Chapter 1: part 1

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• Net

$(A, \preccurlyeq)$: directed set $\overset{def}{=} \left(\forall_{a \in A}\ a \preccurlyeq a,\ \forall_{a,b,c \in A} (a \preccurlyeq b \wedge b \preccurlyeq c) \Rightarrow a \preccurlyeq c, \right.$
$\left. \forall_{a,b \in A} \exists_{c \in A} (a \preccurlyeq c \wedge b \preccurlyeq c) \right)$

$(x_a)_{a \in A}$: net in $\mathcal{X}$ $\overset{def}{\leftrightarrow}$ operator from A to $\mathcal{X}$ ($\mathcal{X}$: nonempty set)

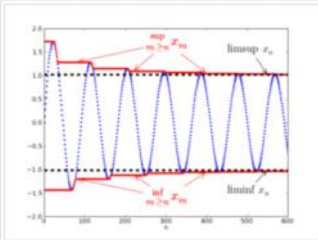
$(y_b)_{b \in B}$: subnet of $(x_a)_{a \in A}$ $\overset{def}{\leftrightarrow}$ $\exists \underbrace{k}_{\text{some operator (not set-valued)}} : B \to A \left(\cdot \forall_{b \in B}\ y_b = x_{k(b)}\ \cdot \forall_{a \in A} \exists_{d \in B} \forall_{b \in B}\ b \succcurlyeq d \Rightarrow k(b) \succcurlyeq a \right)$

• Limit inferior of a net $(\xi_a)_{a \in A}$ in $[-\infty, \infty]$ is:
$\underline{\lim}\ \xi_a = \sup_{c \in A} \inf_{b \in A: b \succcurlyeq c} \xi_b$

Limit superior of a net $(\xi_a)_{a \in A}$ in $[-\infty, \infty]$ is:
$\overline{\lim}\ \xi_a = \inf_{c \in A} \sup_{b \in A: b \succcurlyeq c} \xi_b$

$\underline{\lim}\ \xi_a \le \overline{\lim}\ \xi_a$

Limit inferior and limit superior always exists.



An illustration of limit superior and limit inferior. The sequence x_n is shown in blue. The two red curves approach the limit superior and limit inferior of x_n, shown as dashed black lines. In this case, the sequence *accumulates* around the two limits. The superior limit is the larger of the two, and the inferior limit is the smaller of the two. The inferior and superior limits agree if and only if the sequence is convergent (i.e., when there is a single limit).

* Lemma 1.6 .

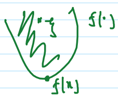
[$(f_i)_{i \in I}$: family of functions from $\mathcal{X}$ to $[-\infty, +\infty]$]

(i) $\operatorname{epi}\left(\sup_{i \in I} f_i\right) = \bigcap_{i \in I} \operatorname{epi} f_i$

(ii) I : finite $\Rightarrow$ $\operatorname{epi} \min_{i \in I} f_i = \bigcup_{i \in I} \operatorname{epi} f_i$

Proof:

(i) $\forall_{(x,\xi)}$



$(x,\xi) \in \operatorname{epi}\left(\sup_{i \in I} f_i\right)$

$\Leftrightarrow \underbrace{\left(\sup_{i \in I} f_i\right)(x)}_{\sup_{i \in I} f_i(x)} \le \xi$

$\Leftrightarrow \forall_{i \in I} \underbrace{f_i(x) \le \xi}_{\Updownarrow \atop (x,\xi) \in \operatorname{epi} f_i} \Leftrightarrow (x,\xi) \in \bigcap_{i \in I} \operatorname{epi} f_i$

(ii) $\forall_{(x,\xi)}$ $(x,\xi) \in \operatorname{epi}\left(\min_{i \in I} f_i\right)$

$\Leftrightarrow \min_{i \in I} f_i(x) \le \xi$ # I : finite $\Leftrightarrow \exists_{i \in I} \forall_{j \in I}\ i \le j$

$\Leftrightarrow \exists_{i \in I} \underbrace{f_i(x) \le \xi}_{\Updownarrow \atop (x,\xi) \in \operatorname{epi} f_i} \Leftrightarrow (x,\xi) \in \bigcup_{i \in I} \operatorname{epi} f_i$

