in\mathcal{H}} \quad g(x) = -\infty \leftrightarrow g \equiv -\infty.$

■

Part 2

7:30 AM

*** Proposition 12.22.**

$[\, f : \mathcal{H} \to]-\infty, +\infty],$

$\gamma \in \mathbb{R}_{++}$

$\mu \in \mathbb{R}_{++} \,]$

(i) $^{\mu}(\gamma f) = \gamma\,(^{\gamma\mu} f)$

(ii) $^{\gamma}(^{\mu} f) = {}^{(\gamma+\mu)} f$

<u>Proof</u>:

Take $x \in \mathcal{H}$

(i) $^{\gamma} f = f \,\square\, \frac{1}{2\gamma}\|\cdot\|^2$

$$^{\mu}(\gamma f)(x) = \left(\gamma f \;\square\, \tfrac{1}{2\mu}\|\cdot\|^2\right)(x)$$

$$= \inf_{y \in \mathcal{H}} \left(\gamma f(y) + \frac{1}{2\mu}\|x-y\|^2\right)$$

$$= \gamma \underbrace{\inf_{y \in \mathcal{H}} \left(f(y) + \frac{1}{2\gamma\mu}\|x-y\|^2\right)}_{\left(f \,\square\, \frac{1}{2(\gamma\mu)}\|\cdot\|^2\right)(x) = {}^{\gamma\mu}(f)}$$

$$= \gamma\,(^{\gamma\mu} f)(x)$$

$\therefore\; ^{\mu}(\gamma f) = \gamma\,(^{\gamma\mu} f)$

(ii)

$^{\gamma}(^{\mu} f)(x)$

$$= \left(^{\mu} f \,\square\, \frac{1}{2\gamma}\|\cdot\|^2\right)(x)$$

$$= \inf_{z \in \mathcal{H}} \left(\underbrace{^{\mu} f(z)}_{\left(f \,\square\, \frac{1}{2\mu}\|\cdot\|^2\right)(z) = \inf_{y\in\mathcal{H}}\left(f(y) + \frac{1}{2\mu}\|z-y\|^2\right)} + \frac{1}{2\gamma}\|z-x\|^2\right)$$

$$= \inf_{z \in \mathcal{H}} \left(\inf_{y \in \mathcal{H}} \left(f(y) + \frac{1}{2\mu}\|z-y\|^2\right) + \underbrace{\frac{1}{2\gamma}\|z-x\|^2}_{\text{constant w.r.t } y}\right)$$

$$= \inf_{z \in \mathcal{H}} \inf_{y \in \mathcal{H}} \left(f(y) + \frac{1}{2\mu}\|z-y\|^2 + \frac{1}{2\gamma}\|z-x\|^2\right) \quad /* \inf_{(a,b)\in\mathcal{H}\times\mathcal{H}} = \inf_{(b,a)\in\mathcal{H}\times\mathcal{H}} */$$

$$= \inf_{y \in \mathcal{H}} \inf_{z \in \mathcal{H}} \left(f(y) + \frac{1}{2\mu}\|z-y\|^2 + \frac{1}{2\gamma}\|z-x\|^2\right)$$

$$= \inf_{y \in \mathcal{H}} f(y) + \inf_{z \in \mathcal{H}} \left(\frac{1}{2\mu}\|z-y\|^2 + \frac{1}{2\gamma}\|z-x\|^2\right)$$

A trick worth remembering

/* define $\alpha = \frac{\mu}{\mu+\gamma} \in (0,1)$, $1-\alpha = \frac{\gamma}{\mu+\gamma} \in (0,1)$

then $\frac{\alpha}{2\alpha\gamma} = \frac{1}{2\gamma}$, $\frac{1-\alpha}{2\alpha\gamma} = \frac{\cancel{\gamma}}{\cancel{\mu+\gamma}} \cdot \frac{1}{2 \cdot \frac{\mu}{\cancel{\mu+\gamma}} \cdot \cancel{\gamma}} = \frac{1}{2\mu}$ */

$$= \inf_{z\in\mathcal{H}} \frac{(1-\alpha)}{2\alpha\gamma}\|z-y\|^2 + \frac{\alpha}{2\alpha\gamma}\|z-x\|^2$$

$$= \frac{1}{2\alpha\gamma} \inf_{z\in\mathcal{H}} (1-\alpha)\|z-y\|^2 + \alpha\|z-x\|^2$$

/* recall, corollary 2.14 : $\|\underbrace{\alpha}_{\in\mathbb{R}} x + (1-\alpha) y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ */

$\|(1-\alpha)(z-y) + \alpha(z-x)\|^2 +$

$\alpha(1-\alpha)\|z-x-z+y\|^2$

$= \|(1-\alpha)z - \alpha(z-y) + \alpha z - \alpha x\|^2 + \alpha(1-\alpha)\|x-y\|^2$

$= \|z - (\alpha x + (1-\alpha)y)\|^2 + \alpha(1-\alpha)\|x-y\|^2$

$$= \frac{1}{2\alpha\gamma} \inf_{z\in\mathcal{H}} \left(\|z - (\alpha x + (1-\alpha)y)\|^2 + \alpha(1-\alpha)\|x-y\|^2\right)$$

$$= \frac{1}{2\alpha\gamma}\alpha(1-\alpha)\|x-y\|^2 + \frac{1}{2\alpha\gamma}\underbrace{\inf_{z\in\mathcal{H}} \|z - (\alpha x + (1-\alpha)y)\|^2}_{0 \;/*\text{ by setting } z := \alpha x + (1-\alpha)y\; */}$$

$$= \frac{1-\alpha}{2\gamma}\|x-y\|^2 \quad /* \;(1-\alpha) = \frac{\gamma}{\gamma+\alpha}\; */$$

$$= \frac{\cancel{\gamma}}{(\gamma+\alpha)} \cdot \frac{1}{2\cancel{\gamma}}\|x-y\|^2 = \frac{1}{2(\gamma+\alpha)}\|x-y\|^2$$

$$= \inf_{y\in\mathcal{H}} \left(f(y) + \frac{1}{2(\gamma+\mu)}\|x-y\|^2\right)$$

$$= \left(f \,\square\, \frac{1}{2(\gamma+\mu)}\|\cdot\|^2\right)(x)$$

$$= {}^{(\gamma+\mu)} f\,(x)$$

$$= \left(f \,\square\, \frac{1}{2(\gamma+\mu)} \|\cdot\|^2 \right)(x)$$

$$= {}^{(\gamma+\mu)}f\,(x)$$

$$\therefore\ {}^{\gamma}\left({}^{\mu}f\right) = {}^{(\gamma+\mu)}f$$

■

Proposition 12.27.

