Part 1

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Proposition 10.6.

$[f: \mathcal{H} \to]-\infty,+\infty]$, proper, $\beta \in \mathbb{R}_{++}]$

f: strongly convex with constant $\beta \Leftrightarrow$

$f - \frac{\beta}{2}\|\cdot\|^2$: convex

Proof:

$f - (\beta/2)\|\cdot\|^2$: convex

$$\Leftrightarrow \forall_{x,y\in \operatorname{dom} f} \ \forall_{\alpha\in]0,1[} \ \underbrace{\left(f - \frac{\beta}{2}\|\cdot\|^2\right)(\alpha x + (1-\alpha)y)}_{f(\alpha x+(1-\alpha)y) - \frac{\beta}{2}\underbrace{\|\alpha x + (1-\alpha)y\|^2}_{\alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2}} \leq \alpha \overbrace{\left(f - \frac{\beta}{2}\|\cdot\|^2\right)x}^{f(x) - \frac{\beta}{2}\|x\|^2} + (1-\alpha)\overbrace{\left(f - \frac{\beta}{2}\|\cdot\|^2\right)y}^{f(y) - \frac{\beta}{2}\|\cdot\|^2}$$

* Corollary 2.14. (Very useful identity)
$\forall_{x,y\in\mathcal{H}} \ \forall_{\alpha\in\mathbb{R}} \ \|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$

$$\Leftrightarrow \forall_{x,y\in \operatorname{dom} f} \ \forall_{\alpha\in]0,1[} \ f(\alpha x+(1-\alpha)y) - \cancel{\frac{\beta}{2}\alpha\|x\|^2} - \cancel{\frac{\beta}{2}(1-\alpha)\|y\|^2} + \frac{\beta}{2}\alpha(1-\alpha)\|x-y\|^2 \leq \alpha f(x) - \cancel{\frac{\beta}{2}\alpha\|x\|^2} + (1-\alpha)f(y) - \cancel{\frac{\beta}{2}(1-\alpha)\|y\|^2}$$

$$\Leftrightarrow \forall_{x,y\in \operatorname{dom} f} \ \forall_{\alpha\in]0,1[} \ f(\alpha x+(1-\alpha)y) + \alpha(1-\alpha)\left(\frac{\beta}{2}\|x-y\|^2\right) \leq \alpha f(x) + (1-\alpha)f(y)$$

$\overset{\operatorname{dom} f}{\Leftrightarrow}$ f: strongly convex with constant β.

■

Corollary 10.11.

[f: proper, convex, exact modulus of convexity ϕ]

f: uniformly convex $\Leftrightarrow$ ϕ: vanishes only at 0
with modulus ψ

Proof:

$(\Rightarrow)$

f: uniformly convex with modulus ψ

$$\Leftrightarrow \forall_{x,y\in \operatorname{dom} f} \ \forall_{\alpha\in]0,1[} \ f(\alpha x+(1-\alpha)y) + \alpha(1-\alpha)\psi(\|x-y\|) \leq \alpha f(x) + (1-\alpha)f(y)$$

$$\Leftrightarrow \forall_{x,y\in \operatorname{dom} f} \ \forall_{\alpha\in]0,1[} \ \psi(\|x-y\|) \leq \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$

$$\Leftrightarrow \forall_{x,y\in \operatorname{dom} f} \ \forall_{\alpha\in]0,1[} \ \psi(\|x-y\|) \leq \inf_{\substack{x,y\in\operatorname{dom} f\\ \|x-y\|=t\\ \alpha\in]0,1[}} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)} = \phi(t)$$

/* $\because$ $\forall_x \ f(x) \leq g(x) \Rightarrow \forall_x \ f(x) \leq \inf_x g(x)$
$\forall_x \forall_z \ f(x, g(z)) \leq h(x,z) \Rightarrow f(x, g(z)) \leq \inf_{x, g(z)=t} h(x,z)$ */

/*
Exact modulus of convexity,

$$\phi(t) = \inf_{\substack{x,y\in\operatorname{dom} f\\ \|x-y\|=t\\ \alpha\in]0,1[}} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$ */

$$\Rightarrow \forall_{x,y\in\operatorname{dom} f} \ \forall_{t:\, t=\|x-y\|} \ \psi(t) \leq \phi(t)$$