■

• Extended real line:

$[-\infty, \infty] = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$

1.7. Topological space: /* see Wiki article on [Topological space] */

$\mathcal{X}$: set

T : family of subsets of $\mathcal{X}$ containing
- $\mathcal{X}$ /* $\mathcal{X}$ is generally open and close at the same time */
- ϕ /* open and closed at the same time */
- arbitrary union of elements of $\mathcal{X}$
- finite intersections " " "

} these elements are open sets

$(T, \mathcal{X})$: topological space — for more specific definition see the neighborhood axioms [Neighborhood definition]

• V : neighborhood of x $\overset{def}{\leftrightarrow}$ $\exists_{U \in T} (x \in U \wedge U \subseteq V)$ /* A neighborhood can be open or close */

$(T, \mathcal{X})$: topological space — for more specific definition see the neighborhood axioms [Neighborhood definition]

- $\underbrace{V}_{\subseteq \mathcal{X}}$: neighborhood of $x \overset{def}{\leftrightarrow} \exists_{U \in T} (x \in U \wedge U \subseteq V)$ /* A neighborhood can be open or close */

  # For a neighborhood you can find a smaller neighborhood in the topology

- $\mathcal{V}(x)$: set of all neighborhoods of x

- $\underbrace{\mathcal{X}}_{\text{topological space}}$: Hausdorff space $\overset{def}{\leftrightarrow} \forall_{x_1, x_2 \in \mathcal{X} : x_1 \neq x_2} \exists_{V_1 \in \mathcal{V}(x_1)} \exists_{V_2 \in \mathcal{V}(x_2)} V_1 \cap V_2 = \emptyset$

- $\underbrace{C}_{\subseteq \mathcal{X}}$: compact $\overset{def}{\leftrightarrow}$ (C: contained in the union of a family of open sets) $\Rightarrow$ (C: contained in the union of a finite subfamily from that family)

- $\underbrace{(x_a)_A}_{\text{net in } \mathcal{X}}$: converges to $x \in \mathcal{X} \overset{def}{\leftrightarrow} \forall_{V \in \mathcal{V}(x)} \exists_{b \in A} \forall_{a \in A} (a \succcurlyeq b \Rightarrow x_a \in V)$

- $\underbrace{x}_{\in \mathcal{X}}$: cluster point of $(x_a)_A \overset{def}{\leftrightarrow} (x_a)_{a \in A}$ lies frequently in every neighborhood of x

  $\overset{def}{\leftrightarrow} \forall_{V \in \mathcal{V}(x)} \forall_{b \in A} \exists_{a \in A : a \succcurlyeq b} x_a \in V$

  /* Pick any neighborhood of x, and any index b∈A of the net, then there will be a subsequent index a ∈ A : a ≽ b, such that xₐ lies in that neighborhood. If this algorithm evaluates true, then we have got ourselves a cluster point x. */

  $\overset{def}{\leftrightarrow} \exists_{(x_{k(b)})_{b \in A}} x_{k(b)} \to x$ /* $(y_b)_{b \in B}$: subnet of $(x_a)_{a \in A} \overset{def}{\leftrightarrow} \exists_{k : B \to A} \left(\begin{array}{l} \cdot\ y_b = x_{k(b)} \\ \cdot\ \forall_{a \in A} \exists_{d \in B} \forall_{b \in B : b \succcurlyeq d} k(b) \succcurlyeq a \end{array} \right)$

*

[Sequential Cluster Point] $\underbrace{x}_{\in \mathcal{X}}$: sequential cluster point of $\underbrace{(x_n)_{n \in \mathbb{N}}}_{\text{sequence in } \mathcal{X}} \overset{def}{\longleftrightarrow} (x_n)_{n \in \mathbb{N}}$ has a subsequence that converges to x

Lemma 1.10.

$\left(\begin{array}{l} C: \text{subset of Hausdorff space} \\ x \in \mathcal{X} \end{array} \right)$

$\underbrace{x \in \bar{C}}_{\text{closure of } C} \leftrightarrow \exists_{(x_a)_{a \in A}} (x_a \to x)$

Lemma 1.12.

[C: compact subset of $\overset{\text{Hausdorff space}}{\mathcal{X}}$] $\Rightarrow$ C: closed, $\forall$subsets of C is compact

* Lemma 1.14.

