Part 1

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Proposition 11.1.

$[f:\mathcal{H}\to[-\infty,+\infty]\,;\ C:\text{nonempty},\subseteq\mathcal{H}]$

(i) f: lower semicontinuous $\Rightarrow \sup f(\overline{C})=\sup f(C)$

(ii) f: convex $\Rightarrow \sup f(\operatorname{conv} C)=\sup f(C)$

(iii) $u\in\mathcal{H}\Rightarrow \begin{cases}\sup\langle\overline{\operatorname{conv}}\,C\mid u\rangle=\sup\langle C\mid u\rangle\\ \inf\langle\overline{\operatorname{conv}}\,C\mid u\rangle=\inf\langle C\mid u\rangle\end{cases}$

(iv) $(f\in\Gamma_0(\mathcal{H});\ C$: convex; $\operatorname{dom} f\cap\operatorname{int} C\neq\varnothing)\Rightarrow \inf f(\overline{C})=\inf f(C)$

Proof:

(i) $C\subseteq\overline{C}$

$\Rightarrow \sup f(C)\leqslant\sup f(\overline{C})$ // optimum objective value improves over a larger set

set $x\in\overline{C}$ /* # Lemma 1.10. /* This is quite useful in membership testing of closure of a set */
[C: subset of a Hausdorff space $\mathcal{X}$,
$x\in\mathcal{X}$]
$x\in\overline{C}\Leftrightarrow\exists_{(x_a)_{a\in A}:\text{net in } C}\ x_a\to x$ */

now a closed set is sequentially closed $\Rightarrow\exists_{(x_n)_{n\in\mathbb{N}}:\text{sequence in } C}\ x_n\to x$

$\Rightarrow\exists\,(x_n)_{n\in\mathbb{N}}\subseteq C$, $x_n\to x$

now f: lower semicontinuous at x /* recall: * $\underset{\mathcal{H}\to[-\infty,+\infty]}{f}$: lower semicontinuous at $x\overset{\text{def}}{\Leftrightarrow}\forall_{(x_a)_{a\in A}:\text{net in }\mathcal{H},\,x_a\to x}\ \underline{\lim} f(x_a)\geqslant f(x)$ */

$\Rightarrow\ \underline{\lim}\, f(x_n)\geqslant f(x)$

/* now: $\underline{\lim}\,\xi_a=\sup_{a\in A}\inf_{b\in A:b\geqslant a}\xi_b$

$\therefore\ \underline{\lim}\,f(x_n)=\sup_{n\in\mathbb{N}}\left(\inf_{m\in\mathbb{N}:m\geqslant n}f(x_m)\right)\leqslant\sup_{n\in\mathbb{N}}f(x_n)=\sup\{f(\tilde{x})\mid\tilde{x}\in\underbrace{(x_n)_{n\in\mathbb{N}}}_{\in C},\ \tilde{x}\in C\}\leqslant\sup\{f(x)\mid x\in C\}=\sup f(C)$

/* as this one is a supremum over a larger set */

$\forall_{x\in\overline{C}}\ f(x)\leqslant\underline{\lim}\,f(x_n)\leqslant\sup f(C)$

$\Rightarrow\ \sup_{x\in\overline{C}}f(x)=\sup f(\overline{C})\leqslant\sup f(C)$ $\ddot\smile$

(ii)

$C\subseteq\operatorname{conv} C$

$\Rightarrow\ \sup f(C)\leqslant\sup f(\operatorname{conv} C)$

take $x\in\operatorname{conv} C$

$\Leftrightarrow\ \exists_{(\alpha_i)_{i\in I}:\text{finite family in }]0,1[,\ \sum_{i\in I}\alpha_i=1}\ \exists_{(x_i)_{i\in I}:\text{finite family in }C}\ x=\sum_{i\in I}\alpha_i x_i$ /* using Proposition 3.4. (Charecterization of convex hull)
[$C\subseteq\mathcal{H}$;
$D=\{\sum_{i\in I}\alpha_i x_i\mid I:\text{finite},\ \{x_i\}_{i\in I}\subseteq C,\ \{\alpha_i\}_{i\in I}\subseteq\,]0,1[,\ \sum_{i\in I}\alpha_i=1\}$: set of all convex combinations of points in C]
$\Rightarrow D=\operatorname{conv} C$ */

now

$f(x)=f\left(\sum_{i\in I}\alpha_i x_i\right)$ /* recall: # Corollary 8.10. [$f:\mathcal{H}\to\,]-\infty,+\infty]$]
(i) f: convex $\Leftrightarrow$
(ii) $\forall_{(\alpha_i)_{i\in I}:\subseteq]0,1[,\ \sum_{i\in I}\alpha_i=1}\ \forall_{(x_i)_{i\in I}\subseteq\operatorname{dom} f}\ f\left(\sum_{i\in I}\alpha_i x_i\right)\leqslant\sum_{i\in I}\alpha_i f(x_i)\Leftrightarrow$
(iii) $\forall_{x\in\mathcal{H}}\ \forall_{y\in\mathcal{H}}\ \forall_{\alpha\in]0,1[}\ f(\alpha x+(1-\alpha)y)\leqslant\alpha f(x)+(1-\alpha)f(y)$ */

$\leqslant\sum_{i\in I}\alpha_i\ \underbrace{f(x_i)}_{\leqslant\sup_{x\in C}f(x)=\sup f(C)}$

$\leqslant\sum_{i\in I}\alpha_i\ \sup f(C)$

$=\sup f(C)\underbrace{\sum_{i\in I}\alpha_i}_{1}=\sup f(C)$

$\therefore\ \forall_{x\in\operatorname{conv} C}\ f(x)\leqslant\sup f(C)$

$\Rightarrow\ \sup_{x\in\operatorname{conv} C}f(x)=\sup f(\operatorname{conv} C)\leqslant\sup f(C)$

$\sup f(C)=\sup f(\operatorname{conv} C)$ $\ddot\smile$

(iii)

first note that $\left.\begin{array}{l}\langle\cdot\mid u\rangle,\\ \langle\cdot\mid u\rangle\end{array}\right\}$ affine function $\Rightarrow$ convex, continuous in $\cdot\Rightarrow$ lower semicontinuous $\Big\}\Rightarrow$ so we can apply (i) and (ii)

$\sup_{x\in\overline{\operatorname{conv}}\,C}\langle x\mid u\rangle=\sup_{x\in\operatorname{conv} C}\langle x\mid u\rangle$ using

$=\sup_{x\in C}\langle x\mid u\rangle$ using

Proposition 11.1.
[$f:\mathcal{H}\to[-\infty,+\infty]$,
C: nonempty subset of $\mathcal{H}$] $\Rightarrow$
(i) f: lower semicontinuous $\Rightarrow\sup f(\overline{C})=\sup f(C)\Leftrightarrow\sup_{x\in\overline{C}}f(x)=\sup_{x\in C}f(x)$
(ii) f: convex $\Rightarrow\sup f(\operatorname{conv} C)=\sup f(C)\Leftrightarrow\sup_{x\in\operatorname{conv} C}f(x)=\sup_{x\in C}f(x)$

$\therefore\ \sup\langle\overline{\operatorname{conv}}\,C\mid u\rangle=\sup\langle C\mid u\rangle$

$= \sup_{x \in C} \langle x | u \rangle$ ← using

(i) f: lower semicontinuous $\Rightarrow$ $\sup f(\bar{C}) = \sup f(C) \Leftrightarrow \sup_{x \in \bar{C}} f(x) = \sup_{x \in C} f(x)$

(ii) f: convex $\Rightarrow$ $\sup f(\text{conv } C) = \sup f(C) \Leftrightarrow \sup_{x \in \text{conv} C} f(x) = \sup_{x \in C} f(x)$

$\therefore$ $\sup \langle \overline{\text{conv}}\, C | u \rangle = \sup \langle C | u \rangle$

as $\sup(\cdot) = -\inf(-\cdot)$, and $-\langle \cdot | u \rangle$ is convex, continuous we have: $\inf \langle \overline{\text{conv}}\, C | u \rangle = \inf \langle C | u \rangle$

(iv) $\bar{C} \supseteq C$

$\Rightarrow$ $\boxed{\inf f(\bar{C}) \leq \inf f(C)}$

take, $x_0 \in \bar{C}$, $x_1 \in \text{dom} f \cap \text{int}\, C$

set, $x_\alpha = (1-\alpha)x_0 + \alpha x_1$ $\forall_{\alpha \in]0,1[}$

/* Proposition 3.35.