[$f \in \Gamma_0(\mathcal{H})$]

Prox_f : firmly nonexpansive

$\mathrm{Id}-\mathrm{Prox}_f$: firmly nonexpansive

Proof:

$\forall_{x,y\in\mathcal{H}}$

$$\begin{cases} p = \mathrm{Prox}_f\, x \\ q = \mathrm{Prox}_f\, y \end{cases}$$

Proposition 12.26. (Defining properties of Prox operator)

[$f\in\Gamma_0(\mathcal{H})$; $x, p\in\mathcal{H}$]

$p = \mathrm{Prox}_f\, x \Leftrightarrow \forall_{y\in\mathcal{H}}\ \langle y-p \mid x-p\rangle + f(p) \leqslant f(y)$

$z = \mathrm{Prox}_f\, v \Leftrightarrow \forall_{r\in\mathcal{H}}\ \langle r-z \mid v-z\rangle + f(z) \leqslant f(r)$

$$\forall_{r\in\mathcal{H}}\ \langle r-p \mid x-p\rangle + f(p) \leqslant f(r) \overset{r:=q}{\Rightarrow} \langle q-p \mid x-p\rangle + f(p) \leqslant f(q)$$

$$\forall_{s\in\mathcal{H}}\ \langle s-q \mid y-q\rangle + f(q) \leqslant f(s) \overset{s:=p}{\Rightarrow} \langle p-q \mid y-q\rangle + f(q) \leqslant f(p)$$

$\oplus$

$$\langle q-p \mid x-p\rangle + \langle p-q \mid y-q\rangle + \cancel{f(p)} + \cancel{f(q)} \leqslant \cancel{f(p)} + \cancel{f(q)}$$

/* this cancellation is valid as $p, q \in \mathrm{dom}\, f$

$p = \mathrm{Prox}_f(x) \Leftrightarrow \forall_{z\in\mathcal{H}}\ \langle z-p \mid x-p\rangle + f(p) \leqslant f(z)$

take $z \in \mathrm{dom} f \setminus \{p\}$ then.

$$\underbrace{\langle z-p \mid x-p\rangle}_{\text{finite}} + f(p) \leqslant \underbrace{f(z)}_{\text{finite}}$$

$\Rightarrow f(p) \leqslant \text{finite} < +\infty \Rightarrow p \in \mathrm{dom}\, f$

similarly $q \in \mathrm{dom}\, f$

*/

$$\Leftrightarrow -\langle p-q \mid x-p\rangle + \langle p-q \mid y-q\rangle \leqslant 0$$

$$\Leftrightarrow -\langle p-q \mid (x-p)-(y-q)\rangle \leqslant 0$$

$$\Leftrightarrow \langle p-q \mid (x-p)-(y-q)\rangle \geqslant 0$$

/* denote $\mathrm{Prox}_f \equiv T$ */

$$\Leftrightarrow \langle Tx-Ty \mid \underbrace{(x-Tx)}_{(\mathrm{Id}-T)x} - \underbrace{(y-Ty)}_{(\mathrm{Id}-T)y}\rangle \geqslant 0$$

$$\therefore \forall_{x\in\mathcal{H}}\ \forall_{y\in\mathcal{H}}\ \langle Tx-Ty \mid (\mathrm{Id}-T)x-(\mathrm{Id}-T)y\rangle \geqslant 0$$

$\Leftrightarrow$ T: firmly nonexpansive

$\Leftrightarrow$ $(\mathrm{Id}-T)$: firmly nonexpansive

/* Equivalence follows from the theorem.

Proposition 4.2. (Different representations of firmly nonexpansive operator)

[D: nonempty subset of $\mathcal{H}$.

$T: D\to\mathcal{H}$]

(i) T: firmly nonexpansive $\Leftrightarrow$ (ii) $(I-T)$: firmly nonexpansive $\Leftrightarrow$ (iii) $(2T-I)$: nonexpansive

$\Leftrightarrow$ (iv) $\forall_{x\in D}\ \forall_{y\in D}\ \|Tx-Ty\|^2 \leqslant \langle x-y \mid Tx-Ty\rangle$

$\Leftrightarrow$ (v) $\forall_{x\in D}\ \forall_{y\in D}\ \langle Tx-Ty \mid (I-T)x-(I-T)y\rangle \geqslant 0$

$\Leftrightarrow$ (vi) $\forall_{x\in D}\ \forall_{y\in D}\ \forall_{\alpha\in[0,1]}\ \|Tx-Ty\| \leqslant \|\alpha(x-y)+(1-\alpha)(Tx-Ty)\|$

*/

■

Proposition 12.26.

[$f\in\Gamma_0(\mathcal{H})$; $x, p\in\mathcal{H}$]

$p = \mathrm{Prox}_f\, x \Leftrightarrow \forall_{y\in\mathcal{H}}\ \langle y-p \mid x-p\rangle + f(p) \leqslant f(y)$

Proof:

($\Rightarrow$)

take any $y\in\mathcal{H}$

$p = \mathrm{Prox}_f\, x$

$\forall_{\alpha\in]0,1[}\ p_\alpha = \alpha y + (1-\alpha)p$, now as $f\in\Gamma_0(\mathcal{H}) \Rightarrow f(p_\alpha) = f(\alpha y + (1-\alpha)p) \leqslant \alpha f(y) + (1-\alpha) f(p)$

now. $\mathrm{Prox}_f\, x$ by definition satisfies the following

$${}^{1}f(x) = \min_{y\in\mathcal{H}} \left(f(y) + \tfrac{1}{2}\|x-y\|^2 \right) = f(\mathrm{Prox}_f\, x) + \tfrac{1}{2}\|x - \mathrm{Prox}_f\, x\|^2$$

$$= f(p) + \tfrac{1}{2}\|x-p\|^2$$

$$\Rightarrow \forall_{\tilde{y}\in\mathcal{H}}\ f(\tilde{y}) + \tfrac{1}{2}\|x-\tilde{y}\|^2 \geqslant f(p) + \tfrac{1}{2}\|x-p\|^2$$

set $\tilde{y} := p_\alpha = \alpha y + (1-\alpha)p$

$$f(p) + \tfrac{1}{2}\|x-p\|^2 \leqslant f(p_\alpha) + \tfrac{1}{2}\|x-p_\alpha\|^2$$

$$\Leftrightarrow f(p) \leqslant f(p_\alpha) + \tfrac{1}{2}\|x-p_\alpha\|^2 - \tfrac{1}{2}\|x-p\|^2$$

$$\leqslant \alpha f(y) + (1-\alpha) f(p) + \tfrac{1}{2}\underbrace{\|x-p_\alpha\|^2} - \tfrac{1}{2}\|x-p\|^2$$

/* $\|x-\alpha y-(1-\alpha)p\|^2 = \|x-\alpha y - p + \alpha p\|^2$

$= \|(x-p)-\alpha(y-p)\|^2 = \|x-p\|^2 + \alpha^2\|y-p\|^2 - 2\alpha\langle x-p \mid y-p\rangle$ */

$$= \alpha f(y) + (1-\alpha) f(p) + \cancel{\tfrac{1}{2}\|x-p\|^2} + \tfrac{\alpha^2}{2}\|y-p\|^2 - \alpha\langle x-p \mid y-p\rangle - \cancel{\tfrac{1}{2}\|x-p\|^2}$$

$= \|(x-p)-\alpha(y-p)\|^2 = \|x-p\|^2+\alpha^2\|y-p\|^2-2\alpha\langle x-p \mid y-p\rangle$ */

$= \alpha f(y)+(1-\alpha)f(p)+\frac{1}{2}\cancel{\|x-p\|^2}+\frac{\alpha^2}{2}\|y-p\|^2-\alpha\langle x-p \mid y-p\rangle-\frac{1}{2}\cancel{\|x-p\|^2}$

$\Rightarrow \quad f(p) \leq \alpha f(y)+(1-\alpha)f(p)+\frac{\alpha^2}{2}\|y-p\|^2-\alpha\langle x-p \mid y-p\rangle$