[C: compact subset of a Hausdorff space $\mathcal{X}$

$(x_a)_{a \in A}$: net in C, x : unique cluster point in C of the net

$x_a \to x$

Proof: per absurdum,

$x_a \not\to x$

/* $x_a \to x \overset{def}{\leftrightarrow} \forall_{V \in \mathcal{V}(x)} \exists_{\gamma \in A} \forall_{\upsilon \in A : \upsilon \succcurlyeq \gamma} x_\upsilon \in V \leftrightarrow (x_a)_{a \in A}$: eventually lies in every neighborhood of x
*/

$\therefore x_a \not\to x \leftrightarrow \exists_{V \in \mathcal{V}(x)} \forall_{\gamma \in A} \exists_{\upsilon \in A : \upsilon \succcurlyeq \gamma} x_\upsilon \notin V$

As a result we can construct a subnet $(x_{k(a)})_{a \in A}$ of $(x_a)_{a \in A}$ such that $(x_{k(a)})_{a \in A}$ lies entirely outside V (e.g. take γ, then find all those $\upsilon : \upsilon \succcurlyeq \gamma : x_\upsilon \notin V$
then some $\tilde{\gamma} \succcurlyeq \gamma$ " " " $\tilde{\upsilon} : \tilde{\upsilon} \succcurlyeq \tilde{\gamma} : x_{\tilde{\upsilon}} \notin V$ and so on)

but as $(x_a)_{a \in A} \subseteq C \Rightarrow (x_{k(a)})_{a \in A} \subseteq C$

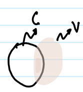
$\therefore (x_{k(a)})_{a \in A} \subseteq C \setminus V$

C: compact $\Rightarrow$ C: closed /* Lemma 1.12 */ — A compact set is closed.

$C \setminus V \subseteq C$

if $C \cap V = \emptyset$, then $C \setminus V = C$: closed

if $C \cap V \neq \emptyset$, then $C \setminus V$: closed



so $C \setminus V$: compact /* closed subset of compact set is compact */ Lemma 1.12

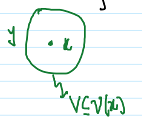
/* recall that:

C: compact $\overset{\text{def}}{\Leftrightarrow}$ every net in C has a cluster point

(i.e., every net in C has a subnet that converges to a point in C */

as $(x_{k(a)})_{a\in A} \subseteq \underbrace{C\setminus V}_{\text{compact}}$, so $(x_{k(a)})_{a\in A}$ must have a cluster point in $C\setminus V$, say $y\in C\setminus V \Rightarrow y\notin V\subseteq \mathcal{V}(x)$ $\Big\} \Rightarrow y \neq \underset{\text{cluster point of } (x_a)_{a\in A}}{x} \in V$

cluster point of $(x_{k(a)})_{a\in A}$ hence cluster point of $(x_a)_{a\in A}$



So, we have found two different cluster points of $(x_a)_{a\in A}$: contradiction.

$\therefore x_a \to x$.

∎

Lemma 1.16.

$[\,(\xi_a)_{a\in A}, (\eta_a)_{a\in A}$: nets in $[-\infty,+\infty]$, $\underline{\lim}\,\xi_a > -\infty$, $\underline{\lim}\,\eta_a > -\infty\,] \Rightarrow \underline{\lim}\,\xi_a + \underline{\lim}\,\eta_a \le \underline{\lim}\,(\xi_a+\eta_a)$

Proof: $a\in A$

$\forall_{c\in A}\; c\ge a \Rightarrow \underbrace{\inf_{\substack{b\in A\\ b\ge a}} \xi_b}_{\le\, \xi_{\tilde b}\;\forall_{\tilde b\in A:\, \tilde b\ge a}} + \underbrace{\inf_{\substack{b\in A\\ b\ge a}} \eta_b}_{\le\, \eta_{\tilde b}\;\forall_{\tilde b\in A:\,\tilde b\ge a}} \le \xi_c+\eta_c \;\therefore\; \forall_{a\in A}\left(\inf_{b\in A:\, b\ge a} \xi_b + \inf_{b\in A:\, b\ge a}\eta_b \le \xi_c+\eta_c \quad \forall_{c\in A:\, c\ge a}\right) \Leftrightarrow \forall_{a\in A}\left(\inf_{b\in A:\, b\ge a}\xi_b + \inf_{b\in A:\, b\ge a}\eta_b \le \inf_{\substack{c\in A:\\ c\ge a}}(\xi_c+\eta_c)\right)$