C: convex subset of $\mathcal{H}$ $\Rightarrow$ $\forall_{x \in \text{int} C}$ $\forall_{y \in \bar{C}}$ $[x, y[\subseteq C$ /* $[x,y[= \{(1-\alpha)x + \alpha y \mid 0 \leq \alpha < 1\}$ */ */

using this we have: $x_\alpha = (1-\alpha)x_0 + \alpha x_1 \in C$

$\therefore \forall_{\alpha \in]0,1[}$ $x_\alpha = (1-\alpha)x_0 + \alpha x_1 \in C$

$\Rightarrow \forall_{\alpha \in]0,1[}$ $f(\underbrace{x_\alpha}_{\in C}) \geq \inf_{x \in C} f(x)$

$\Rightarrow \underline{\lim}_{\alpha \downarrow 0} f(x_\alpha) \geq \inf_{x \in C} f(x)$

/*

Fact 1.1.1. / these are extendable to nets as well */
(comes handy in dealing with sequences)

- $\forall_{n \in \mathbb{N}}$ $a_n \leq a \Rightarrow \overline{\lim}\, a_n \leq a$
- $\forall_{n \in \mathbb{N}}$ $b \leq b_n \Rightarrow b \leq \underline{\lim}\, b_n$
- $\forall_{n \in \mathbb{N}}$ $a_1 \leq a_n \leq a_2 \Rightarrow$ $a_1 \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq a_2$; $\underline{\lim}\, a_n$ exists in finite norm; $\overline{\lim}\, a_n$ exists in finite norm
- $\forall_{n \in \mathbb{N}}$ $a_n \leq b_n \Rightarrow (\underline{\lim}\, a_n \leq \underline{\lim}\, b_n,\ \overline{\lim}\, a_n \leq \overline{\lim}\, b_n)$
- $\forall_{n \in \mathbb{N}}$ $a_n \geq 0$, $b_n \geq 0 \Rightarrow \overline{\lim}\, a_n b_n \leq \overline{\lim}\, a_n \overline{\lim}\, b_n$

*/

/*

*Proposition 9.14.

[$f \in \Gamma_0(\mathcal{H})$, $\tilde{x} \in \mathcal{H}$, $\tilde{y} \in \text{dom} f$

$\forall_{\alpha \in]0,1[}$ $\tilde{x}_\alpha := (1-\alpha)\tilde{x} + \alpha \tilde{y}$]

$\lim_{\alpha \downarrow 0} f(\tilde{x}_\alpha) = f(\tilde{x})$

*/ using this: $\lim_{\alpha \downarrow 0} f(x_\alpha) = f(x_0)$

$\therefore$ $\underline{\lim}_{\alpha \downarrow 0} f(x_\alpha) = \lim_{\alpha \downarrow 0} f(x_\alpha) = f(x_0)$

$f(x_0) \geq \inf_{x \in C} f(x)$

So, $\forall_{x_0 \in \bar{C}}$ $f(x_0) \geq \inf_{x \in C} f(x)$ $\Leftrightarrow$ $\inf_{x_0 \in \bar{C}} f(x_0) \geq \inf_{x \in C} f(x)$

$\therefore$ $\boxed{\inf f(\bar{C}) \geq \inf f(C)}$

So, $\inf f(\bar{C}) = \inf f(C)$

■

* Proposition 11.4.

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex]

x: local minimizer of f $\Leftrightarrow$ x: minimizer of f.

Proof:

($\Leftarrow$): trivial

($\Rightarrow$): x: local minimizer of f

$\Leftrightarrow \exists_{\rho \in \mathbb{R}_{++}}$ $f(x) = \min f(\underbrace{B(x; \rho)}_{\text{closed ball of center } x \text{ and radius } \rho})$

pick $y \in \text{dom} f \setminus B(x; \rho) \Rightarrow f(y) < +\infty$, $\|x - y\| > \rho \rightarrow 1 > \frac{\rho}{\|x-y\|} > 0 \Rightarrow -1 < -\frac{\rho}{\|x-y\|} < 0$

$\Rightarrow$ $0 < \underbrace{1 - \frac{\rho}{\|x-y\|}}_{\alpha} < 1$

$\therefore \alpha \in]0,1[$

construct $z = \alpha x + (1-\alpha) y$

note that $\|z - x\| = \|\alpha x + (1-\alpha)y - x\|$

$= \|(\alpha - 1)x + (1-\alpha)y\|$

$= \|-(1-\alpha)x + (1-\alpha)y\|$

$= \|(1-\alpha)x - (1-\alpha)y\|$

$= (1-\alpha)\|x - y\|$

$= \left(1 - \left(1 - \frac{\rho}{\|x-y\|}\right)\right)\|x-y\|$

$= \frac{\rho}{\|x-y\|}\|x-y\| = \rho$

$= \left(1-\left(1-\frac{\rho}{\|x-y\|}\right)\right)\|x-y\|$

$= \frac{\rho}{\|x-y\|}\|x-y\| = \rho$

$\therefore z \in B(x;\rho)$

now, $f(x) = \min f(B(x;\rho))$

$\Rightarrow \forall_{\tilde{x}\in B(x;\rho)} f(x) \leq f(\tilde{x})$ — recall $y \in \operatorname{dom} f \setminus B(x;\rho)$

as $z \in B(x;\rho)$ $f(x) \leq f(z) = f(\alpha x + (1-\alpha)y)$

$\leq \alpha f(x) + (1-\alpha) f(y)$

$\Rightarrow (1-\alpha) f(x) \leq (1-\alpha) f(y)$

$\Rightarrow f(x) \leq f(y)$

$\therefore \forall_{y \in \operatorname{dom} f \setminus B(x;\rho)} f(x) \leq f(y)$ \
by given, $\forall_{\tilde{x}\in B(x;\rho)} f(x) \leq f(\tilde{x})$ $\Big\}$ $\forall_{y\in \operatorname{dom} f} f(x) \leq f(y) \Leftrightarrow x$: minimizer of f.

/* Note, we donot have to worry about $y \in \mathcal{H}\setminus \operatorname{dom} f$ as $f(y) = +\infty$ then */

∎

Proposition. 11.5.

[$f: \mathcal{H} \to]-\infty,+\infty]$, proper, convex

C: subset of $\mathcal{H}$

x: minimizer of f over C, $x \in \operatorname{int} C$]

x: minimizer of f.

Proof:

$x \in \operatorname{int} C \Rightarrow C \neq \emptyset$

$\Rightarrow \exists_{\rho \in \mathbb{R}_{++}} B(x;\rho) \subseteq C$

as x: minimizer of f over C

$\Rightarrow x$: minimizer of f over $B(x;\rho)$

$\therefore f(x) = \inf f(B(x;\rho))$

so x: local minimizer of f

$\Rightarrow x$: global minimizer of f.

∎

/* f: quasiconvex $\Leftrightarrow (\operatorname{lev}_{\leq \xi} f)_{\xi\in\mathbb{R}}$: convex set */

* Proposition 11.6.

[$f: \mathcal{H} \to]-\infty,+\infty]$, proper, quasiconvex]

Argmin f: convex

Proof:

$\operatorname{Argmin} f = \{x \in \mathcal{H} \mid \forall_{y\in\mathcal{H}} f(x) \leq f(y)\}$

$= \{x\in\mathcal{H} \mid \forall_{y\in \operatorname{dom} f} f(x) \leq f(y), \overbrace{\forall_{y\in\mathcal{H}\setminus \operatorname{dom} f} f(x) \leq f(y) = +\infty}^{\text{redundant}}\}$

$= \{x\in\mathcal{H} \mid \forall_{y \in \operatorname{dom} f} f(x) \leq f(y) \in \mathbb{R}\}$

$\Updownarrow$

$f(x) \leq \inf_{y\in \operatorname{dom} f} f(y) = \alpha \in \mathbb{R}$ /* $\forall_{y\in \operatorname{dom} f}$ $f(y) < +\infty$ ($f: \mathcal{H} \to]-\infty,+\infty]$) $\Rightarrow$ $f(\operatorname{dom} f)$: bounded $\Rightarrow$ $\inf f(\operatorname{dom} f)$ = finite */

$= \{x\in\mathcal{H} \mid f(x) \leq \alpha, \alpha \in \mathbb{R}\} = \operatorname{lev}_{\underbrace{\alpha}_{\in\mathbb{R}}} f$: convex by definition.

∎

* Proposition 11.7. (Existence of atleast one minimizer in an optimization problem)

[$f: \mathcal{H} \to]-\infty,+\infty]$, quasiconvex

C: convex subset of $\mathcal{H}$, $C \cap \operatorname{dom} f \neq \emptyset$

One of the following holds:

(i) $f+\iota_C$: strictly quasiconvex

(ii) f : convex, $C\cap \operatorname{Argmin} f=\emptyset$, C: strictly convex
$\Updownarrow$ def
$\forall_{x,y\in C}\ x\neq y \Rightarrow \frac{x+y}{2}\in \operatorname{int} C$

]

$\Rightarrow f$: atmost one minimizer over C.