$\Rightarrow \psi \leq \phi$

now by definition,

* Definition 10.5. (Uniformly convex)

[$f: \mathcal{H} \to]-\infty,+\infty]$, proper]

f: uniformly convex with modulus $\psi: \mathbb{R}_+ \to [0,+\infty] \overset{\text{def}}{\Leftrightarrow}$

- ψ: increasing
- ψ: vanishes only at 0
- $\forall_{x,y\in\operatorname{dom} f} \ \forall_{\alpha\in]0,1[} \ f(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\psi(\|x-y\|) \leq \alpha f(x) + (1-\alpha)f(y)$ . . . (10.1)

- $\cancel{\phi}\ \psi$: increasing
- $\cancel{\phi}\ \psi$: vanishes only at 0
- $\forall_{x,y\in \mathrm{dom} f}\ \forall_{\alpha\in]0,1[}\quad f(\alpha x+(1-\alpha)y)+\alpha(1-\alpha)\cancel{\phi}\,\psi(\|x-y\|)\leq \alpha f(x)+(1-\alpha)f(y)\quad \ldots\ (10\cdot 1)$

as ψ: vanishes only at zero, strictly positive elsewhere and increasing

ϕ: vanishes at zero, increasing /* * Proposition 10·10· (Properties of exact modulus of convexity)

[$f:\mathcal{H}\to]-\infty,+\infty]$, proper, convex, ϕ: exact modulus of convexity] $\Rightarrow$

- $\phi(0)=0$
- $\forall_{t\in\mathbb{R}_+}\ \forall_{\gamma\in[1,+\infty]}\ \phi(\gamma t)\geq \gamma^2\phi(t)$
- ϕ: increasing. */

so. $\forall_t\quad \psi(0)=0=\phi(0)$

$\forall_{t\neq 0}\quad 0<\psi(t)\leq\phi(t)$

$\Rightarrow \phi$: vanishes only at zero

($\Leftarrow$)

ϕ: vanishes only at zero

now: $\underbrace{\phi(t)}_{=\|x-y\|} = \inf\limits_{\substack{x,y\in\mathrm{dom} f\\ \|x-y\|=t\\ \alpha\in]0,1[}} \dfrac{\alpha f(x)+(1-\alpha)f(y)-f(\alpha x+(1-\alpha)y)}{\alpha(1-\alpha)} \leq \dfrac{\alpha f(x)+(1-\alpha)f(y)-f(\alpha x+(1-\alpha)y)}{\alpha(1-\alpha)}\quad \forall_{\alpha\in]0,1[}\ \forall_{x,y\in \mathrm{dom} f:\ \|x-y\|=t}$

$\Rightarrow \forall_{x,y\in\mathrm{dom} f}\ \forall_{\alpha\in]0,1[}\quad f(\alpha x+(1-\alpha)y)+\alpha(1-\alpha)\phi(\|x-y\|)\leq \alpha f(x)+(1-\alpha)f(y)$

also. ϕ: increasing

$\therefore$ By definition $f(\cdot)$: uniformly convex with modulus ϕ.

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*Proposition 10·12·

[$f:\mathcal{H}\to]-\infty,+\infty]$, proper, convex, exact modulus of convexity ϕ

$\psi:\mathbb{R}_+\to[0,+\infty],\ t\mapsto \inf\limits_{\substack{x\in\mathrm{dom} f\\ y\in\mathrm{dom} f\\ \|x-y\|=t}} \frac{1}{2}f(x)+\frac{1}{2}f(y)-f\left(\frac{1}{2}x+\frac{1}{2}y\right)$

] $\Rightarrow$

(i) $2\psi\leq\phi\leq 4\psi$

(ii) f: uniformly convex $\Leftrightarrow$ ψ: vanishes only at 0.