/* set $\tilde b := c : c \ge a$ */

$\Leftrightarrow \underbrace{\sup_{a\in A}\inf_{b\in A:\, b\ge a}\xi_b}_{\underline{\lim}\,\xi_b} + \underbrace{\sup_{a\in A}\inf_{b\in A:\, b\ge a}\eta_b}_{\underline{\lim}\,\eta_b} \le \underbrace{\sup_{a\in A}\inf_{\substack{c\in A:\\ c\ge a}}(\xi_c+\eta_c)}_{\underline{\lim}\,(\xi_c+\eta_c)}$

$\therefore \underline{\lim}\,\xi_b + \underline{\lim}\,\eta_b \le \underline{\lim}\,(\xi_b+\eta_b)$

∎

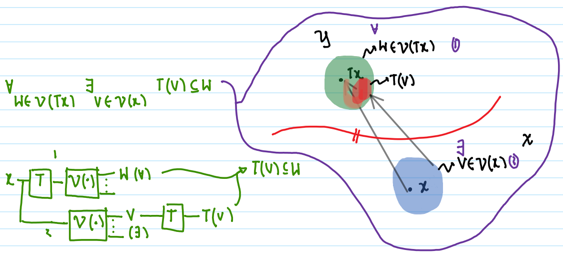
*Continuity: [sec: Continuity]

$(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$: topological spaces

$(\mathcal{Y}, \mathcal{T}_{\mathcal{Y}})$ " "

$\underset{\mathcal{X}\to\mathcal{Y}}{T}$: continuous at $x\in\mathcal{X}$ $\overset{\text{def}}{\Leftrightarrow} \forall_{W\in\mathcal{V}(Tx)}\; \exists_{V\in\mathcal{V}(x)}\; T(V)\subseteq W$

#



T: continuous $\overset{\text{def}}{\leftrightarrow} (T\text{: continuous at } x)_{\forall x\in\mathcal{X}}$

*fact:

$(\mathcal{X},\mathcal{T}_{\mathcal{X}})$: topological space
$(\mathcal{Y},\mathcal{T}_{\mathcal{Y}})$: topological space
$T: \mathcal{X}\to\mathcal{Y}$
$\mathcal{B}_{\mathcal{Y}}$: base of $\mathcal{T}_{\mathcal{Y}}$ /* $\underbrace{\mathcal{B}}_{\text{subfamily of } \mathcal{T}}$: base of $\mathcal{T}$ $\overset{\text{def}}{\Leftrightarrow} \forall_{x\in\mathcal{X}}\,\forall_{V\in\mathcal{V}(x)}\,\exists_{\tilde B\in\mathcal{B}} (x\in\tilde B \wedge \tilde B\subseteq V)$ */

then,

T: continuous $\leftrightarrow \forall_{\tilde B\in\mathcal{B}_{\mathcal{Y}}} \underbrace{T^{-1}(\tilde B)}_{T^{-1}(\tilde B)=\{x \mid Tx\in\tilde B\}}\in\mathcal{T}_{\mathcal{X}}$

* fact 1.19

$\mathcal{X},\mathcal{Y}$: Hausdorff space
$T: \mathcal{X}\to\mathcal{Y}$
$x\in\mathcal{X}$

[def: continuity using nets]

T: continuous at $x \leftrightarrow \left(\forall_{(x_a)_{a\in A}:\text{net in } \mathcal{X},\; x_a\to x}\; Tx_a\to Tx\right)$

/* When a continuous mapping takes a convergent net, the output net is also convergent with the limit being the operator acting on the *input sequence limit point* */

very intuitive

*Lemma 1.20:

$\mathcal{X},\mathcal{Y}$: Hausdorff spaces
$T: \mathcal{X}\to\mathcal{Y}$, continuous
C: compact, $\subseteq\mathcal{X}$

$\Rightarrow \underbrace{T(C)}_{\{Tx\,:\,x\in C\}}$: compact

Proof:

$(y_a)_{a\in A}$: any net in $T(C)$

$\Rightarrow \forall_{a\in A}\; y_a\in T(C)=\{Tx : x\in C\}$

$\leftrightarrow \forall_{a\in A}\,\exists_{x_a\in C}\; y_a = Tx_a$ $\therefore$ Any net in $T(C)$ has form $(Tx_a)_{a\in A}$

given, C: compact # recall: $[\,C\subseteq \underbrace{\mathcal{X}}_{\text{Hausdorff space}}\,]$ C: compact $\leftrightarrow$ every net in C has a subnet that converges to a point in C

$\Updownarrow$

$\exists_{(x_{k(b)})_{b\in B}}$: subnet of $(x_a)_{a\in A}$ $x_{k(b)}\to \underbrace{x}_{\in C}$

given, T: continuous

recall fact 1.19:

$\mathcal{X},\mathcal{Y}$: Hausdorff space

$(x_{k(b)})_{b \in B}$: subnet of $(x_a)_{a \in A}$ $\in C$

given, T: continuous

$\updownarrow$

$\forall_{x \in \mathcal{X}} \ \forall_{(x_a)_{a \in A} : x_a \to x} \ Tx_a \to Tx$

recall fact 1.19:

$\boxed{\begin{array}{l} \mathcal{X}, \mathcal{Y} : \text{Hausdorff space} \\ T : \mathcal{X} \to \mathcal{Y} \\ x \in \mathcal{X} \end{array}}$

T: continuous at $x \leftrightarrow \left(\forall_{(x_a)_{a \in A} : \text{net in } \mathcal{X},\ x_a \to x} \ Tx_a \to Tx \right)$

as the subnet $(Tx_{k(b)})_{b \in B}$ comes from $(Tx_a)_{a \in A}$ $\Rightarrow \left(Tx_{k(b)} \to \underbrace{Tx}_{\in T(C)}\right)$ /* $\because$ fact 1.9. says

$\left(\underbrace{(x_a)_{a \in A}}_{\text{net in Hausdorff space}} : x_a \to x,\ (x_{k(b)})_{b \in B} : \text{subnet of } (x_a)_{a \in A} \right) \Rightarrow \left(x_{k(b)} \to x \right)$ */

So, we have shown:

$\forall_{(Tx_a)_{a \in A} : \text{net in } T(C)} \ \exists_{(Tx_{k(b)}) : \text{subnet of } (Tx_a)_{a \in A}} \ Tx_{k(b)} \to \underbrace{Tx}_{\in T(C)}$

$T(C)$: compact

Chapter 1 : Part 2

6:52 AM

1.10. Lower semicontinuity [sec: Lower Semicontinuity]

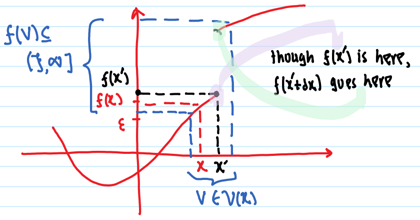
Definition.

$\mathcal{X}$: Hausdorff space
$f: \mathcal{X} \to [-\infty, \infty]$
$x \in \mathcal{X}$

/* limit inferior of a net $(\xi_a)_{a\in A}$ in $[-\infty,\infty]$, $\underline{\lim}\, \xi_a = \sup_{c \in A} \inf_{\substack{b\in A \\ c \preceq b}} \xi_b$ */

f: lower semicontinuous at $x \overset{\text{def}}{\leftrightarrow} \forall_{(x_a)_{a\in A} : \text{net in } \mathcal{X},\, x_a \to x} \quad \underline{\lim}\, f(x_a) \geq f(x)$

$\overset{\text{def}}{\leftrightarrow} \forall_{\xi \in (-\infty, f(x))} \; \exists_{V \in \mathcal{V}(x)} \quad f(V) \subseteq (\xi, \infty]$



f: upper semicontinuous at $x \leftrightarrow -f$: lower semicontinuous at x

f: continuous at $x \leftrightarrow f$: both upper and lower semicontinuous at x

* Domain of continuity : $\operatorname{cont} f = \{x \in \mathcal{X} : f(x) \in \mathbb{R} \text{ and } f: \text{continuous at } x\}$

$\subseteq \operatorname{int} \operatorname{dom} f$

* nice to know :
  - A function cannot be continuous and real valued ($\in(-\infty,\infty)$) on the boundary of its domain
  - A continuous function can take values $-\infty$ and $+\infty$.