Proof:
Assume C: not a singleton, otherwise proof is trivial

$\mu=\inf f(C)$ /* as $C\cap \operatorname{dom} f\neq\emptyset \Rightarrow \exists_{z\in C\cap \operatorname{dom} f}\ f(z)<+\infty$, so $\inf f(C)$: finite */

Assume $\exists_{x,y\in C\cap\operatorname{dom} f:\, x\neq y}\ f(x)=f(y)=\mu$ /* Per absurdum */

note that, $x\in C\cap \operatorname{lev}_{\leqslant\mu} f$
$y\in C\cap \operatorname{lev}_{\leqslant\mu} f$

/* $x\in C\cap\operatorname{dom} f \Leftrightarrow x\in C,\ f(x)<+\infty$
$f(x)=\mu \Rightarrow x: f(x)\leqslant\mu<+\infty \Leftrightarrow x\in \operatorname{lev}_{\leqslant\mu} f$
$\therefore x\in C\cap \operatorname{lev}_{\leqslant\mu} f$
Similarly, $y\in C\cap\operatorname{lev}_{\leqslant\mu} f$ */

as f: convex $\Rightarrow \operatorname{lev}_{\leqslant\mu} f$: convex
C: convex

$\therefore C\cap \operatorname{lev}_{\leqslant\mu} f$: convex /* intersection of convex sets are always convex */

$\underbrace{\tfrac{1}{2}x+\tfrac{1}{2}y}_{=z\ (\text{say})}\in C\cap\operatorname{lev}_{\leqslant\mu} f \Rightarrow f(z)\leqslant\mu$ but $\mu=\inf f(C)\leqslant f(z)$

So, $f(z)=\mu$ (i.e., z: another different minimizer of $f|_C$)
$=\inf f(C)$

(i)
recall, $\tilde f$: strictly quasiconvex $\overset{\text{def}}{\Leftrightarrow} \forall_{\tilde x\in\operatorname{dom}}\ \forall_{\tilde y\in \operatorname{dom} f}\ \forall_{\alpha\in]0,1[}\ \tilde x\neq\tilde y \Rightarrow \tilde f(\alpha\tilde x+(1-\alpha)\tilde y)<\max\{\tilde f(\tilde x),\tilde f(\tilde y)\}$

$f(x)=f(y)=\underset{\frac12 x+\frac12 y}{\overset{\|}{f(z)}}=\inf f(C)$: finite $\Rightarrow x,y,z\in\operatorname{dom}(f+\iota_C)$

also, $x\neq y$

so, $\underbrace{(f+\iota_C)(\overset{z}{\overset{\|}{\tfrac12 x+\tfrac12 y}})}_{\underbrace{f(z)}_{\mu}+\underbrace{\iota_C(z)}_{0}=f(z)} < \max\{\underbrace{(f+\iota_C)(x)}_{\underset{\mu}{f(x)}+\underbrace{\iota_C(x)}_{0}},\ \underbrace{(f+\iota_C)(y)}_{\underset{\mu}{f(y)}+\underbrace{\iota_C(y)}_{0}}\}$

$\Rightarrow \mu<\max\{\mu,\mu\}=\mu$: contradiction

$\therefore$ There cannot be more than one minimizer.

(ii)
$z=\frac12 x+\frac12 y\in C\cap \operatorname{lev}_{\leqslant\mu} f$, $f(z)=\inf f(C)$

as $x,y\in C$, $x\neq y \Rightarrow \frac{x+y}{2}=z\in\operatorname{int} C$ /* C: strictly convex set
so, applying the definition */

/* Recall

✱ Proposition 11.5. /* The statement is not trivial at all */
[$f:\mathcal{H}\to]-\infty,+\infty]$, proper, convex ✓
C: subset of $\mathcal{H}$ ✓
z ~~x~~: minimizer of f over C, ~~x~~ $z\in\operatorname{int} C$ ✓]
$\Rightarrow$ z ~~x~~: minimizer of f.
*/

z: minimizer of $f \Leftrightarrow z\in\operatorname{Argmin} f$

$z\in C\cap\operatorname{Argmin} f$

$\therefore C\cap\operatorname{Argmin} f\neq\emptyset$: contradiction (According to given,
$C\cap\operatorname{Argmin} f=\emptyset$)

$\therefore f$ has atmost one minimizer over C.

■

*Theorem 11.9. (Existence of minimizers)

$[f : \mathcal{H} \to]-\infty, +\infty]$, lower semicontinuous, quasiconvex

C: closed convex subset of $\mathcal{H}$

$\exists_{\xi \in \mathbb{R}}$ $C \cap \operatorname{lev}_{\leq \xi} f$: nonempty, bounded $] \Rightarrow$

f: has a minimizer over C.

Proof: Recall

/*

* Proposition 10.23.

$[f : \mathcal{H} \to]-\infty, +\infty]$, quasiconvex $]$

(i) f: weakly sequentially lower semicontinuous $\Leftrightarrow$

(ii) f: sequentially lower semicontinuous $\Leftrightarrow$

(iii) f: lower semicontinuous $\Leftrightarrow$

(iv) f: weakly lower semicontinuous */

f: lower semicontinuous, quasiconvex $\Rightarrow$ f: weakly lower semicontinuous.

C: closed convex

$\operatorname{lev}_{\leq \xi} f$: closed convex /* A quasiconvex function has convex lower level set by definition

A lower semicontinuous f has closed lower level set as

Lemma 1.24. ★

$[\mathcal{X}$: Hausdorff space,

$f : \mathcal{X} \to [-\infty, \infty]]$

f: lower semicontinuous $\Leftrightarrow$ $\operatorname{epi} f$: closed in $\mathcal{X} \times \mathbb{R}$ $\Leftrightarrow$ $\forall_{\xi \in \mathbb{R}}$ $\operatorname{lev}_{\leq \xi} f$: closed in $\mathcal{X}$ */

$\therefore C \cap \operatorname{lev}_{\leq \xi} f$: closed, convex

/* closedness and convexness are preserved under intersection */

So, $C \cap \operatorname{lev}_{\leq \xi} f$: closed, convex, $\underbrace{\text{bounded}}_{\text{given}}$ $\Rightarrow$ $C \cap \operatorname{lev}_{\leq \xi} f$: weakly compact, weakly sequentially compact. /* using :

Theorem 3.33.

$[C$: bounded, closed, convex subset of $\mathcal{H}]$

C: weakly compact, weakly sequentially compact /* C: weakly compact $\overset{\text{def}}{\leftrightarrow}$ Every net in has a weak cluster point in C

C: weakly sequentially compact $\overset{\text{def}}{\leftrightarrow}$ Every sequence in C has a weak sequential cluster point in C

x : (strong) cluster point of $(x_a)_{a \in A}$ $\overset{\text{def}}{\leftrightarrow}$ $(x_a)_{a \in A}$ has a subnet that (strongly) converges to $x \in \mathcal{X}$

x: weak cluster point of $(x_a)_{a \in A}$ $\overset{\text{def}}{\leftrightarrow}$ $(x_a)_{a \in A}$ has a subnet that weakly converges to $x \in \mathcal{X}$ */

*/

$\Leftrightarrow$ $C \cap \operatorname{lev}_{\leq \xi} f$: bounded, weakly closed

/* Lemma 2.29: C: weakly compact $\Leftrightarrow$ C: bounded, weakly closed */

Note that

$$\begin{pmatrix} \min_{x \in \mathcal{H}} & f(x) \\ \text{s.t.} & x \in C \end{pmatrix} = \left(\min_{x \in \mathcal{H}} . \ f(x) + \iota_C(x) \right)$$

$$= \left(\min_{x \in \mathcal{H}} . \ f(x) + \iota_C(x) + \iota_{\operatorname{lev}_{\leq \xi} f}(x) \right)$$

/* Given, $C \cap \operatorname{lev}_{\leq \xi} f$: nonempty, bounded

$\Leftrightarrow$ $\exists_{x \in C}$ $f(x) \leq \overbrace{\xi}^{\in \mathbb{R}}$

So we can search for the minimizer in $\operatorname{lev}_{\leq \xi} f$ where $\iota_C(x)$ will evaluate to zero */

$$= \begin{pmatrix} \min_{x \in \mathcal{H}} . & f(x) \\ \text{s.t.} & x \in C \cap \operatorname{lev}_{\leq \xi} f \end{pmatrix}$$

So, minimizing f over C is equivalent to minimizing f over $C \cap \operatorname{lev}_{\leq \xi} f$.

/* Recall,

- Lemma. 2.23. $\mathcal{H}^{\text{weak}}$: Hausdorff space

- * Theorem 1.28. (Weierstrass) /* fundamental tool in proving the existence of solutions of optimization problem */

  $[\mathcal{X}$: Hausdorff space,

  $f : \mathcal{X} \to [-\infty, +\infty]$, lower semicontinuous

  C: compact subset of $\mathcal{X}$

  $C \cap \operatorname{dom} f \neq \emptyset \ ] \Rightarrow$

  f: achieves its infimum over C.