Proof:

(i) $t\in\mathbb{R}_+$

$\phi(0)=0$

* Definition 10·9· (Exact modulus of convexity)

[$f:\mathcal{H}\to]-\infty,+\infty]$, proper, convex]

Exact modulus of convexity of f is:

$\phi:\mathbb{R}_+\to[0,+\infty],\ t\mapsto \inf\limits_{\substack{x,y\in\mathrm{dom} f\\ \|x-y\|=t\\ \alpha\in]0,1[}} \dfrac{\alpha f(x)+(1-\alpha)f(y)-f(\alpha x+(1-\alpha)y)}{\alpha(1-\alpha)}$

* Proposition 10·10· (Properties of exact modulus of convexity)

[$f:\mathcal{H}\to]-\infty,+\infty]$, proper, convex, ϕ: exact modulus of convexity] $\Rightarrow$

- $\phi(0)=0$
- $\forall_{t\in\mathbb{R}_+}\ \forall_{\gamma\in[1,+\infty]}\ \phi(\gamma t)\geq\gamma^2\phi(t)$
- ϕ: increasing.

if $t=0$, then

$\|x-y\|=0\Leftrightarrow x=y$

then

$\psi(0)=\inf\limits_{\substack{x,y\in\mathrm{dom} f\\ x=y}} \underbrace{\frac{1}{2}f(x)+\frac{1}{2}f(y)-f\left(\frac{1}{2}x+\frac{1}{2}y\right)}_{\frac{1}{2}f(x)+\frac{1}{2}f(x)-f(x)=0}=0$

$\therefore$ at $t=0$, $\psi(0)=\phi(0)=0$.

assume dom$f$: not a singleton /* recall f: convex $\Rightarrow$ dom f: convex */

/* dom$f$: singleton $\Rightarrow$ trivially the claim holds as $x=y\to t=\|x-y\|=0$, for which case $\psi(0)=\phi(0)=0$ */

$t>0$

$\alpha\in]0,\frac{1}{2}]$,

$x_0,y_0\in\mathrm{dom} f:\ \|x_0-y_0\|=t$ /* if no such point x,y exists, trivially we have the claim (i) as $\phi(t)=\psi(t)=+\infty$ */

now,

$f(\alpha x_0+(1-\alpha)y_0)$

$=f\left(2\alpha\cdot\frac{1}{2}x_0+2\alpha\cdot\frac{1}{2}y_0+y_0-2\alpha y_0\right)$

now,

$f(\alpha x_0+(1-\alpha)y_0)$

$=f\left(2\alpha\cdot\frac{1}{2}x_0+2\alpha\cdot\frac{1}{2}y_0+y_0-2\alpha y_0\right)$

$=f\left(2\alpha\left(\frac{1}{2}x_0+\frac{1}{2}y_0\right)+(1-2\alpha)y_0\right)$ /* $2\alpha\in]0,1]$, so, 2α is a convex parameter */

$\leq 2\alpha\, f\left(\frac{1}{2}x_0+\frac{1}{2}y_0\right)+(1-2\alpha)f(y_0)$

/* now by definition,

$$\psi(t) = \inf_{\substack{x,y\in \mathrm{dom} f\\ \|x-y\|=t}} \frac{1}{2}f(x)+\frac{1}{2}f(y)-f\left(\frac{1}{2}x+\frac{1}{2}y\right) \leq \frac{1}{2}f(x)+\frac{1}{2}f(y)-f\left(\frac{1}{2}x+\frac{1}{2}y\right) \quad \forall_{\substack{x,y\in\mathrm{dom}f\\ \|x-y\|=t}}$$

$\rightarrow \quad \psi(\|x-y\|)\leq \frac{1}{2}f(x)+\frac{1}{2}f(y)-f\left(\frac{1}{2}x+\frac{1}{2}y\right) \quad \forall_{x,y\in\mathrm{dom} f}$

$\Rightarrow \forall_{x,y\in\mathrm{dom}f} \quad f\left(\frac{1}{2}x+\frac{1}{2}y\right)\leq \frac{1}{2}f(x)+\frac{1}{2}f(y)-\psi(\|x-y\|)$

$\therefore x_0,y_0\in\mathrm{dom}f \Rightarrow \quad f\left(\frac{1}{2}x_0+\frac{1}{2}y_0\right)\leq\frac{1}{2}f(x_0)+\frac{1}{2}f(y_0)-\psi(\|x_0-y_0\|)$

$\Rightarrow 2\alpha\, f\left(\frac{1}{2}x_0+\frac{1}{2}y_0\right)\leq \alpha f(x_0)+\alpha f(y_0)-2\alpha\psi(\|x_0-y_0\|)$ */