$\Leftrightarrow \quad \cancel{\alpha} f(p) \leq \cancel{\alpha} f(y)+\frac{\cancel{\alpha^2}^{\,\alpha}}{2}\|y-p\|^2-\cancel{\alpha}\langle x-p \mid y-p\rangle \quad [\because \alpha\in\,]0,1[\,]$

$\therefore \forall_{\alpha\in\,]0,1[} \quad f(p)+\langle x-p \mid y-p\rangle \leq f(y)+\frac{\alpha}{2}\|y-p\|^2$

$\alpha\downarrow 0 \Rightarrow \quad \langle y-p \mid x-p\rangle+f(p)\leq f(y) \quad \forall_{y\in\mathcal{H}}$

($\Leftarrow$)

$\forall_{y\in\mathcal{H}} \quad \langle y-p \mid x-p\rangle+f(p)\leq f(y)$

$\Leftrightarrow \quad f(p)\leq f(y)-\langle y-p \mid x-p\rangle$

$\Leftrightarrow \quad f(p)+\frac{1}{2}\|x-p\|^2 \leq f(y)+\frac{1}{2}\|x-p\|^2+\langle x-p \mid p-y\rangle$

$\leq f(y)+\underbrace{\frac{1}{2}\|x-p\|^2+\langle x-p \mid p-y\rangle+\underbrace{\frac{1}{2}\|p-y\|^2}_{\geq 0}}$ so adding it will make r.h.s bigger

$\frac{1}{2}\left(\|x-p\|^2+2\langle x-p \mid p-y\rangle+\|p-y\|^2\right)=\frac{1}{2}\left(\|x-p+p-y\|^2\right)=\frac{1}{2}\|x-y\|^2$

$= f(y)+\frac{1}{2}\|x-y\|^2$

$\therefore \forall_{y\in\mathcal{H}} \quad f(p)+\frac{1}{2}\|x-p\|^2 \leq f(y)+\frac{1}{2}\|x-y\|^2$

$\Leftrightarrow \quad f(p)+\frac{1}{2}\|x-p\|^2 = \min_{y\in\mathcal{H}} f(y)+\frac{1}{2}\|x-y\|^2$

$\Leftrightarrow \quad p = \mathrm{Prox}_f\, x$

$\blacksquare$

* Proposition 12.29.

$[\![\; f\in\Gamma_0(\mathcal{H})$

$\gamma\in\mathbb{R}_{++} \;]\!]$

${}^{\gamma}f:\mathcal{H}\to\mathbb{R}$, Fréchet differentiable, /* $[\![x\in\mathcal{H},\ C\in\mathcal{V}(x),\ T:C\to\mathcal{K}]\!]$ T: Fréchet differentiable at $x \overset{\text{def}}{\Leftrightarrow} \exists\ DT(x)\in\underbrace{\mathcal{B}(\mathcal{H},\mathcal{K})}_{\text{space of bounded linear operator from } \mathcal{H} \text{ to } \mathcal{K} \text{ with domain } \mathcal{H}}$ $\displaystyle\lim_{0\neq\|y\|\to 0}\frac{\|T(x+y)-Tx-DT(x)y\|}{\|y\|}=0$

$\nabla({}^{\gamma}f)=\frac{1}{\gamma}\left(\mathrm{Id}-\mathrm{Prox}_{\gamma f}\right)$: γ^{-1} Lipschitz continuous.

<u>Proof</u>:

take, $x,y\in\mathcal{H}$: $x\neq y$

$p=\mathrm{Prox}_{\gamma f}\,x,\quad q=\mathrm{Prox}_{\gamma f}\,y$

/* ${}^{\gamma}f(x)=f\left(\mathrm{Prox}_{\gamma f}x\right)+\frac{1}{2\gamma}\|x-\mathrm{Prox}_{\gamma f}x\|^2$ */

so,

${}^{\gamma}f(x)=f(p)+\frac{1}{2\gamma}\|x-p\|^2$

${}^{\gamma}f(y)=f(q)+\frac{1}{2\gamma}\|y-q\|^2$

${}^{\gamma}f(y)-{}^{\gamma}f(x)=\underline{f(q)-f(p)}+\frac{1}{2\gamma}\left(\|y-q\|^2-\|x-p\|^2\right)$

/* Proposition 12.26 :

ঞ $=\mathrm{Prox}_f$(দী) $\Leftrightarrow \forall_{\text{বি}\in\mathcal{H}}$ ⟨বি − ঞ | দী − ঞ⟩ + f(ঞ) ≤ f(বি) /* [illegible] */

$p=\mathrm{Prox}_{\gamma f}\,x \Leftrightarrow \forall_{v\in\mathcal{H}} \quad \langle v-p \mid x-p\rangle+\gamma f(p)\leq\gamma f(v)$

$v:=q$

$\langle q-p \mid x-p\rangle+\gamma f(p)\leq\gamma f(q)$

$\Leftrightarrow \quad \gamma f(q)-\gamma f(p)\geq\langle q-p \mid x-p\rangle$

$\Leftrightarrow \quad f(q)-f(p)\geq\frac{1}{\gamma}\langle q-p \mid x-p\rangle$

*/

${}^{\gamma}f(y)-{}^{\gamma}f(x)\geq\frac{1}{\gamma}\langle q-p \mid x-p\rangle+\frac{1}{2\gamma}\left(\|y-q\|^2-\|x-p\|^2\right)$

$=\frac{1}{2\gamma}\left(2\langle q-p \mid x-p\rangle+\|y-q\|^2-\|x-p\|^2\right)$ /* now

$\boxed{\begin{aligned}&\|(y-q)-(x-p)\|^2+2\langle y-x \mid x-p\rangle\\ &=\|y-q\|^2+\|x-p\|^2\underbrace{-2\langle y-q \mid x-p\rangle+2\langle y-x \mid x-p\rangle}\end{aligned}}$

$$= \frac{1}{2\gamma}\left(2\langle q-p \mid x-p\rangle + \|y-q\|^2 - \|x-p\|^2\right)$$

/* now

$$\boxed{\begin{aligned} &\|(y-q)-(x-p)\|^2 + 2\langle y-x \mid x-p\rangle \\ &= \|y-q\|^2 + \|x-p\|^2 \underbrace{- 2\langle y-q \mid x-p\rangle + 2\langle y-x \mid x-p\rangle}_{+2\langle y-x-y+q \mid x-p\rangle} \\ &= \|y-q\|^2 + \|x-p\|^2 + 2\langle q-x \mid x-p\rangle \\ &= \|y-q\|^2 - \|x-p\|^2 + \underbrace{2\|x-p\|^2 + 2\langle q-x \mid x-p\rangle}_{\substack{2\langle x-p \mid x-p\rangle + 2\langle q-x \mid x-p\rangle \\ = 2\langle \cancel{x}-p+q-\cancel{x} \mid x-p\rangle}} \\ &= \|y-q\|^2 - \|x-p\|^2 + 2\langle q-p \mid x-p\rangle \end{aligned}}$$

*/

$$= \frac{1}{2\gamma}\left(\underbrace{\|(y-q)-(x-p)\|^2}_{\substack{\geq 0 \text{, so removing} \\ \text{it will give smaller term}}} + 2\langle y-x \mid x-p\rangle\right)$$

$$\geq \frac{1}{\gamma}\langle y-x \mid x-p\rangle$$

$$\therefore \forall_{x,y\in\mathcal{H}:x\neq y} \quad {}^{\gamma}f(y) - {}^{\gamma}f(x) \geq \frac{1}{\gamma}\langle y-x \mid x-\underbrace{p}_{\operatorname{Prox}_{\gamma f}x}\rangle$$