*/

in our case we have:

$\mathcal{H}^{\text{weak}}$: Hausdorff space

$f : \mathcal{H} \to [-\infty, +\infty]$, lower semicontinuous

$C \cap \operatorname{lev}_{\leq \xi} f$: weakly compact

$(C \cap \operatorname{lev}_{\leq \xi} f) \cap \operatorname{dom} f \neq \emptyset$ /* as $\operatorname{lev}_{\leq \xi} f \subseteq \operatorname{dom} f$, $\Rightarrow$ $\operatorname{dom} f \cap \operatorname{lev}_{\leq \xi} f = \operatorname{lev}_{\leq \xi} f \neq \emptyset$

given $C \cap \operatorname{lev}_{\leq \xi} f \neq \emptyset$ */

$\therefore f$ achieves a minimum over $C \cap \operatorname{lev}_{\leq \xi} f$

$\Leftrightarrow f$ " " " " C.

■

Proposition 11.11. (Coercivity of a function in terms of lower level set)

$[f : \mathcal{H} \to [-\infty, +\infty]]$

f: coercive $\Leftrightarrow$ $(\operatorname{lev}_{\leq \xi} f)_{\xi \in \mathbb{R}}$: bounded

Proof:

($\Rightarrow$) Proof by contrapositive

$$\neg\left((\mathrm{lev}_{\leq\xi} f)_{\xi\in\mathbb{R}}\right)\text{: bounded} \Rightarrow f\text{: not coercive} \leftrightarrow \lim_{\|x\|\to+\infty} f(x) = +\infty$$

Per absurdum,

$$\exists_{\xi\in\mathbb{R}} \quad \mathrm{lev}_{\leq\xi} f\text{: unbounded}$$

$$\Rightarrow \exists_{(x_n)_{n\in\mathbb{N}} \subseteq \mathrm{lev}_{\leq\xi} f} \quad \|x_n\| \to +\infty$$

$$f(x_n) \leq \xi \in \mathbb{R}$$

$$\leftrightarrow \quad \|x_n\| \to +\infty, \left(f(x_n) \leq \xi \Rightarrow f(x_n) \not\to +\infty\right)$$

/* using

$\bullet\ \forall_{n\in\mathbb{N}}\ a_1 \leq a_n \leq a_2 \Rightarrow a_1 \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq a_2$

$\underline{\lim}\, a_n$ exists in finite norm

$\overline{\lim}\, a_n$ exists in finite norm */

$$\Rightarrow \underline{\lim}_{\|x_n\|\to+\infty} f(x_n) \leq \overline{\lim}_{\|x_n\|\to+\infty} f(x_n) \leq \xi < +\infty$$

as, $\underline{\lim} \leq \lim \leq \overline{\lim}$

$$\lim_{\|x\|\to+\infty} f(x) \neq +\infty$$

$$f\text{: not coercive.}$$

($\Leftarrow$) /* $(\mathrm{lev}_{\leq\xi} f)_{\xi\in\mathbb{R}}$: bounded $\Rightarrow$ f: coercive */

$$\forall_{\xi\in\mathbb{R}} \quad \mathrm{lev}_{\leq\xi} f = \{x\in\mathcal{H} \mid f(x)\leq\xi\}\text{: bounded} \Rightarrow \exists_{t\in\mathbb{R}_+} \quad \forall_{x\in\mathrm{lev}_{\leq\xi} f \leftrightarrow f(x)\leq\xi} \quad \|x\|\leq t$$

$$\therefore \forall_{\xi\in\mathbb{R}} \exists_{t\in\mathbb{R}_+} \forall_x \quad (f(x)\leq\xi \Rightarrow \|x\|\leq t)$$

contrapositive

$$(\|x\| > t \Rightarrow f(x) > \xi)$$

take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$: $\|x_n\|\to+\infty \overset{\text{def}}{\leftrightarrow} \forall_{a\in\mathbb{R}} \exists_{N\in\mathbb{N}} \forall_{n\in\mathbb{N}: n\geq N} \quad \|x_n\| > a$

$$\overset{a:=t}{\Rightarrow} \exists_{N\in\mathbb{N}} \forall_{n\in\mathbb{N}: n\geq N} \quad \|x_n\| > t \Rightarrow f(x_n) > \xi$$

$$\therefore \exists_{N\in\mathbb{N}} \forall_{n\in\mathbb{N}: n\geq N} \quad f(x_n) > \xi$$

$$\Rightarrow \inf_{n\geq N} f(x_n) \geq \xi$$

but ξ was arbitrary positive number

$$\therefore \forall_{\xi\in\mathbb{R}_+} \forall_{(x_n)_{n\in\mathbb{N}}: \|x_n\|\to+\infty} \quad \exists_N \quad \inf_{n\geq N} f(x_n) \geq \xi$$

$$\Downarrow$$

$$f(x_n)\to+\infty$$

$$\therefore \lim_{\|x\|\to+\infty} f(x) = +\infty \overset{\text{def}}{\leftrightarrow} f\text{: coercive.}$$

■

Proposition 11.12.

[$\mathcal{H}$: finite-dimensional

$f \in \Gamma_0(\mathcal{H})$]

f: coercive $\leftrightarrow \exists_{\xi\in\mathbb{R}} \quad \mathrm{lev}_{\leq\xi} f$: nonempty, bounded

Proof:

($\Rightarrow$)

* Proposition 11.11. (Coercivity of a function in terms of lower level set)

[$f: \mathcal{H}\to[-\infty,+\infty]$] f: coercive $\leftrightarrow (\mathrm{lev}_{\leq\xi} f)_{\xi\in\mathbb{R}}$: bounded

using this:

$\exists_{\xi\in\mathbb{R}} \quad \mathrm{lev}_{\leq\xi} f$: nonempty, bounded

as long as $\mathrm{dom} f$: nonempty we can find a nonempty sublevel set.

($\Leftarrow$)

$\exists_{\xi\in\mathbb{R}} \mathrm{lev}_{\leq\xi} f$: nonempty bounded $\leftrightarrow$ $\mathrm{lev}_{\leq\xi} f = \{x\in\mathcal{H} \mid f(x)\leq\xi\}$: nonempty bounded $\leftrightarrow$ $\exists_{x'\in\mathcal{H}} f(x')\leq\xi$

$\exists_{\eta>0} \forall_{x\in\mathcal{H}: f(x)\leq\xi} \|x\|\leq\eta$

$\Rightarrow \forall_{\xi'\leq\xi} \quad \mathrm{lev}_{\leq\xi'} f$: bounded /* note that we are focusing on boundedness only as we want to use Proposition 11.11 */

$\exists_{\xi \in \mathbb{R}} \mathrm{lev}_{\leq \xi} f$: nonempty bounded $\Leftrightarrow$ $\mathrm{lev}_{\leq \xi} f = \{x \in \mathcal{H} \mid f(x) \leq \xi\}$: nonempty bounded $\Leftrightarrow$ $\exists_{x' \in \mathcal{H}} f(x') \leq \xi$; $\exists_{\eta > 0} \forall_{x \in \mathcal{H}: f(x) \leq \xi} \|x\| \leq \eta$

$\Rightarrow \forall_{\xi' \leq \xi} \mathrm{lev}_{\leq \xi'} f$: bounded /* note that we are focusing on boundedness only as we want to use Proposition 11.11 */

/* $\forall_{y \in \mathrm{lev}_{\leq \xi'} f}$

$f(y) \leq \xi' \Rightarrow f(y) \leq \xi' \leq \xi \Rightarrow y \in \mathrm{lev}_{\leq \xi} f \Rightarrow \|y\| \leq \eta$ $\therefore$ $\mathrm{lev}_{\leq \xi'} f$: bounded */

now take $\eta \in]\xi, +\infty[$ we want to show that $\mathrm{lev}_{\leq \eta} f$: bounded

Per absurdum let us assume: $\mathrm{lev}_{\leq \eta} f$: unbounded /* now to be unbounded it has to be nonempty;

also $f \in \Gamma_0(\mathcal{H}) \Rightarrow \mathrm{lev}_{\leq \eta} f$: convex */

/* *Corollary 6.51. [$\mathcal{H}$: finite dimensional, C: nonempty convex subset of $\mathcal{H}$] $\Rightarrow$ C: bounded $\Leftrightarrow$ $\mathrm{rec}\, C = \{0\}$ */

/* $\mathrm{rec}\, C = \{x \in \mathcal{H} \mid x + C \subseteq C\}$ */

$\mathrm{lev}_{\leq \eta} f$: convex, nonempty, bounded

$\Rightarrow$ $\mathrm{rec}\ \mathrm{lev}_{\leq \eta} f \neq \{0\}$ $\Rightarrow$ $\mathrm{rec}\ \mathrm{lev}_{\leq \eta} f$: unbounded as this is a cone // $\mathbb{R}_{++}$cone = cone

/* $\{\tilde{x} \in \mathcal{H} \mid \tilde{x} + \mathrm{lev}_{\leq \eta} f \subseteq \mathrm{lev}_{\leq \eta} f\} \neq \{0\}$