$f(\alpha x_0+(1-\alpha)y_0)\leq \alpha f(x_0)+\alpha f(y_0)-2\alpha\psi(\|x_0-y_0\|)+(1-2\alpha)f(y_0)$

$= \alpha f(x_0)+(1-\alpha)f(y_0)-2\alpha\psi(\underbrace{\|x_0-y_0\|}_{t})$

$=\alpha f(x_0)+(1-\alpha)f(y_0)-2\alpha\,\psi(t)$

$\leq \alpha f(x_0)+(1-\alpha)f(y_0)-2\alpha\psi(t)+\underbrace{2\alpha^2}_{\geq 0}\underbrace{\psi(t)}_{}$ nonnegative function by construction, $\psi:\mathbb{R}_+\to[0,+\infty[$

$=\alpha f(x_0)+(1-\alpha)f(y_0)-2\alpha(1-\alpha)\psi(t)$

$\Rightarrow f(\alpha x_0+(1-\alpha)y_0)\leq \alpha f(x_0)+(1-\alpha)f(y_0)-2\alpha(1-\alpha)\psi(t)$

$\Rightarrow \psi(t)\leq \dfrac{\alpha f(x_0)+(1-\alpha)f(y_0)-f(\alpha x_0+(1-\alpha)y_0)}{2\alpha(1-\alpha)}$

$\Rightarrow 2\psi(t)\leq \dfrac{\alpha f(x_0)+(1-\alpha)f(y_0)-f(\alpha x_0+(1-\alpha)y_0)}{\alpha(1-\alpha)}$

$\therefore \forall_{x_0,y_0\in\mathrm{dom}f}, t=\|x_0-y_0\|, \alpha\in]0,1/2]$

$2\psi(t)\leq \dfrac{\alpha f(x_0)+(1-\alpha)f(y_0)-f(\alpha x_0+(1-\alpha)y_0)}{\alpha(1-\alpha)}$

/* Because the R-H-S is symmetric in x_0,y_0, so whatever is constructed in terms of $\alpha\in]0,1/2]$, can be created using $\alpha\in]0,1[$ (just by swapping x_0,y_0); say $\alpha=\frac{1}{4}$, then $2\psi(t)\leq \dfrac{\frac{1}{4}f(x_0)+\frac{3}{4}f(y_0)-f\left(\frac{1}{4}x_0+\frac{3}{4}y_0\right)}{\frac{1}{4}\cdot\frac{3}{4}}$, if we take $\tilde{\alpha}=\frac{3}{4}\in]0,1[$,

$2\psi(t)\leq \dfrac{\tilde{\alpha}f(y_0)+(1-\tilde{\alpha})f(x_0)-f(\tilde{\alpha}f(y_0)+(1-\tilde{\alpha})f(x_0))}{\tilde{\alpha}(1-\tilde{\alpha})}$

now swap x_0,y_0 as it is upto us:

$2\psi(t)\leq \left(\tilde{\alpha}f(x_0)+(1-\tilde{\alpha})f(y_0)-f(\tilde{\alpha}f(x_0)+(1-\tilde{\alpha})f(y_0))\right)/\tilde{\alpha}(1-\tilde{\alpha}) \quad \forall\tilde{\alpha}\in]0,1[$ */

$\therefore \forall_{x_0,y_0\in\mathrm{dom}f}, t=\|x_0-y_0\|, \alpha\in]0,1[$

$2\psi(t)\leq \dfrac{\alpha f(x_0)+(1-\alpha)f(y_0)-f(\alpha x_0+(1-\alpha)y_0)}{\alpha(1-\alpha)}$

$$\Rightarrow 2\psi(t)\leq \inf_{\substack{x_0,y_0\in\mathrm{dom}f\\ t=\|x_0-y_0\|\\ \alpha\in]0,1[}} \frac{\alpha f(x_0)+(1-\alpha)f(y_0)-f(\alpha x_0+(1-\alpha)y_0)}{\alpha(1-\alpha)}$$