/* in free notation

$$\forall_{z,w\in\mathcal{H}:z\neq w} \quad {}^{\gamma}f(w) - {}^{\gamma}f(z) \geq \frac{1}{\gamma}\langle w-z \mid z-\operatorname{Prox}_{\gamma f} z\rangle$$

so, $z := y$, $w := x$

$$\forall_{y,x\in\mathcal{H}:y\neq x} \quad {}^{\gamma}f(x) - {}^{\gamma}f(y) \geq \frac{1}{\gamma}\langle x-y \mid y-\overbrace{\operatorname{Prox}_{\gamma f} y}^{q}\rangle$$

$$\leftrightarrow \quad {}^{\gamma}f(y) - {}^{\gamma}f(x) \leq \frac{1}{\gamma}\langle y-x \mid y-q\rangle \quad */$$

$$\forall_{x,y\in\mathcal{H}:x\neq y} \quad 0 \leq \underbrace{{}^{\gamma}f(y) - {}^{\gamma}f(x)}_{\leq \frac{1}{\gamma}\langle y-x \mid y-q\rangle} - \frac{1}{\gamma}\langle y-x \mid x-p\rangle$$

$$\leq \frac{1}{\gamma}\langle y-x \mid y-q\rangle - \frac{1}{\gamma}\langle y-x \mid x-p\rangle$$

$$= \frac{1}{\gamma}\langle y-x \mid (y-q)-(x-p)\rangle$$

$$= \frac{1}{\gamma}\langle y-x \mid (y-x)-(q-p)\rangle$$

$$= \frac{1}{\gamma}\left(\underbrace{\langle y-x \mid y-x\rangle}_{\|y-x\|^2} - \langle y-x \mid q-p\rangle\right)$$

/* set $\operatorname{Prox}_{\gamma f} = T$: firmly nonexpansive then by Proposition 4.2. (iv)

$$\forall_{x\in\mathcal{H}}\forall_{y\in\mathcal{H}} \quad \|\underbrace{Ty}_{q} - \underbrace{Tx}_{p}\|^2 \leq \langle y-x \mid \underbrace{Ty}_{q} - \underbrace{Tx}_{p}\rangle$$

$$\leftrightarrow \quad \|q-p\|^2 \leq \langle y-x \mid q-p\rangle$$

$$\leftrightarrow \quad -\|q-p\|^2 \geq -\langle y-x \mid q-p\rangle \quad */$$

$$\leq \frac{1}{\gamma}\left(\|y-x\|^2 - \underbrace{\|q-p\|^2}_{\substack{\leq 0 \text{, so removing it would give} \\ \text{a larger term}}}\right)$$

$$\leq \frac{1}{\gamma}\|y-x\|^2$$

so we have got

$$\forall_{x,y\in\mathcal{H}:x\neq y} \quad 0 \leq {}^{\gamma}f(y) - {}^{\gamma}f(x) - \left\langle y-x \,\middle|\, \frac{1}{\gamma}(x-p)\right\rangle \leq \frac{1}{\gamma}\|y-x\|^2$$

$$\leftrightarrow \quad 0 \leq \frac{{}^{\gamma}f(y) - {}^{\gamma}f(x) - \left\langle y-x \,\middle|\, \frac{1}{\gamma}(x-p)\right\rangle}{\|y-x\|} \leq \frac{1}{\gamma}\|y-x\|$$

set $y-x = h \neq 0$ then $y = x+h$, which makes the above as follows:

$$0 \leq \frac{{}^{\gamma}f(x+h) - {}^{\gamma}f(x) - \left\langle h \,\middle|\, \frac{1}{\gamma}(x-\operatorname{Prox}_{\gamma f}x)\right\rangle}{\|h\|} \leq \frac{1}{\gamma}\|h\|$$

thus,

$$\lim_{0\neq\|h\|\to 0} \frac{{}^{\gamma}f(x+h) - {}^{\gamma}f(x) - \left\langle \overbrace{\frac{1}{\gamma}(x-\operatorname{Prox}_{\gamma f}x)}^{(D^{\gamma}f)(x)} \,\middle|\, h\right\rangle}{\|h\|} = 0 \quad \text{/* checking with the definition */}$$

$$\therefore \nabla({}^{\gamma}f)(\cdot) = (D^{\gamma}f)^{\overset{\text{transpose}}{T}}(\cdot) = \frac{1}{\gamma}\left(\cdot - \operatorname{Prox}_{\gamma f}\cdot\right)^{TT} = \underbrace{\frac{1}{\gamma}\left(\cdot - \operatorname{Prox}_{\gamma f}\cdot\right)}_{\text{column vector}}$$

$$\text{so. } \nabla({}^{\gamma}f) = \frac{1}{\gamma}\left(\mathrm{Id} - \operatorname{Prox}_{\gamma f}\right)$$

■

Proposition 12.31.

[K: nonempty closed convex cone in $\mathcal{H}$

$q = \frac{1}{2}\|\cdot\|^2$]

$$\nabla(q \cdot P_K) = \nabla\left(\frac{1}{2}d^2_{K^{\ominus}}\right) = P_K$$

Proof:

/* $C^{\ominus}$: polar cone of $C \overset{\text{def}}{\leftrightarrow} C^{\ominus} = \{u\in\mathcal{H} \mid \sup\langle C \mid u\rangle \leq 0\}$ */

Recall Moreaus Theorem (Theorem 6.29)

/*

* Theorem 6.29. (Moreau) /* relates projection, cone and polar cone */

⟦ K: nonempty convex cone in $\mathcal{H}$

$x \in \mathcal{H}$ ⟧

(i) $x = P_K x + P_{K^\ominus} x$ /* this is amazing, any vector in $\mathcal{H}$ can be decomposed into two vectors associated with the projection on a cone and its polar */

(ii) $P_K x \perp P_{K^\ominus} x$

(iii) $\|x\|^2 = d_K^2(x) + d_{K^\ominus}^2(x)$

and Corollary 12.30.

* Corollary 12.30.

⟦ C: nonempty closed convex subset of $\mathcal{H}$ ⟧

d_C^2: Fréchet differentiable /* d_C: distance function to a set: $d_C(x) = \inf d(x, C)$ */

$\nabla d_C^2 = 2(\mathrm{Id} - P_C)$

*/

$$\forall_{x \in \mathcal{H}} \ \|x\|^2 = d_K^2(x) + d_{K^\ominus}^2(x)$$

$$\Leftrightarrow d_K^2(x) = \|x\|^2 - d_{K^\ominus}^2(x)$$

$$\Rightarrow \nabla_x d_K^2(x) = \nabla_x\left(\|x\|^2 - d_{K^\ominus}^2(x)\right)$$

$$= 2x - \nabla_x d_{K^\ominus}^2(x)$$

$$\Leftrightarrow \nabla_x d_K^2(x) = 2(\mathrm{Id} - P_K)(x) = 2x - \nabla_x d_{K^\ominus}^2(x)$$

$$\Rightarrow 2(x - P_K x) = 2x - \nabla_x d_{K^\ominus}^2(x)$$

$$\Leftrightarrow 2P_K x = \nabla_x d_{K^\ominus}^2(x)$$

$$\Leftrightarrow \tfrac{1}{2}\nabla_x d_{K^\ominus}^2(x) = P_K(x)$$

$$\therefore P_K = \nabla\left(\tfrac{1}{2} d_{K^\ominus}^2\right)$$

∎

Proposition 12.32.