$\therefore$ $\forall_{\tilde{y}}: \tilde{y} \in \mathrm{rec}\ \mathrm{lev}_{\leq \eta} f$ $\therefore$ so, $\tilde{y} + x' \in \mathrm{lev}_{\leq \eta} f$

$\Updownarrow$

$\tilde{y} + \underbrace{\mathrm{lev}_{\leq \eta} f}_{\ni x'} \subseteq \mathrm{lev}_{\leq \eta} f$

/* $x' \in \mathrm{lev}_{\leq \xi} f \Leftrightarrow f(x') \leq \xi \leq \eta \Rightarrow x' \in \mathrm{lev}_{\leq \eta} f$ */

now $\mathrm{rec}\ \mathrm{lev}_{\leq \eta} f$ is a cone, so $\forall_{\lambda \in \mathbb{R}_{++}} \lambda \tilde{y} \in \mathrm{rec}\ \mathrm{lev}_{\leq \eta} f$

$\lambda \tilde{y} + x' \in \mathrm{lev}_{\leq \eta} f$ $\forall_{\lambda \in \mathbb{R}_{++}}$

now take $\lambda \in]1, +\infty[$

$\forall_{\lambda \in]1,+\infty[}$ $\Downarrow$ $\frac{1}{\lambda} \in]0,1[$

$x' + \tilde{y} = x' - \frac{1}{\lambda}x' + \frac{1}{\lambda}x' + \frac{1}{\lambda}\lambda\tilde{y} = (1 - \frac{1}{\lambda})\underbrace{x'}_{\in \mathrm{lev}_{\leq \eta} f} + \frac{1}{\lambda}\underbrace{(x' + \lambda\tilde{y})}_{\in \mathrm{lev}_{\leq \eta} f}$ $\therefore$ convex combination of x', $x' + \lambda\tilde{y}$

also. $\mathrm{lev}_{\leq \eta} f \subseteq \mathrm{dom}\, f$ and $f \in \Gamma_0(\mathcal{H})$

so,

$f(x' + \tilde{y}) = f\left((1 - \frac{1}{\lambda})\underbrace{x'}_{\in \mathrm{dom} f} + \frac{1}{\lambda}\underbrace{(x' + \lambda\tilde{y})}_{\in \mathrm{dom} f}\right) \leq (1 - \frac{1}{\lambda}) f(x') + \frac{1}{\lambda}\underbrace{f(x' + \lambda\tilde{y})}_{\leq \eta}$

$\leq (1 - \frac{1}{\lambda}) f(x') + \frac{1}{\lambda}\eta$

$\Rightarrow$ $f(x' + \tilde{y}) - f(x') \leq \frac{1}{\lambda}(\eta - f(x'))$

$\Leftrightarrow$ $\forall_{\lambda \in]1,+\infty[}$ $\lambda(f(x' + \tilde{y}) - f(x')) \leq \underbrace{\eta - f(x')}_{\geq 0 \text{ and finite}}$ /* $x' \in \mathrm{lev}_{\leq \eta} f \Rightarrow f(x) \leq \eta \Rightarrow \eta - f(x) \geq 0$ and $\eta - f(x)$: finite */ ($f \in \Gamma_0(\mathcal{H})$ are from $\mathcal{H}$ to $]-\infty, +\infty]$)

$\Rightarrow$ $\forall_{\lambda \in]1,+\infty[}$ $f(x' + \tilde{y}) - f(x') \leq \frac{1}{\lambda}(\eta - f(x')) \Leftrightarrow f(x' + \lambda\tilde{y}) - f(x') \leq \inf_{\lambda \in]1,+\infty[} \frac{1}{\lambda}(\eta - f(x'))$

$= (\eta - f(x')) \underbrace{\inf_{\lambda \in]1,+\infty[} \frac{1}{\lambda}}_{=0} = 0$

$\therefore$ $f(x' + \tilde{y}) - f(x') \leq 0$

$\Rightarrow$ $f(x' + \tilde{y}) \leq \underbrace{f(x') \leq \xi}$

/* $\because x' \in \mathrm{lev}_{\leq \xi} f$ */

$\therefore$ $\forall_{\tilde{y} \in \mathrm{rec}\ \mathrm{lev}_{\leq \eta} f}$ $f(x' + \tilde{y}) \leq \xi$

$\Updownarrow$

$x' + \tilde{y} \in \mathrm{lev}_{\leq \xi} f$

$\Rightarrow$ $\underbrace{x' + \underbrace{\mathrm{rec}\ \mathrm{lev}_{\leq \eta} f}_{\text{unbounded}}}_{\text{unbounded}} \subseteq \underbrace{\mathrm{lev}_{\leq \xi} f}_{\text{bounded}}$

so, we have contained an unbounded set in a bounded set.

$\Downarrow$

contradiction $\Rightarrow$ $\mathrm{lev}_{\leq \eta} f$: can not be unbounded

$\therefore$ $\mathrm{lev}_{\leq \eta} f$: bounded

so, we have proven that:

$\forall$ $\mathrm{lev}_{\leq \eta} f$: bounded /* *Proposition 11.11. (Coercivity of a function in terms of lower level set)

So, we have proven that:

$\forall_{\eta \in \mathbb{R}}$ $\operatorname{lev}_{\leq \eta} f$: bounded /* *Proposition 11.11. (Coercivity of a function in terms of lower level set) [$f: \mathcal{H} \to [-\infty, +\infty]$] f : coercive $\Leftrightarrow$ $(\operatorname{lev}_{\leq \xi} f)_{\xi \in \mathbb{R}}$: bounded */

$\Leftrightarrow$ f : coercive. ■

*Proposition 11.17. (Asympotic center)

⟦ $(z_n)_{n \in \mathbb{N}}$: bounded sequence in $\mathcal{H}$

C : nonempty closed convex subset of $\mathcal{H}$

$T: C \to C$, nonexpansive

$f: \mathcal{H} \to \mathbb{R}: x \mapsto \overline{\lim} \|x - z_n\|^2$ ⟧

(i) f : strongly convex with constant 2.

(ii) f : supercoercive

(iii) $f + \iota_C$: strongly convex, supercoercive,
its unique minimizer z_C is called the asympotic center of $(z_n)_{n \in \mathbb{N}}$ relative to C.

(iv) $(z \in \mathcal{H}, z_n \to z) \Rightarrow \forall_{x \in \mathcal{H}}$ $f(x) = \|x - z\|^2 + f(z)$, $z_C = P_C z$

(v) $(z_n)_{n \in \mathbb{N}}$: Fejer monotone w.r.t. $C \Rightarrow P_C z_n = z_C$

(vi) $\forall_{n \in \mathbb{N}}$ $z_{n+1} = T z_n \Rightarrow z_C \in \operatorname{Fix} T$

(vii) $z_n - T z_n \to 0 \Rightarrow z_C \in \operatorname{Fix} T$.

*Corollary 11.18.

⟦ C : nonempty closed convex subset of $\mathcal{H}$

$T: C \to C$, nonexpansive

$z_0 \in C$, $\forall_{n \in \mathbb{N}}$ $z_{n+1} = T z_n$ /* Classical Banach-Picard iteration */ ⟧

(i) $\operatorname{Fix} T \neq \emptyset$ $\Leftrightarrow$

(ii) $\forall_{z_0 \in C}$ $(z_n)_{n \in \mathbb{N}}$: bounded $\Leftrightarrow$

(iii) $\exists_{z_0 \in C}$ $(z_n)_{n \in \mathbb{N}}$: bounded.

Proof:

(i) $\Rightarrow$ (ii)

T : nonexpansive $\Rightarrow$ T : quasinonexpansive.

Recall Example 5.3, Proposition 5.4.(i)

/*

*Example 5.3. ★ (Used later)

[$\cancel{D}$ C : nonempty subset of $\mathcal{H}$ ✓

$T: \cancel{D} \to \cancel{D}$ $C \to C$, quasinonexpansive ✓ /* T : quasinonexpansive $\Leftrightarrow \forall_{x \in D} \forall_{y \in \operatorname{Fix} T}$ $\|Tx - \underbrace{Ty}_{y}\| \leq \|x - y\|$ */

$\operatorname{Fix} T \neq \emptyset$ ✓ [Given] ✓

$\cancel{x_0} \in \cancel{D}$ $z_0 \in C$, $\forall_{n \in \mathbb{N}}$ $x_{n+1} = T x_n$ /* Banach-Picard iteration */

] $(z_n)_{n \in \mathbb{N}}$

$\cancel{(x_n)_{n \in \mathbb{N}}}$: Fejer monotone w.r.t. $\operatorname{Fix} T$ ✓

*Proposition 5.4. (Some key properties of Fejer monotone sequences)

[$\cancel{C}$ $\operatorname{Fix} T$: nonempty subset of $\mathcal{H}$ ✓

$\cancel{(x_n)_{n \in \mathbb{N}}}$ $(z_n)_{n \in \mathbb{N}}$: Fejer monotone sequence w.r.t $\cancel{C}$ $\operatorname{Fix} T$ ✓ $\subseteq \mathcal{H}$] $\Rightarrow$