$= \varphi(t)$ /* recall, $\varphi(t)= \inf_{\substack{t=\|x_0-y_0\|\\ \alpha\in]0,1[}} \dfrac{\alpha f(x_0)+(1-\alpha)f(y_0)-f(\alpha x_0+(1-\alpha)y_0)}{\alpha(1-\alpha)}$ */

so, $2\psi(t)\leq\varphi(t)$

and again,

$$\varphi(t)=\inf_{x_0,y_0\in\mathrm{dom}f} \frac{\alpha f(x_0)+(1-\alpha)f(y_0)-f(\alpha x_0+(1-\alpha)y_0)}{\alpha(1-\alpha)} \leq \left[\inf_{\substack{x_0,y_0\in\mathrm{dom}f\\ t=\|x_0-y_0\|}} \frac{\alpha f(x_0)+(1-\alpha)f(y_0)-f(\alpha x_0+(1-\alpha)y_0)}{\alpha(1-\alpha)}\right]_{\alpha=\frac{1}{2}}$$

and again,

$$\phi(t) = \inf_{\substack{x_0, y_0 \in \operatorname{dom} f \\ t = \|x_0 - y_0\| \\ \alpha \in]0,1[}} \frac{\alpha f(x_0) + (1-\alpha) f(y_0) - f(\alpha x_0 + (1-\alpha) y_0)}{\alpha(1-\alpha)} \leq \left[\inf_{\substack{x_0, y_0 \in \operatorname{dom} f \\ t = \|x_0 - y_0\|}} \frac{\alpha f(x_0) + (1-\alpha) f(y_0) - f(\alpha x_0 + (1-\alpha) y_0)}{\alpha(1-\alpha)} \right]_{\alpha = \frac{1}{2}}$$

$$= \inf_{\substack{x_0, y_0 \in \operatorname{dom} f \\ t = \|x_0 - y_0\|}} \frac{\frac{1}{2} f(x_0) + \frac{1}{2} f(y_0) - f(\frac{1}{2} x_0 + \frac{1}{2} y_0)}{\frac{1}{4}}$$

$$= 4 \underbrace{\inf_{\substack{x_0, y_0 \in \operatorname{dom} f \\ t = \|x_0 - y_0\|}} \left(\frac{1}{2} f(x_0) + \frac{1}{2} f(y_0) - f(\frac{1}{2} x_0 + \frac{1}{2} y_0) \right)}_{\psi(t) \quad /* \text{By definition} */}$$

$$= 4\psi(t)$$

$$\therefore \forall_{\substack{t : t = \|x_0 - y_0\|, \\ x_0 \in \operatorname{dom} f, \\ y_0 \in \operatorname{dom} f}} \quad 2\psi(t) \leq \phi(t) \leq 4\psi(t)$$

$$\Leftrightarrow \quad 2\psi \leq \phi \leq 4\psi$$

(ii) from (i)

$$\forall_{\substack{t : t = \|x_0 - y_0\|, \\ x_0 \in \operatorname{dom} f \\ y_0 \in \operatorname{dom} f}} \quad 2\psi(t) \leq \phi(t) \leq 4\psi(t)$$

When $t = 0$, $2\psi(0) \leq \phi(0) \leq 4\psi(0)$

Assume ψ: vanishes only at 0

$\Leftrightarrow$ $\psi(0) = 0$, $0 \leq \phi(0) \leq 0$

$\Leftrightarrow$ $\phi(0) = 0$ at 0 only

$\Leftrightarrow$ f: uniformly convex with modulus ϕ

/* * Corollary 10.11.
[f: proper, convex, exact modulus of convexity ϕ]
f: uniformly convex $\Leftrightarrow$ ϕ: vanishes only at 0 */

■

Proposition 10.15.

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper, convex;
C: nonempty compact convex, $\subseteq \operatorname{dom} f$, f: strictly convex on C
$f|_C$: continuous] $\Rightarrow$

f: uniformly convex on C

Proof:

$g := f + \iota_C \Rightarrow \operatorname{dom} g = C$

define:

$\psi : \mathbb{R}_+ \to [0, +\infty]$

$$t \mapsto \inf_{\substack{x, y \in \operatorname{dom} g \\ \|x - y\| = t}} \frac{1}{2} g(x) + \frac{1}{2} g(y) - g\left(\frac{1}{2} x + \frac{1}{2} y\right)$$

So, $$\psi(t) = \inf_{\substack{x, y \in \operatorname{dom} g \\ \|x - y\| = t}} \frac{1}{2} g(x) + \frac{1}{2} g(y) - g\left(\frac{1}{2} x + \frac{1}{2} y\right)$$