⟦ $f \in \Gamma_0(\mathcal{H})$

$x \in \mathcal{H}$ ⟧

$$\left({}^{\gamma}f(x)\right)_{\gamma \in \mathbb{R}_{++}} : \begin{cases} \bullet \text{ decreasing} \\ \bullet \ {}^{\gamma}f(x) \uparrow f(x) \text{ as } \gamma \downarrow 0 \\ \bullet \ {}^{\gamma}f(x) \downarrow \inf f(\mathcal{H}) \text{ as } \gamma \uparrow +\infty \end{cases}$$

Proof:

Recall that ${}^{\gamma}f$ is the power 2-norm infimal convolution

So, ${}^{\gamma}f$ will satisfy all the properties of power norm infimal convolution:

(Power norm infimal convolution of a general function and its properties)

* Proposition 12.9. ★★★

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper

$p \in [1, +\infty[$

$\forall_{\gamma \in \mathbb{R}_{++}} \ g_\gamma = f \,\square\, \left(\frac{1}{\gamma p}\|\cdot\|^p\right)$: power norm infimal convolution]

$\forall_{\gamma \in \mathbb{R}_{++}} \forall_{x \in \mathcal{H}}$

dom $g_\gamma = \mathcal{H}$

$\mu \in]\gamma, +\infty[\Rightarrow \inf f(\mathcal{H}) \le g_\mu(x) \le g_\gamma(x) \le f(x)$

$\inf g_\gamma(\mathcal{H}) = \inf f(\mathcal{H})$

$g_\mu(x) \downarrow \inf f(\mathcal{H})$ as $\mu \uparrow +\infty$

g_γ: bounded on every ball in $\mathcal{H}$

/* • β-Pasch-Hausdorff envelope is the power 1-norm infimal convolution for $\beta > 0$.

• Moreau envelope is power 2-norm infimal convolution */

in this theorem we can replace g_γ with ${}^{\gamma}f$.

as $\forall_{\mu > \gamma} \ g_\mu(x) \le g_\gamma(x)$

$$\Rightarrow \forall_{\mu > \gamma} \ {}^{\mu}f(x) \le {}^{\gamma}f(x)$$

$\therefore \left({}^{\gamma}f(x)\right)_{\gamma \in \mathbb{R}_{++}}$: decreasing ☺

$g_\mu(x) \downarrow \inf f(\mathcal{H})$ as $\mu \to +\infty$

$\therefore {}^{\mu}f(x) \downarrow \inf f(\mathcal{H})$ as $\mu \to +\infty$ ☺

So, we are left with proving ${}^{\gamma}f(x) \uparrow f(x)$ as $\gamma \downarrow 0$

set $\mu = \sup_{\gamma \in \mathbb{R}_{++}} {}^{\gamma}f(x)$.

From Proposition 12.9 (ii):

$$\forall_{\gamma \in \mathbb{R}_{++}} \forall_{\tilde{\mu} \in]\gamma, +\infty[} \ \inf f(\mathcal{H}) \le {}^{\tilde{\mu}}f(x) \le {}^{\gamma}f(x) \le f(x) \Rightarrow \forall_{\gamma \in \mathbb{R}_{++}} \ {}^{\gamma}f(x) \le f(x) \Rightarrow \sup_{\gamma \in \mathbb{R}_{++}} {}^{\gamma}f(x) \le \sup_{\gamma \in \mathbb{R}_{++}} f(x) \Rightarrow \mu \le f(x)$$

$\gamma \in \mathbb{R}_{++}$

From Proposition 12.9 (ii):

$\forall_{\gamma \in \mathbb{R}_{++}} \forall_{\tilde{\mu} \in]\gamma, +\infty[} \quad \inf f(\mathcal{H}) \leq {}^{\tilde{\mu}}f(x) \leq {}^{\gamma}f(x) \leq f(x) \Rightarrow \forall_{\gamma \in \mathbb{R}_{++}} \; {}^{\gamma}f(x) \leq f(x) \Rightarrow \underbrace{\sup_{\gamma \in \mathbb{R}_{++}} {}^{\gamma}f(x)}_{\mu} \leq \underbrace{\sup_{\gamma \in \mathbb{R}_{++}} f(x)}_{f(x)} \Rightarrow \mu \leq f(x)$

$\therefore \forall_{\gamma \in \mathbb{R}_{++}} \forall_{\tilde{\mu} \in]\gamma, +\infty[} \quad \inf f(\mathcal{H}) \leq {}^{\tilde{\mu}}f(x) \leq {}^{\gamma}f(x) \leq \mu \leq f(x) \longrightarrow \underline{\lim}_{\gamma \to 0} {}^{\gamma}f(x) \leq \overline{\lim}_{\gamma \to 0} {}^{\gamma}f(x) \leq \mu \leq f(x) \quad (1)$

as ${}^{\gamma}f(x)$: decreasing in γ, the sup will be achieved as $\gamma \downarrow 0$ /* A bounded set has a finite supremum */

assume $\mu < +\infty$ /* otherwise, $\mu = +\infty \leq f(x) \Rightarrow f(x) = +\infty \Rightarrow x \notin \operatorname{dom} f$ the claim holds trivially */

Now if we show,

$\underline{\lim}_{\gamma \downarrow 0} {}^{\gamma}f(x) \geq f(x)$ /* it is enough as we have already shown (1) then from (1) and $\underline{\lim}_{\gamma \downarrow 0} {}^{\gamma}f(x) \geq f(x) \Rightarrow \underline{\lim}_{\gamma \downarrow 0} {}^{\gamma}f(x) \leq \overline{\lim}_{\gamma \downarrow 0} {}^{\gamma}f(x) \leq \mu \leq f(x) \leq \underline{\lim}_{\gamma \downarrow 0} {}^{\gamma}f(x)$ which means: $\lim_{\gamma \downarrow 0} {}^{\gamma}f(x) = f(x)$ */

/* trick move: as $\gamma \downarrow 0$, we can confine γ to $]0,1[$ instead of $\mathbb{R}_{++}$ in our analysis */

First, recall from definition of ${}^{\gamma}f$:

$${}^{\gamma}f(x) = f(\operatorname{Prox}_{\gamma f} x) + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2$$

so,

$$\sup_{\tilde{\gamma} \in \mathbb{R}_{++}} {}^{\tilde{\gamma}}f(x) = \mu \geq {}^{\gamma}f(x) = f(\operatorname{Prox}_{\gamma f} x) + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2$$

denote $g := f + \frac{1}{2}\|x - \cdot\|^2$

$\therefore \operatorname{lev}_{\leq \mu} g = \{y \in \mathcal{H} \mid g(y) = f(y) + \frac{1}{2}\|x - y\|^2 \leq \mu\}$

suppose $\gamma \in]0,1[$ then,

$\frac{1}{2} \leq \frac{1}{2} \cdot \frac{1}{\gamma}$

$\Rightarrow \frac{1}{2}\|x-y\|^2 \leq \frac{1}{2\gamma}\|x-y\|^2$

$\Leftrightarrow \underbrace{f(y) + \frac{1}{2}\|x-y\|^2}_{g(y)} \leq f(y) + \frac{1}{2}\frac{1}{\gamma}\|x-y\|^2$