(i) $\cancel{(x_n)_{n \in \mathbb{N}}}$ $(z_n)_{n \in \mathbb{N}}$: bounded

(ii) $\forall_{x \in C} (\|x_n - x\|)_{n \in \mathbb{N}}$: converges /* note that it does not necessarily mean convergence in C */

(iii) $(d_C(x_n))_{n \in \mathbb{N}}$: decreasing and converges

*/

So, $\forall_{z_0 \in C}$ $(z_n)_{n \in \mathbb{N}}$: bounded ☺

(ii) $\Rightarrow$ (iii) : obvious. ☺

(iii) $\Rightarrow$ (i) :

(ii) ⇒ (iii) : obvious. ☺

(iii) ⇒ (i) :

given $\exists_{z_0 \in C}$ $(z_n)_{n\in\mathbb{N}}$: bounded

Recall Proposition 11.17·(vi) says:

[$(z_n)_{n\in\mathbb{N}}$: bounded sequence, ✓

C : nonempty closed convex subset of $\mathcal{H}$ ✓

$T: C \to C$, nonexpansive ✓

$\forall_{n\in\mathbb{N}}$ $z_n = Tz_n$ ✓]

⇒

$z_c : \underbrace{(P_C z_n \to z_c)}$, $z_c \in \operatorname{Fix} T$

/* convergent point of the shadow sequence */

⇒ $\operatorname{Fix} T \neq \emptyset$

■

* Proposition 11.19.

[$f: \mathcal{H} \to]-\infty, +\infty]$, coercive proper function]

⇒

$\forall_{(x_n)_{n\in\mathbb{N}}}$: minimizing sequence of f $\quad (x_n)_{n\in\mathbb{N}}$: bounded

Proof: first note that $f: \mathcal{H} \to]-\infty, +\infty]$, $\overset{\operatorname{dom} f \neq \emptyset,\ -\infty \notin f(\mathcal{H})}{\text{proper}}$ ⇒ $\inf f(\mathcal{H})$: finite

/* Minimizing sequence :

[$f: \mathcal{X} \to [-\infty, +\infty]$; $(x_n)_{n\in\mathbb{N}}$: sequence in $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\}$]

$(x_n)_{n\in\mathbb{N}}$: minimizing sequence of f $\overset{\text{def}}{\leftrightarrow}$ $f(x_n) \to \inf f(\mathcal{X})$ */

* Proposition 11.11. (Coercivity of a function in terms of lower level set)

[$f: \mathcal{H} \to [-\infty, +\infty]$] f : coercive ↔ $(\operatorname{lev}_{\leq \xi} f)_{\xi \in \mathbb{R}}$: bounded

$\forall_{\xi \in \mathbb{R}}$ $(\operatorname{lev}_{\leq \xi} f)$: bounded

⇔ $\operatorname{lev}_{\leq \xi} f = \{x \in \mathcal{H} \mid f(x) \leq \xi\}$: bounded

per absurdum assume $(x_n)_{n\in\mathbb{N}}$: not bounded /* $(x_n)_{n\in\mathbb{N}}$: sequence in C, bounded $\overset{\text{def}}{\leftrightarrow}$ $\exists_{m\in\mathbb{N}}$ $\forall_{n\in\mathbb{N}}$ $x_n \in C \cap B(0;m)$ */

so, $\|x_n\| \to +\infty$

now, f : coercive $\overset{\text{def}}{\leftrightarrow}$ $\lim\limits_{\|x\| \to \infty} f(x) = +\infty$

⇒ $\lim\limits_{\|x_n\| \to \infty} f(x_n) = +\infty$ ⇒ $f(x_n) \to +\infty$

but $f(x_n) \to \inf f(\mathcal{H}) < +\infty$ } : contradiction

∴ Any minimizing sequence of f : bounded.

■

Part 2

6:50 PM

*Proposition 11.25.

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper, lower semicontinuous, quasiconvex

$(x_n)_{n\in\mathbb{N}}$: minimizing sequence of f

$\exists_{\xi \in]\inf f(\mathcal{H}), +\infty[}$ $C = \mathrm{lev}_{\leq \xi} f$: bounded]

(i) $(x_n)_{n\in\mathbb{N}}$: has a weak sequential cluster point,

and such point is a minimizer

(ii) $(f + \iota_C)$: strictly quasiconvex $\Rightarrow$ f: has unique minimizer, $x_n \rightharpoonup x$

(iii) $(f + \iota_C)$: uniformly quasiconvex $\Rightarrow$ f: has a unique minimizer, $x_n \to x$

Proof: Assume $(x_n)_{n\in\mathbb{N}} \subseteq C = \mathrm{lev}_{\leq\xi} f$ without loss of generality /* $\mathrm{lev}_{\leq \xi} f$: closed, convex, bouned */

as f: lower semicontinuous ⇓ ; f: quasiconvex ⇓ ; given ⇑

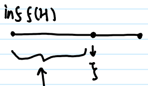
/* Explanation: Rudin 3.2.(a)

$\{p_n\}$ converges to $p \in X \Leftrightarrow$ any neighborhood of p contains p_n for all but finitely many n

now, $(x_n)_{n\in\mathbb{N}} \subseteq C = \mathrm{lev}_{\leq\xi} f = \{x \in \mathcal{H} \mid f(x) \leq \xi\}$

$\Leftrightarrow \forall_{n\in\mathbb{N}}$ $f(x_n) \leq \xi$

Assume $\exists_{n\in\mathbb{N}}$ $f(x_n) > \xi$



$(f(x_n))_{n\in\mathbb{N}}$: convergent sequence in $\mathbb{R}$, take the open interval, which will contain all but a finite numbers of points of $(f(x_n))_{n\in\mathbb{N}}$. Discard those outside and rename the new (sub) sequence $f(x_n)_{n\in\mathbb{N}}$: $f(x_n) \leq \xi$ $\forall_{n\in\mathbb{N}}$ $\Leftrightarrow \forall_{n\in\mathbb{N}}$ $x_n \in \mathrm{lev}_{\leq\xi} f$

$\Leftrightarrow (x_n)_{n\in\mathbb{N}} \subseteq C = \mathrm{lev}_{\leq\xi} f$

*/

(i)

Recall Lemma 2.37. [[$(x_n)_{n\in\mathbb{N}}$: bounded sequence in $\mathcal{H}$]] $\Rightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}}}$: subsequence of $(x_n)_{n\in\mathbb{N}}$ (x_{k_n}) : weakly convergent

$\Leftrightarrow (x_n)_{n\in\mathbb{N}}$ has a weak sequential cluster point

as, $(x_n)_{n\in\mathbb{N}} \subseteq C = \mathrm{lev}_{\leq\xi} f$: bounded $\Rightarrow (x_n)_{n\in\mathbb{N}}$: has a weak sequential cluster point

$\Leftrightarrow \exists_{(x_{k_n})_{n\in\mathbb{N}}}$: subsequence of $(x_n)_{n\in\mathbb{N}}$ $\exists_{x\in\mathcal{H}}$ $x_{k_n} \rightharpoonup x$

as, given that $(x_n)_{n\in\mathbb{N}}$: minimizing sequence of f $\overset{\text{def}}{\Leftrightarrow}$ $f(x_n) \to \inf f(\mathcal{H})$

$\Rightarrow f(x_{k_n}) \to \inf f(\mathcal{H})$ /* As, $(f(x_{k_n}))_{n\in\mathbb{N}}$: subsequence of $\underbrace{(f(x_n))_{n\in\mathbb{N}}}_{\text{minimizing sequence}}$

for a convergent sequence, any subsequence converges to the same point */

$\therefore (x_{k_n})_{n\in\mathbb{N}}$: minimizing sequence.

/* As a result, in face any convergent subsequence of the minimizing sequence will be another minimizing subsequence */

$x_{k_n} \rightharpoonup x \in \mathcal{H}$

Recall:

*Proposition 11.20. ☆

[[$f: \mathcal{H} \to]-\infty, +\infty]$, proper lower semicontinuous quasiconvex function

~~$(x_n)_{n\in\mathbb{N}}$~~ $(x_{k_n})_{n\in\mathbb{N}}$: minimizing sequence of f, converges weakly to $x \in \mathcal{H}$]]

$f(x) = \inf f(\mathcal{H})$

$\therefore f(x) = \inf f(\mathcal{H})$

$\Leftrightarrow x$: minimizer of f by definition.