* Proposition 10.12. /* Finding bounds for exact modulus of convexity */
[$f : \mathcal{H} \to]-\infty, +\infty]$, proper, convex, exact modulus of convexity ϕ

$\psi : \mathbb{R}_+ \to [0, +\infty] : t \mapsto \inf_{\substack{x, y \in \operatorname{dom} f \\ \|x - y\| = t}} \frac{1}{2} f(x) + \frac{1}{2} f(y) - f(\frac{1}{2} x + \frac{1}{2} y)$] $\Rightarrow$

(i) $2\psi \leq \phi \leq 4\psi$

(ii) f: uniformly convex $\Leftrightarrow$ ψ: vanishes only at 0.

* Proposition 10.10. (Properties of exact modulus of convexity)

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper, convex, ϕ : exact modulus of convexity] $\Rightarrow$

- $\phi(0) = 0$
- $\forall_{t \in \mathbb{R}_+} \forall_{\gamma \in [1, +\infty]} \ \phi(\gamma t) \geq \gamma^2 \phi(t)$
- ϕ : increasing.

So, at $t = 0 \Rightarrow 2\psi(0) \leq \phi(0) \leq 4\psi(0)$

(ii) f: uniformly convex $\Leftrightarrow$ ψ: vanishes only at 0.

$\cdot\ \varphi$: increasing.

So, at $t=0 \Rightarrow 2\psi(0) \leq \underbrace{\varphi(0)}_{=0} \leq 4\psi(0)$

$\Leftrightarrow \boxed{\psi(0) \leq 0} \leq 2\psi(0)$
as ran $\psi \subseteq [0,+\infty]$ $\Bigg\} \Rightarrow \psi(0)=0 \Rightarrow$ so ψ has atleast one zero possibly many

We want to prove that ψ vanishes only at zero

take such a t that $\psi(t)=0$, and per absurdum take $t=\|x-y\| \neq 0 \Leftrightarrow x \neq y$

$$\inf_{\substack{x,y \in \operatorname{dom} g = C \\ \|x-y\|=t}} \tfrac{1}{2}g(x)+\tfrac{1}{2}g(y)-g\left(\tfrac{1}{2}x+\tfrac{1}{2}y\right)=0 \quad //\text{finite}$$

now using

***Fact 1.8.1. (Existence of a minimizing sequence for finite infimum)**
$\inf f(C) \in \mathbb{R} \Rightarrow \exists_{(x_n)_{n\in\mathbb{N}} \subseteq C}\ f(x_n) \to \inf f(C)$

we have:

$\exists_{\substack{(x_n)_{n\in\mathbb{N}},\,(y_n)_{n\in\mathbb{N}}:\ x_n, y_n \in \operatorname{dom} g = C, \\ \|x_n - y_n\| = t}}\ \tfrac{1}{2}g(x_n)+\tfrac{1}{2}g(y_n)-g\left(\tfrac{1}{2}x_n+\tfrac{1}{2}y_n\right) \to 0$

now C: compact $\Rightarrow$ so every sequence in C has a convergent subsequence

/* C: compact $\Leftrightarrow$
$\forall_{(x_a)_{a\in A}:\text{net in } C}\ \exists_{(x_{k(b)})_{b\in B}:\text{ subnet of } (x_a)_{a\in A};\ x\in C}\ x_{k(b)} \to x$ */

as $(x_n)_{n\in\mathbb{N}} \subseteq C$; $(y_n)_{n\in\mathbb{N}} \subseteq C$ $\quad \exists_{x\in C,\, y\in C}\ \exists_{\substack{(x_{k_n})_{n\in\mathbb{N}} \subseteq C, \\ (y_{k_n})_{n\in\mathbb{N}} \subseteq C}}\ x_{k_n} \to x \in C,\ y_{k_n} \to y \in C$

$\tfrac{1}{2}g(x_{k_n})+\tfrac{1}{2}g(y_{k_n})-g\left(\tfrac{1}{2}x_{k_n}+\tfrac{1}{2}y_{k_n}\right) \to 0$