$\therefore \forall_{\gamma \in]0,1[} \forall_{y \in \mathcal{H}} \quad g(y) \leq f(y) + \frac{1}{2}\frac{1}{\gamma}\|x-y\|^2$

setting $y := \operatorname{Prox}_{\gamma f} x$

$\forall_{\gamma \in]0,1[} \quad g(\operatorname{Prox}_{\gamma f} x) \leq f(\operatorname{Prox}_{\gamma f} x) + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2 = {}^{\gamma}f(x) \leq \mu$

$\Rightarrow \forall_{\gamma \in]0,1[} \quad \operatorname{Prox}_{\gamma f}(x) \in \operatorname{lev}_{\leq \mu} g$

now note that

g: coercive /* from Corollary 11.15

* Corollary 11.15. *

[$\tilde{f}, \tilde{g} \in \Gamma_0(\mathcal{H})$] /* set $\tilde{g} := f$, $\tilde{f} := \frac{1}{2}\|x - \cdot\|^2$

$\operatorname{dom}\tilde{f} \cap \operatorname{dom}\tilde{g} \neq \emptyset$ /* holds */

One of the following holds:

(i) $\tilde{f}$: supercoercive

(ii) $\tilde{f}$: coercive, $\tilde{g}$: bounded below]

/* $\tilde{f} := \frac{1}{2}\|x - \cdot\|^2$: supercoercive as $\lim_{\|y\| \to +\infty} \frac{\tilde{f}(y)}{\|y\|} = +\infty$ $\therefore \tilde{f} = \frac{1}{2}\|x - \cdot\|^2$: supercoercive */

- $\tilde{f} + \tilde{g}$: coercive, has a minimizer over $\mathcal{H}$
- $\tilde{f}$ or $\tilde{g}$: strictly convex $\Rightarrow$ $f + g$: has exactly one minimizer over $\mathcal{H}$.

so, $g := f + \frac{1}{2}\|x - \cdot\|^2$: coercive, has a minimizer over $\mathcal{H}$

now

g: coercive $\Leftrightarrow$ $(\operatorname{lev}_{\leq \xi} g)_{\xi \in \mathbb{R}}$: bounded /* Prop 11.11 */

$\Rightarrow (\operatorname{lev}_{\leq \mu} g)$: bounded

as, $\operatorname{Prox}_{\gamma f}(x) \in \operatorname{lev}_{\leq \mu} g$, so $\overbrace{\sup_{\gamma \in]0,1[} \|\operatorname{Prox}_{\gamma f} x\|}^{V}$ will be finite, i.e., $\sup_{\gamma \in]0,1[} \|\operatorname{Prox}_{\gamma f} x\| = V < +\infty$

Again, $f \in \Gamma_0(\mathcal{H}) \Rightarrow f$: has a continuous affine minorant /* recall:

* Definition: (continuous affine minorant of a function)

[$f: \mathcal{H} \to]-\infty, +\infty]$, $u \in \mathcal{H}$]

f: has a continuous affine minorant with slope u $\overset{\text{def}}{\Leftrightarrow}$ $f - \langle \cdot \mid u \rangle$: bounded below.

$\overset{\text{def}}{\Leftrightarrow} \exists_{u \in \mathcal{H}} \exists_{\eta \in \mathbb{R}} \quad f \geq \underbrace{\langle u \mid \cdot \rangle + \eta}_{\text{continuous affine term}}$

* Theorem 9.19.

[$f \in \Gamma_0(\mathcal{H})$] f: possesses a continuous affine minorant. */

$\Rightarrow \exists_{u \in \mathcal{H}} \exists_{\eta \in \mathbb{R}} \quad f \geq \langle \cdot \mid u \rangle + \eta$

$\Leftrightarrow \exists_{u \in \mathcal{H}} \exists_{\eta \in \mathbb{R}} \forall_{\tilde{x} \in \mathcal{H}} \quad f(\tilde{x}) \geq \langle \tilde{x} \mid u \rangle + \eta$

set $\tilde{x} := \operatorname{Prox}_{\gamma f} x$

$\Rightarrow f(\operatorname{Prox}_{\gamma f} x) \geq \langle \operatorname{Prox}_{\gamma f} x \mid u \rangle + \eta$

$\forall_{\gamma \in]0,1[} \quad \mu \geq f(\operatorname{Prox}_{\gamma f} x) + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2$

$\geq \langle \operatorname{Prox}_{\gamma f} x \mid u \rangle + \eta + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2$ /* Cauchy Schwartz: $\overbrace{|\langle x \mid y \rangle|}^{\max\{\langle x|y \rangle, -\langle x|y \rangle\}} \leq \|x\|\,\|y\| \Rightarrow -\langle x \mid y \rangle \leq \|x\|\,\|y\|$ $\Leftrightarrow \langle x \mid y \rangle \geq -\|x\|\,\|y\|$ */

$\geq -\|\operatorname{Prox}_{\gamma f} x\|\,\|u\| + \eta + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2$

$\geq \underbrace{\left(\inf_{\gamma \in]0,1[} -\|\operatorname{Prox}_{\gamma f} x\|\right)}\|u\| + \eta + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2$

$\boxed{-\sup_{\gamma \in]0,1[} \|\operatorname{Prox}_{\gamma f} x\| \quad \text{/* } \sup f = -\inf -f \text{ */} \quad = -V}$

$= -\underbrace{V}_{\in [0,+\infty) = \mathbb{R}_+}\|u\| + \eta + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2$

$\Rightarrow \forall_{\gamma \in]0,1[} \quad \mu \geq -V\|u\| + \eta + \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2$

$\Leftrightarrow \forall \quad \frac{1}{2\gamma}\|x - \operatorname{Prox}_{\gamma f} x\|^2 \leq \mu + V\|u\| - \eta$

$\Rightarrow \forall_{\gamma \in]0,1[} \quad \mu \geq -\nu\|u\| + \eta + \frac{1}{2\gamma}\|x - \mathrm{Prox}_{\gamma f} x\|^2$

$\Leftrightarrow \forall_{\gamma \in]0,1[} \quad \frac{1}{2\gamma}\|x - \mathrm{Prox}_{\gamma f} x\|^2 \leq \mu + \nu\|u\| - \eta$

$\Leftrightarrow \forall_{\gamma \in]0,1[} \quad \|x - \mathrm{Prox}_{\gamma f} x\|^2 \leq 2\gamma \underbrace{(\mu + \nu\|u\| - \eta)}_{\text{finite}}$

so, $\|x - \mathrm{Prox}_{\gamma f} x\|^2 \to 0$, as $\gamma \to 0$

so, $\mathrm{prox}_{\gamma f} x \to x$ as $\gamma \to 0$

now $f \in \Gamma_0(\mathcal{H}) \Leftrightarrow f$: convex, proper, lower semicontinuous

$\Rightarrow f$: lower semicontinuous /* so, $x_a \to x \Rightarrow f(x) \leq \underline{\lim} f(x_a)$ */

$^{\gamma}f(x) = f(\mathrm{Prox}_{\gamma f} x) + \frac{1}{2\gamma}\|x - \mathrm{prox}_{\gamma f} x\|^2$

$\Rightarrow \underline{\lim}_{\gamma \downarrow 0} {}^{\gamma}f(x) = \underline{\lim}_{\gamma \downarrow 0} f(\mathrm{prox}_{\gamma f} x) + \underbrace{\frac{1}{2\gamma}\|x - \mathrm{prox}_{\gamma f} x\|^2}_{\geq 0 \text{, so removing will give a smaller number}}$ /* $f(\overset{]a,b[}{\gamma}, x) = g(\gamma, x) \Rightarrow \lim_{\gamma \to a} f(\gamma, x) = \lim_{\gamma \to a} g(\gamma, x)$ */