(ii)

Recall:

* Proposition 11.7. (Existence of almost one minimizer in an optimization problem)

[$f: \mathcal{H} \to]-\infty, +\infty]$. quasiconvex ✓

C: convex subset of $\mathcal{H}$, $C \cap \operatorname{dom} f \neq \emptyset$ ✓ /* there are atleast one point in C where f is finite */

/* $C = \operatorname{lev}_{\leq \xi} f$: convex, closed, $\operatorname{lev}_{\leq \xi} f \subseteq \operatorname{dom} f$ */

One of the following holds:

(i) $f + \iota_C$: strictly quasiconvex ✓

(ii) f: convex, $\underbrace{C \cap \operatorname{Argmin} f = \emptyset}_{\text{this means}}$, C: strictly convex set /* C: strictly convex set $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{x,y \in C}$ $x \neq y \Rightarrow \frac{x+y}{2} \in \operatorname{int} C$ */

]

$\Rightarrow$ f: has atmost one minimizer over C. ✓

so, f: has atmost one minimizer over C

in (i) we have shown the existence of minimizer

$\therefore$ f: posseses a unique minimizer

in (i) we have shown that $\forall_{(x_{k_n})_{n \in \mathbb{N}}}$: convergent subsequence of $(x_n)_{n \in \mathbb{N}}$ $x_{k_n} \rightharpoonup x$: unique minimizer of f.

recall:

because any weak sequential cluster point is the unique minimizer of f.

* lemma 2.38. ★★

[$(x_n)_{n \in \mathbb{N}}$: sequence in $\mathcal{H}$]

$(x_n)_{n \in \mathbb{N}}$: converges weakly $\Leftrightarrow$ $(x_n)_{n \in \mathbb{N}}$: bounded ✓, posseses at most one weak sequential cluster point. ✓

$\Uparrow$

as $(x_n)_{n \in \mathbb{N}} \subseteq \operatorname{lev}_{\leq \xi} f$: bounded

$\therefore$ so, $(x_n)_{n \in \mathbb{N}}$: converges weakly

$(x_n)_{n \in \mathbb{N}}$: will also converge weakly to the minimizer of f, as otherwise the subsequence cannot go to the minimizer !

/* As if a net converges to some point, so does any subsequences Fact 1.9 */

$\therefore$ f: posseses a unique minimizer of x, $x_n \rightharpoonup x$.

(iii)

$f + \iota_C$: uniformly quasiconvex $\Rightarrow$ $f + \iota_C$: strictly quasiconvex $\Rightarrow$ f: has a unique minimizer over C.

/* uniform quasiconvexity $\Rightarrow$ strict quasiconvexity */

recall:

/*

* Definition 10.25.

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper]

(i) f : strictly quasiconvex $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{x \in \operatorname{dom} f}$ $\forall_{y \in \operatorname{dom} f}$ $\forall_{\alpha \in]0,1[}$ $x \neq y \Rightarrow f(\alpha x + (1-\alpha) y) < \max\{f(x), f(y)\}$

(ii) f : uniformly quasiconvex with modulus ϕ $: \mathbb{R}_+ \to [0, +\infty]$ $\overset{\text{def}}{\Leftrightarrow}$
- ϕ : increasing
- ϕ : vanishes only at zero
- $\forall_{x \in \operatorname{dom} f}$ $\forall_{y \in \operatorname{dom} f}$ $\forall_{\alpha \in]0,1[}$ $f(\alpha x + (1-\alpha) y) + \alpha(1-\alpha)\phi(\|x-y\|) \leq \max\{f(x), f(y)\}$

*/

fix $\alpha \in]0,1[$.

as, $(x_n)_{n \in \mathbb{N}} \subseteq \operatorname{lev}_{\leq \xi} f$, $\underset{\in \operatorname{Argmin} f}{x} \in \operatorname{lev}_{\leq \xi} f$ and $\operatorname{lev}_{\leq \xi} f \subseteq \operatorname{dom} f$

So, $\forall_{n \in \mathbb{N}}$ $\forall_{\alpha \in]0,1[}$ $f(\alpha x_n + (1-\alpha) x) + \alpha(1-\alpha)\phi(\|x_n - x\|) \leq \max\{f(x_n), f(x)\} = f(x_n)$ /* as $f(x) = \inf f(\mathcal{H}) \leq f(y)$ $\forall_y$ */

now $f(x) + \alpha(1-\alpha)\phi(\|x_n - x\|) = \inf f(\mathcal{H}) + \alpha(1-\alpha)\phi(\|x_n - x\|)$

$\forall_{n\in\mathbb{N}}$ $\forall_{\alpha\in]0,1[}$

now, $\underbrace{f(x)}_{\inf f(\mathcal{H})} + \alpha(1-\alpha)\,\phi(\|x_n-x\|) = \inf f(\mathcal{H}) + \alpha(1-\alpha)\,\phi(\|x_n-x\|)$

/* $\leq f(y)\ \forall_{y\in\mathcal{H}}$
set $y := \alpha x_n + (1-\alpha)x$ */

$\leq f(\alpha x_n + (1-\alpha)x) + \alpha(1-\alpha)\,\phi(\|x_n-x\|)$

$\leq f(x_n)$

so.
$\forall_{n\in\mathbb{N}}\ \forall_{x_n}\ \forall_{\alpha\in]0,1[}\quad f(x) + \alpha(1-\alpha)\,\phi(\|x_n-x\|) \leq f(x_n)$

Fact 1.3.1
/* $\forall_{n\in\mathbb{N}}\ a_n \leq b_n \Rightarrow \left(\underline{\lim}\, a_n \leq \underline{\lim}\, b_n,\ \overline{\lim}\, a_n \leq \overline{\lim}\, b_n\right)$ */

$\Rightarrow \forall_{\alpha\in]0,1[}$
$\underline{\lim}\ f(x) + \alpha(1-\alpha)\,\phi(\|x_n-x\|) \leq \underline{\lim}\ f(x_n) = \lim f(x_n) = \inf f(\mathcal{H}) = f(x)$ // from
$\overline{\lim}\ f(x) + \alpha(1-\alpha)\,\phi(\|x_n-x\|) \leq \overline{\lim}\ f(x_n) = f(x)$

$\Rightarrow \forall_{\alpha\in]0,1[}\ \cancel{f(x)} + \underline{\lim}\ \alpha(1-\alpha)\,\phi(\|x_n-x\|) \leq \cancel{f(x)}\ ;\ \cancel{f(x)} + \overline{\lim}\ \alpha(1-\alpha)\,\phi(\|x_n-x\|) \leq \cancel{f(x)}$

$\Rightarrow \alpha(1-\alpha)\ \underline{\lim}\ \phi(\|x_n-x\|) \leq 0,\quad \alpha(1-\alpha)\ \overline{\lim}\ \phi(\|x_n-x\|) \leq 0$

$\Rightarrow \quad \underline{\lim}\ \phi(\|x_n-x\|) \leq 0,\ \overline{\lim}\ \phi(\|x_n-x\|) \leq 0$

but $\phi : \mathbb{R}_+ \to [0,+\infty]$

$\therefore\ \underline{\lim}\ \phi(\|x_n-x\|) = \overline{\lim}\ \phi(\|x_n-x\|) = \lim\ \phi(\|x_n-x\|) = 0$

/* but ϕ vanishes only at zero */

$\Rightarrow \quad \|x_n - x\| \to 0$ /* in (ii) we have shown that: $x_n \rightharpoonup x$ i.e., $(x_n - x) \rightharpoonup 0$ */

/* (one charecterization of strong convergence using weak convergence) * *
corollary 2.42: $[(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H};\ x\in\mathcal{H}]$ $x_n \to x \Leftrightarrow (x_n \rightharpoonup x \wedge \|x_n\| \to \|x\|)$ */

$\Rightarrow \quad x_n - x \to 0 \quad$ so, $x_n \to x$

$\therefore$ f: posseses a unique minimizer x. and $x_n \to x$. ■

*Proposition 11.27.

[
$f \in \Gamma_0(\mathcal{H})$

C: bounded closed convex subset of $\mathcal{H}$, $C \cap \operatorname{dom} f \neq \emptyset$
$C \cap \operatorname{Argmin} f = \emptyset$
uniformly convex set $\overset{\text{def}}{\Leftrightarrow}$ $\exists\ \phi : [0, \operatorname{diam} C] \to \mathbb{R}_+$
- increasing function
- vanishes only at zero
- $\forall_{x\in C}\ \forall_{y\in C}\ B\left(\frac{x+y}{2}; \phi(\|x-y\|)\right) \subseteq C$ (11.10)

$(x_n)_{n\in\mathbb{N}}$: minimizing sequence of $f + \iota_C$
] $\Rightarrow$

f: has a unique minimizer of x over C, $x_n \to x$.

Proof: without loss of generality, assume that $(x_n)_{n\in\mathbb{N}} \subseteq C$ /* same logic as */
if C = singleton $\Rightarrow$ trivial
consider $C \neq$ singleton
recall

* Proposition 11.14.