/* If a sequence converges to something, any of its subsequence converges to the same thing */

also $\|x_{k_n}-y_{k_n}\| = t \Rightarrow t \leq \|x_{k_n}-y_{k_n}\| \leq t$

now, $\left(x_{k_n} \to x,\ y_{k_n} \to y\right) \Rightarrow x_{k_n}-y_{k_n} \to x-y$

$\Rightarrow \|x_{k_n}-y_{k_n}\| \to \|x-y\|$ /* using (one characterization of strong convergence) *
Corollary 2.42. [$(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$; $x \in \mathcal{H}$] $x_n \to x \Leftrightarrow \left(x_n \rightharpoonup x \wedge \|x_n\| \to \|x\|\right)$ */

/* using Fact 1.3.1.
$\forall_{n\in\mathbb{N}}\ a_1 \leq a_n \leq a_2 \Rightarrow a_1 \leq \underline{\lim}\, a_n \leq \overline{\lim}\, a_n \leq a_2$ */

$t \leq \|x-y\| \leq t$

$\therefore \|x-y\| = t$

now $g|_C = (f+\underbrace{\iota_C}_{0})|_C = f|_C$: continuous by given

/*
***Fact 1.19. * Definition of continuity using nets**
[$\mathcal{X}, \mathcal{Y}$: Hausdorff spaces
$T: \mathcal{X} \to \mathcal{Y}$
$x \in \mathcal{X}$]
T: continuous at $x \Leftrightarrow \forall_{\underbrace{(x_a)_{a\in A}}_{\subseteq \mathcal{X}}:\ x_a \to x}\ Tx_a \to Tx$.
*/

$\therefore \tfrac{1}{2}g(x_{k_n})+\tfrac{1}{2}g(y_{k_n})-g\left(\tfrac{1}{2}x_{k_n}+\tfrac{1}{2}y_{k_n}\right) \to \tfrac{1}{2}g(x)+\tfrac{1}{2}g(y)-g\left(\tfrac{1}{2}x+\tfrac{1}{2}y\right)$

Both of the limits must be the same thing

$\tfrac{1}{2}g(x)+\tfrac{1}{2}g(y)-g\left(\tfrac{1}{2}x+\tfrac{1}{2}y\right)=0$

$\Leftrightarrow \tfrac{1}{2}g(x)+\tfrac{1}{2}g(y)=g\left(\tfrac{1}{2}x+\tfrac{1}{2}y\right)$

but $g|_C = f|_C$: strictly convex on C by given

$\therefore \forall_{x,y\in C:\ x\neq y}\ g\left(\tfrac{1}{2}x+\tfrac{1}{2}y\right) < \tfrac{1}{2}g(x)+\tfrac{1}{2}g(y)$

$x=y$

$\therefore t=\|x-y\|=0$ contradiction with $\therefore \psi$: vanishes only at zero
$\Updownarrow$
f: uniformly convex on C

/*
* Proposition 10.12. /* Finding bounds for exact modulus of convexity */
[f: ...] ... exact modulus of convexity ...

contradiction with

$\therefore\ t = \|x - y\| = 0$ $\qquad$ $\therefore\ \psi$: vanishes only at zero

$\Updownarrow$

f : uniformly convex on C

/*

* Proposition 10.12. /* Finding bounds for exact modulus of convexity */

[$f : \mathcal{H} \to]-\infty, +\infty]$, proper, convex, exact modulus of convexity φ

$\psi : \mathbb{R}_+ \to [0, +\infty] : t \mapsto \inf\limits_{\substack{x, y \in \operatorname{dom} f \\ \|x - y\| = t}} \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\left(\frac{1}{2}x + \frac{1}{2}y\right)$] $\Rightarrow$

(i) $2\psi \le \varphi \le 4\psi$

(ii) f : uniformly convex $\Leftrightarrow$ ψ : vanishes only at 0.

*/

$\boxtimes$

Part 2

10:17 AM

Proposition. 10.22.