$\geq \underline{\lim}_{\gamma \downarrow 0} f(\mathrm{Prox}_{\gamma f} x)$

$= \lim_{\gamma \downarrow 0} f(\mathrm{Prox}_{\gamma f} x)$ /* $\underline{\lim}\, \xi_a = \lim \xi_a = \overline{\lim}\, \xi_a$ when the limit exists */

$\geq f(x)$ /* $\mathrm{Prox}_{\gamma f} x \to x$ as $\gamma \to 0 \Rightarrow \underline{\lim}_{\gamma \downarrow 0} f(\mathrm{Prox}_{\gamma f} x) \geq f x$ */

implies:

as f : lower semicontinuous

$\therefore \lim_{\gamma \downarrow 0} {}^{\gamma}f(x) = f(x)$

[Goal reached ⍨]

∎

Part 3

8:37 AM

(Power norm infimal convolution of a general function and its properties)

*Proposition 12.9. * * *

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper

$p \in [1, +\infty]$

$\forall_{\gamma \in \mathbb{R}_{++}}$ $g_\gamma = f \,\square \left(\frac{1}{\gamma p} \|\cdot\|^p \right)$;power norm infimal convolution]

$\Rightarrow \forall_{\gamma \in \mathbb{R}_{++}} \forall_{x \in \mathcal{H}}$

• $\operatorname{dom} g_\gamma = \mathcal{H}$

• $\mu \in]\gamma, +\infty[\Rightarrow \inf f(\mathcal{H}) \leq g_\mu(x) \leq g_\gamma(x) \leq f(x)$

• $\inf g_\gamma(\mathcal{H}) = \inf f(\mathcal{H})$

• $g_\mu(x) \downarrow \inf f(\mathcal{H})$ as $\mu \uparrow +\infty$

• g_γ: bounded on every ball in $\mathcal{H}$.

Proof:

take $x \in \mathcal{H}$, $\gamma \in \mathbb{R}_{++}$

(i) $g_\gamma = f \,\square\, \frac{1}{\gamma p} \|\cdot\|^p$

$\operatorname{dom} g_\gamma = \operatorname{dom} f + \underbrace{\operatorname{dom} \frac{1}{\gamma p} \|\cdot\|^p}_{\mathcal{H}}$

$= \underbrace{\operatorname{dom} f}_{\subseteq \mathcal{H}} + \mathcal{H} = \mathcal{H}$

/* *Proposition 12.6.

[[$f, g, h : \mathcal{H} \to]-\infty, +\infty]$]] $\Rightarrow$

(i) f, g: have continuous affine minorant with slope $u \in \mathcal{H}$

$\Rightarrow f \,\square\, g$: has continuous affine minorant with slope u, $-\infty \notin (f \,\square\, g)(\mathcal{H})$

(ii) $\operatorname{dom}(f \,\square\, g) = \operatorname{dom} f + \operatorname{dom} g$

(iii) $f \,\square\, g = g \,\square\, f$ /* symmetry */

(iv) (f, g, h: have continuous affine minorant with same slope)

$\Rightarrow f \,\square\, (g \,\square\, h) = (f \,\square\, g) \,\square\, h$ /* Associativity */

*/

(ii) /* required info

Fact 1.7.1. [$f, g : \mathcal{H} \to [-\infty, +\infty]$, $\forall_{x \in \mathcal{H}}$ $f(x) \leq g(x) \leftrightarrow f \leq g$] $\sup f(\mathcal{H}) \leq \sup g(\mathcal{H})$, $\inf f(\mathcal{H}) \leq \inf g(\mathcal{H})$

Fact 1.7.2. [$f, g : \mathcal{H} \to [-\infty, +\infty]$, $\forall_{x, y \in \mathcal{H}}$ $f(x) \leq g(y)$] $\sup f(\mathcal{H}) \leq \inf g(\mathcal{H})$

*/

take $\mu \in]\gamma, +\infty[$

$\leftrightarrow \gamma < \mu < +\infty$

$\Rightarrow \frac{1}{\gamma} > \frac{1}{\mu}$

$\Rightarrow \frac{1}{\gamma} \cdot \underbrace{\frac{1}{p} \|\cdot\|^p}_{\text{positive}} \geq \frac{1}{\mu} \frac{1}{p} \|\cdot\|^p$ /* equality when $\cdot = 0$ */

$$\Rightarrow \forall_{x,y\in\mathcal{H}} \quad f(y)+\frac{1}{\gamma}\frac{1}{p}\|x-y\|^p \geq f(y)+\frac{1}{\mu}\cdot\frac{1}{p}\|x-y\|^p$$

$$\Rightarrow \forall_{x\in\mathcal{H}} \quad \underbrace{\inf_{y\in\mathcal{H}} f(y)+\frac{1}{\gamma}\frac{1}{p}\|x-y\|^p}_{\left(f\,\square\,\frac{1}{\gamma}\cdot\frac{1}{p}\|\cdot\|^p\right)(x)=g_\gamma(x)} \geq \underbrace{\inf_{y\in\mathcal{H}} f(y)+\frac{1}{\mu p}\|x-y\|^p}_{\left(f\,\square\,\frac{1}{\mu p}\|\cdot\|^p\right)(x)=g_\mu(x)}$$

$$\Rightarrow \forall_{x\in\mathcal{H}} \quad g_\gamma(x)\geq g_\mu(x) \qquad (1)$$

now

$$\forall_{x\in\mathcal{H}}\forall_{y\in\mathcal{H}} \quad f(y)\leq f(y)+\underbrace{\frac{1}{\gamma p}\|x-y\|^p}_{\substack{\text{positive number}\\ \text{so adding it would}\\ \text{make the side larger}}}$$

$$\Rightarrow \forall_{x\in\mathcal{H}} \quad \underbrace{\inf_{y\in\mathcal{H}} f(y)}_{\inf f(\mathcal{H})} \leq \underbrace{\inf_{y\in\mathcal{H}}\left(f(y)+\frac{1}{\gamma p}\|x-y\|^p\right)}_{g_\gamma(x)} \overset{\text{set } y:=x}{\leq} f(x)+\frac{1}{\gamma p}\|x-x\|^p = f(x) \qquad (2)$$

now γ in (2) was arbitrary so, $\forall_{\gamma\in\mathbb{R}_{++}} \quad \inf f(\mathcal{H})\leq g_\gamma(x)\leq f(x)$

and $\forall_{\mu\in]\gamma,+\infty[} \quad \inf f(\mathcal{H})\leq g_\mu(x)\leq f(x)$

But $(1)\Rightarrow \quad g_\mu(x)\leq g_\gamma(x)$ $\quad\Rightarrow$

$$\forall_{x\in\mathcal{H}}\ \forall_{\gamma\in\mathbb{R}_{++}}\ \forall_{\mu\in]\gamma,+\infty[} \quad \inf f(\mathcal{H})\leq g_\mu(x)\leq g_\gamma(x)\leq f(x)$$