[$f \in \Gamma_0(\mathcal{H})$, ✓

C: closed convex subset of $\mathcal{H}$, $C \cap \operatorname{dom} f \neq \emptyset$ ✓

One of the following holds:

(i) f: coercive

(ii) C: bounded ✓] $\Rightarrow$

f: has a minimizer over C. ✓ /* So, a coercive objective function is guaranteed to provide a minimizer even when the constraint set is not bounded */

So, f has a minimizer over C.

recall:

* Proposition 11.7. (Existence of atmost one minimizer in an optimization problem)

[$f: \mathcal{H} \to]-\infty, +\infty]$, quasiconvex ✓ ($f \in \Gamma_0(\mathcal{H})$)

C: convex subset of $\mathcal{H}$, $C \cap \operatorname{dom} f \neq \emptyset$ ✓ /* there are atleast one point in C where f is finite */

One of the following holds:

(i) ~~$f + \iota_C$: strictly quasiconvex~~

(ii) f: convex, $C \cap \operatorname{Argmin} f = \emptyset$, C: strictly convex set /* C: strictly convex set $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{x,y \in C}$ $x \neq y \Rightarrow \frac{x+y}{2} \in \operatorname{int} C$ */

(C: uniformly convex) /* uniform convexity implies strict convexity, both in functions and in sets */

this means

]

$\Rightarrow$ f: has atmost one minimizer over C. ✓

$\therefore$ So, f: has almost one minimizer over C.

f: has a unique minimizer over C.

Now let us show the convergence

recall:

* Proposition 11.25.

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper, lower semicontinuous, quasiconvex ✓ ($f \in \Gamma_0(\mathcal{H})$)

$(x_n)_{n \in \mathbb{N}}$: minimizing sequence of ~~f~~ $f + \iota_C$ ✓

$\exists_{\xi \in]\inf \cancel{f(\mathcal{H})} (f+\iota_C)(\mathcal{H}), +\infty[}$ $\cancel{C} D = \operatorname{lev}_{\leq \xi} f$: bounded ✓] /* Explanation: C: bounded $\Rightarrow \exists_{\xi \in \mathbb{R}} \forall_{x \in C}$ $f(x) \leq \xi$

$D = \operatorname{lev}_{\leq \xi} f + \iota_C = \{x \in \mathcal{H} \mid f(x) + \iota_C(x) \leq \xi\} = \{x \in \mathcal{H} \mid f(x) \leq \xi, x \in C\}$

$= \{x \in C \mid f(x) \leq \xi\} \subseteq C$ */ as C: bounded, D must be as well

(i) $(x_n)_{n \in \mathbb{N}}$: has a weak sequential cluster point $\Leftrightarrow \exists_x \exists_{(x_{k_n})_{n \in \mathbb{N}}}$ $x_{k_n} \rightharpoonup x$

every such point is a minimizer of f. ✓

(ii) ($f + \iota_C$: strictly quasiconvex) $\Rightarrow$ f: has a unique minimizer, $x_n \rightharpoonup x$

(iii) ($f + \iota_C$: uniformly quasiconvex) $\Rightarrow$ f: has a unique minimizer, $x_n \to x$

$\therefore$ $(x_n)_{n \in \mathbb{N}}$: has a weak sequential cluster point, where such point is a unique minimizer of $f + \iota_C$

$\Rightarrow$ " : " " unique weak sequential cluster point which happens to be the unique minimizer x of $f + \iota_C$.

using

* Lemma 2.38.

[$(x_n)_{n \in \mathbb{N}}$: sequence in $\mathcal{H}$]

$(x_n)_{n \in \mathbb{N}}$: converges weakly $\Leftrightarrow$ $(x_n)_{n \in \mathbb{N}}$: bounded, posseses at most one weak sequential cluster point.

($(x_n)_{n \in \mathbb{N}} \subseteq C$: bounded)

we have:

$x_n \rightharpoonup x$.

$C \cap \operatorname{Argmin} f = \emptyset$

but, $x \in C$, $x \in \operatorname{Argmin}(f + \iota_C)$, $x \notin \operatorname{Argmin} f$

recall:

* Proposition 11.5. /* The statement is

[$f: \mathcal{H} \to]-\infty, +\infty]$, proper, convex

C: subset of $\mathcal{H}$

x: minimizer of f over C, $x \in \operatorname{int} C$]

$\Rightarrow$ x: minimizer of f.

using contrapositive

$x \notin \operatorname{int} C$, but $x \in C$

$\therefore x \in C \setminus \operatorname{int} C = \operatorname{bdry} C$

definition of uniform convexity tells us that $\operatorname{int} C \neq \emptyset$

/* recall:

* Corollary 7.6. [C: nonempty closed convex subset of $\mathcal{H}$ ✓

one of the following holds:

(i) $\operatorname{int} C \neq \emptyset$ ✓

* Corollary 7.6: [C: nonempty closed convex subset of $\mathcal{H}$
one of the following holds:
(i) $\text{int}\, C \neq \emptyset$ ✓
(ii) C: closed affine subspace
(iii) $\mathcal{H}$: finite dimensional] $\Rightarrow$ $\text{spts}\, C = \text{bdry}\, C$ ✓
*/

$\therefore x \in \text{spts}\, C$

denote u: $\|u\|=1$ which is the associated normal vector u /* [C: nonempty subset of $\mathcal{H}$; $x \in C$; $u \in \mathcal{H}\setminus\{0\}$]
x: support point of C with normal vector $u \overset{\text{def}}{\Leftrightarrow} \langle x|u\rangle \geq \sup\langle C|u\rangle$ */

$\forall_{n\in\mathbb{N}}\ z_n := \frac{x_n+x}{2} + \phi(\|x_n - x\|)u$

now, $x_n \in C$, $x \in C$ then

$z_n \overset{?}{\in} B\left(\frac{x_n+x}{2}; \phi(\|x_n - x\|)\right) \subseteq C$

$\Leftrightarrow \|\underbrace{z_n}_{} - \frac{x_n+x}{2}\| \overset{?}{\leq} \phi(\|x_n - x\|)$

$= \frac{x_n+x}{2} + \phi(\|x_n-x\|)u$

$\Leftrightarrow \|\phi(\|x_n-x\|)u\| = \underbrace{\|u\|}_{=1}\,\|\phi(\|x_n-x\|)\| \overset{\checkmark}{\leq} \phi(\|x_n-x\|)$

$\therefore z_n \in C$

Recall

* Definition 7.1 (Support point)
[C: nonempty subset of $\mathcal{H}$; $x \in C$; $u \in \mathcal{H}\setminus\{0\}$]
x: support point of C with normal vector $u \overset{\text{def}}{\Leftrightarrow} \langle x|u\rangle \geq \sup\langle C|u\rangle$
supporting hyperplane of C at support point $x = \{y \in \mathcal{H} \mid \langle y|u\rangle = \langle x|u\rangle\}$
spts C = set of support points of C,
$\overline{\text{spts}}$ C = set of closure of spts C

as $x \in \text{spts}\, C$, $u: \|u\|=1 \neq \{0\}$ we have:

$\langle x|u\rangle \geq \sup\langle C|u\rangle$ — $\langle x|u\rangle$ — $\sup\langle C|u\rangle$

$\Rightarrow \forall_{\tilde{x}\in C}\ \langle x|u\rangle \geq \langle \tilde{x}|u\rangle$

$\because z_n \in C \ \therefore\ \langle x|u\rangle \geq \langle \underbrace{z_n}_{\frac{x_n+x}{2}+\phi(\|x_n-x\|)u}|u\rangle = \left\langle \frac{x_n+x}{2} + \phi(\|x_n-x\|)u \middle| u\right\rangle$

$\Rightarrow \left\langle \frac{x_n+x}{2} + \phi(\|x_n-x\|)u \middle| u\right\rangle - \langle x|u\rangle \leq 0$

$\Leftrightarrow \left\langle \frac{x_n+x}{2} + \phi(\|x_n-x\|)u - x \middle| u\right\rangle = \left\langle \frac{x_n-x}{2} + \phi(\|x_n-x\|)u \middle| u\right\rangle$ /* $\langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle$ */

$= \left\langle \frac{x_n-x}{2} \middle| u\right\rangle + \underbrace{\phi(\|x_n-x\|)}_{\text{number}}\underbrace{\langle u|u\rangle}_{=\|u\|^2=1}$ /* $\langle cx|y\rangle = c\langle x|y\rangle$ */

≤ 0

$\Leftrightarrow \left\langle \frac{x_n-x}{2} \middle| u\right\rangle + \phi(\|x_n-x\|) \leq 0$

$\Leftrightarrow \phi(\|x_n-x\|) \leq \left\langle \frac{x-x_n}{2} \middle| u\right\rangle$

so, $\forall_n\ \phi(\|x_n-x\|) \leq \left\langle \frac{x-x_n}{2} \middle| u\right\rangle = \frac{1}{2}\langle x-x_n|u\rangle$

But $x_n \rightharpoonup x \overset{\text{def}}{\Leftrightarrow} \forall_u \langle x_n - x|u\rangle \to 0$

$\phi(\|x_n-x\|) \to 0$ /* ϕ: vanishes only at 0 */

$\Leftrightarrow \|x_n - x\| \to 0$

$\Leftrightarrow x_n \to x$.

☺