$[\ (f_i)_{i\in I}$: family of quasiconvex functions from $\mathcal{H}$ to $[-\infty,+\infty]\ ]$

$\sup_{i\in I} f_i$: quasiconvex

Proof:

$\forall_{\xi\in\mathbb{R}}$

$\operatorname{lev}_{\leq\xi} \sup_{i\in I} f_i$

$= \{x\in\mathcal{H} \mid \sup_{i\in I} f_i \leq \xi\}$

$\Updownarrow$

$\forall_{i\in I}\ f_i \leq \xi$

$= \bigcap_{i\in I} \{x\in\mathcal{H} \mid f_i \leq \xi\}$

$= \left(\bigcap_{i\in I} \underbrace{\operatorname{lev}_{\leq\xi} f_i}_{\text{: convex /* By definition */}}\right)$: convex /* As intersection of convex sets are always convex */

$\forall_{\xi\in\mathbb{R}} \left(\sup_{i\in I} f_i\right)$: convex $\overset{\text{def}}{\Leftrightarrow} \sup_{i\in I} f_i$: quasiconvex

/* $\because$ By definition, f : quasiconvex $\overset{\text{def}}{\Leftrightarrow} \forall_{\xi\in\mathbb{R}} \operatorname{lev}_{\leq\xi} f$: convex

$\blacksquare$

*** Proposition 10.23.**

$[\ f:\mathcal{H}\to\,]-\infty,+\infty]$, quasiconvex $]$

(i) f: weakly sequentially lower semicontinuous $\Leftrightarrow$

(ii) f: sequentially lower semicontinuous $\Leftrightarrow$

(iii) f: lower semicontinuous $\Leftrightarrow$

(iv) f: weakly lower semicontinuous

Proof:

- f : quasiconvex $\overset{\text{def}}{\Leftrightarrow} \forall_{\xi\in\mathbb{R}} \operatorname{lev}_{\leq\xi} f$: convex

assume

f : lower semicontinuous $\Rightarrow \forall_{\xi\in\mathbb{R}} \operatorname{lev}_{\leq\xi} f$: closed

/* using

Lemma 1.24.

$[\ \mathcal{X}$: Hausdorff space,

$f:\mathcal{X}\to[-\infty,\infty]\ ]$

f : lower semicontinuous $\Leftrightarrow \operatorname{epi} f$: closed in $\mathcal{X}\times\mathbb{R} \Leftrightarrow \forall_{\xi\in\mathbb{R}} \operatorname{lev}_{\leq\xi} f$: closed in $\mathcal{X}$

But /* Theorem 3.32. /* this is a very important theorem which says that for a convex set all the different types of closedness coincides */

$[\ C$: convex subset of $\mathcal{H}\ ]$

C : weakly sequentially closed $\Leftrightarrow C$: sequentially closed $\Leftrightarrow C$: closed $\Leftrightarrow C$: weakly closed */

but/* Theorem 3.32. /* this is a very important theorem which says that for a convex set all the different types of closedness coincides */

⟦ C : convex subset of $\mathcal{H}$ ⟧

C : weakly sequentially closed $\Leftrightarrow$ C : sequentially closed $\Leftrightarrow$ C : closed $\Leftrightarrow$ C : weakly closed */

$\forall_{\xi \in \mathbb{R}}$ $\operatorname{lev}_{\leq \xi} f$: weakly sequentially closed $\Leftrightarrow$ f : weakly sequentially lower semicontinuous

$\Updownarrow$

$\operatorname{lev}_{\leq \xi} f$: sequentially closed $\Leftrightarrow$ f : sequentially lower continuous

$\Updownarrow$

$\operatorname{lev}_{\leq \xi} f$: closed $\Leftrightarrow$ f : lower semicontinuous

$\Updownarrow$

$\operatorname{lev}_{\leq \xi} f$: weakly closed $\Leftrightarrow$ f : weakly lower semicontinuous

*Lemma 1.35.

⟦ $\mathcal{X}$: Hausdorff space

$f : \mathcal{X} \to [-\infty, +\infty]$ ⟧

(i) f : sequentially lower semicontinuous $\Leftrightarrow$

(ii) epi f : sequentially closed $\Leftrightarrow$

(iii) $\forall_{\xi \in \mathbb{R}}$ $\operatorname{lev}_{\leq \xi} f$: sequentially closed in $\mathcal{X}$

So, we have

f : weakly sequentially lower semicontinuous $\Leftrightarrow$

f : sequentially lower semicontinuous $\Leftrightarrow$

f : lower semicontinuous $\Leftrightarrow$

f : weakly lower semicontinuous

■