(iii) from (ii)

$$\forall_{\gamma\in\mathbb{R}_{++}}\ \forall_{x\in\mathcal{H}} \quad \inf f(\mathcal{H})\leq g_\gamma(x)\leq f(x)$$

$$\Rightarrow \forall_{\gamma\in\mathbb{R}_{++}}\ \forall_{x\in\mathcal{H}} \quad g_\gamma(x)\leq f(x), \text{ and } \inf f(\mathcal{H})\leq \inf_{x\in\mathcal{H}} g_\gamma(x) = \inf g_\gamma(\mathcal{H})$$

$$\Rightarrow \forall_{\gamma\in\mathbb{R}_{++}} \quad \underbrace{\inf_{x\in\mathcal{H}} g_\gamma(x)}_{\inf g_\gamma(\mathcal{H})}\leq \underbrace{\inf_{x\in\mathcal{H}} f(x)}_{\inf f(\mathcal{H})} \text{ and } \inf f(\mathcal{H})\leq \inf g_\gamma(\mathcal{H})$$

$$\Rightarrow \forall_{\gamma\in\mathbb{R}_{++}} \quad \inf f(\mathcal{H})\leq \inf g_\gamma(\mathcal{H})\leq \inf f(\mathcal{H})$$

$$\therefore \forall_{\gamma \in \mathbb{R}_{++}} \quad \inf g_\gamma(\mathcal{H}) = \inf f(\mathcal{H})$$

(iv)

$\forall_{y \in \mathcal{H}}$

$$\forall_{\mu \in \mathbb{R}_{++}} \quad g_\mu(x) = \inf_{y \in \mathcal{H}} \left(f(y) + \frac{1}{\mu p} \|x-y\|^p \right) \leq f(y) + \frac{1}{\mu p} \|x-y\|^p$$

recall the definition of functional limit superior:

Definition 1.23.2. (Functional limit superior)

[$\mathcal{X}$: Hausdorff space; $f: \mathcal{X} \to [-\infty,+\infty]$; $x \in \mathcal{X}$] $\quad \overline{\lim}_{y \to x} f(y) = \inf_{V \in \mathcal{V}(x)} \sup f(V)$

$$= \max_{\substack{(x_a)_{a \in A}:\text{ net in } \mathcal{X}, \\ x_a \to x}} \overline{\lim} f(x_a)$$

so,

$$\overline{\lim}_{\mu \uparrow +\infty} g_\mu(x) = \max_{\substack{(\tilde{x}_a)_{a \in A}:\text{net in } \mathbb{R}, \tilde{x}_a \to +\infty}} \overline{\lim} \underbrace{g_{\tilde{x}_a}(x)}_{\inf_{y \in \mathcal{H}} \left(f(y) + \frac{1}{\tilde{x}_a p}\|x-y\|^p\right)}$$

$$= \max_{\substack{(\tilde{x}_a)_{a \in A}:\text{net in } \mathbb{R}, \\ \tilde{x}_a \to +\infty}} \underbrace{\overline{\lim} \inf_{y \in \mathcal{H}} \left(f(y) + \overbrace{\frac{1}{\tilde{x}_a p}}^{\to 0} \|x-y\|^p \right)}_{\inf_{y \in \mathcal{H}} f(y) = \inf f(\mathcal{H})}$$

$$= \max_{\substack{(\tilde{x}_a)_{a \in A}:\text{net in } \mathbb{R}, \\ \tilde{x}_a \to +\infty}} \inf f(\mathcal{H}) = \inf f(\mathcal{H})$$

now recall the definition of limit inferior for functions:

Definition 1.23.1. (Functional limit inferior) **

[$\mathcal{X}$: Hausdorff space; $f: \mathcal{X} \to [-\infty,+\infty]$; $x \in \mathcal{X}$] $\quad \underline{\lim}_{y \to x} f(y) = \sup_{V \in \mathcal{V}(x)} \inf f(V)$

$$= \min_{\substack{(x_a)_{a \in A}:\text{ net in } \mathcal{X}, \\ x_a \to x}} \underline{\lim} f(x_a)$$

Lemma 1.23.

so,

$$\underline{\lim}_{\mu \uparrow +\infty} g_\mu(x) = \min_{\substack{(\tilde{x}_a)_{a \in A}:\text{net in } \mathbb{R}, \tilde{x}_a \to +\infty}} \lim \underbrace{g_{\tilde{x}_a}(x)}_{\inf_{y \in \mathcal{H}} \left(f(y) + \frac{1}{\tilde{x}_a p}\|x-y\|^p\right)}$$

$$= \min_{\substack{(\tilde{x}_a)_{a \in A}:\text{net in } \mathbb{R}, \\ \tilde{x}_a \to +\infty}} \underbrace{\underline{\lim} \inf_{y \in \mathcal{H}} \left(f(y) + \overbrace{\frac{1}{\tilde{x}_a p}}^{\to 0} \|x-y\|^p \right)}_{\parallel}$$

$= \min_{\substack{(\tilde{x}_a)_{a \in A}:\text{net in } \mathbb{R},\\ \tilde{x}_a \to +\infty}} \underbrace{\quad \; y \in \mathcal{H} \quad x_a p}_{\substack{\| \\ \inf_{y\in\mathcal{H}} f(y) = \inf f(\mathcal{H})}}$

$$= \min_{\substack{(\tilde{x}_a)_{a\in A}:\text{net in } \mathbb{R},\\ \tilde{x}_a \to +\infty}} \quad \inf f(\mathcal{H}) = \inf f(\mathcal{H})$$

$$\therefore \varliminf_{\mu \uparrow +\infty} g_\mu(x) = \varlimsup_{\mu\uparrow+\infty} g_\mu(x) = \lim_{\mu\uparrow+\infty} g_\mu(x) = \inf f(\mathcal{H})$$

(v) $z \in \operatorname{dom} f$. $\rho \in \mathbb{R}_{++}$

$$\forall_{y \in B(x;\rho)} \quad g_\gamma(y) = \inf_{z\in\mathcal{H}} f(z) + \frac{1}{\gamma p}\|y-z\|^p$$

$$\leq f(z) + \frac{1}{\gamma p}\|y-z\|^p$$

/* we use one step at a time trick:

- $p=1$

$\|y-z\|^1 = \|y-x+x-z\|^1 \leq \|y-x\|+\|x-z\|$

- $p>1$ /*recall Example 8.21. **
[$p \in]1,+\infty[$] $\Rightarrow$
  - $\|\cdot\|^p$: strictly convex
  - $\forall_{x\in\mathcal{H}} \forall_{y\in\mathcal{H}} \|x+y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p)$ */

$\|y-z\|^p = \|(y-x)+(x-z)\|^p \leq 2^{p-1}(\|y-x\|^p + \|x-z\|^p)$ */

$$\leq f(z) + \frac{1}{\gamma p} 2^{p-1}(\|y-x\|^p + \|x-z\|^p)$$

/* now $y \in B(x;\rho)$
$\Leftrightarrow \|y-x\| \leq \rho$ */

$$\leq \underbrace{f(z)}_{\substack{\text{finite}\\ \text{as } z\in\operatorname{dom} f}} + \underbrace{\frac{1}{\gamma p} 2^{p-1}}_{\text{finite}} (\underbrace{\rho^p}_{\text{finite}} + \underbrace{\|x-z\|^p}_{\substack{\text{fixed, so finite}\\ \text{on the ball } B(x;\rho)}}) < +\infty$$

$$\therefore \forall_{y\in B(x;\rho)} \quad g_\gamma(y) < +\infty$$

$\Leftrightarrow g_\gamma$: bounded above on every ball in $\mathcal{H}$. ☒