$\mathcal{X}$: Hausdorff space
$f: \mathcal{X} \to [-\infty, \infty]$
$x \in \mathcal{X}$

  then $\underline{\lim}_{y \to x} f(y) = \sup_{V \in \mathcal{V}(x)} \inf f(V)$

  $= \left(\left(V \in \mathcal{V}(x) \,//\, f(\cdot) \,//\, \inf_x (\cdot) \right) // \sup_{V \in \mathcal{V}(x)} (\cdot) \right)$

Lemma 1.23.

$\mathcal{X}$: Hausdorff space
$f: \mathcal{X} \to [-\infty, \infty]$
$x \in \mathcal{X}$
$N(x)$: set of all nets in $\mathcal{X}$ converging to x

/* recall limit inferior of a net $(\xi_a)_{a\in A}$ in $[-\infty,+\infty]$,

$\underline{\lim}\, \xi_a = \sup_{c\in A} \inf_{\substack{b \in A \\ b \succeq c}} \xi_b = \left(\xi_{b : b\in A,\, b \succeq c} \,//\, \inf(\cdot) \,//\, \sup_{c\in A} (\cdot) \right)$

*/

$$\underline{\lim}_{y\to x} f(y) = \underbrace{\min_{(x_a)_{a\in A} \in N(x)} \underbrace{\underline{\lim}\, f(x_a)}_{\text{limit inferior for one net } (x_a)_{a\in A} \text{ in the neighborhood of } N(x)}}_{\text{minimum over all such nets in } N(x)}$$

[Lemma 1.24] Lemma 1.24. /* Relates lower semicontinuity with epigraph and sublevel set */

$\mathcal{X}$: Hausdorff space.
$f: \mathcal{X} \to [-\infty, \infty]$

$\Rightarrow$

(i) f: lower semicontinuous /* i.e. $\left(f: \text{lower semicontinuous at } x\right)_{\forall x \in \mathcal{X}}$ */

$\Updownarrow$

(i) f : lower semicontinuous /* i.e. $\left(f \text{: lower semicontinuous at } x\right)_{\forall x \in \mathcal{X}}$ */

$\Updownarrow$

(ii) $\operatorname{epi} f$: closed in $\mathcal{X} \times \mathbb{R}$

$\Updownarrow$

(iii) $\forall_{\xi \in \mathbb{R}}$ $\operatorname{lev}_{\leq \xi} f$: closed in $\mathcal{X}$

Proof: (i) $\Rightarrow$ (ii):

/* C : closed $\Leftrightarrow \forall_{(x_a)_{a\in A} : \subseteq C,\, x_a \to x}\ x \in C$ */

take $(x_a, \xi_a)_{a \in A} : \subseteq \operatorname{epi} f$, $(x_a, \xi_a) \to (x, \xi) \in \mathcal{X} \times \mathbb{R}$

as $(x_a, \xi_a) \in \operatorname{epi} f \Leftrightarrow f(x_a) \leq \xi_a$

$x_a \to x$, f : lower semicontinuous $\Rightarrow f(x) \leq \underline{\lim}\, f(x_a)$

* Definition 1.21.
[$\mathcal{X}$: Hausdorff space, $f : \mathcal{X} \to [-\infty, +\infty]$, $x \in \mathcal{X}$]
f : lower semicontinuous at $x \overset{\text{def}}{\Leftrightarrow} \forall_{(x_a)_{a\in A} : \text{net in } \mathcal{X},\, x_a \to x}\ \underline{\lim}\, f(x_a) \geq f(x)$ # definition in terms of a net

$\leq \underline{\lim}\, \xi_a$ /* $\because x_a \geq y_a \Rightarrow \underline{\lim}\, x_a \geq \underline{\lim}\, y_a$ */

$= \lim \xi_a = \xi$

$\therefore f(x) \leq \xi \Leftrightarrow (x, \xi) \in \operatorname{epi} f$

so, $\operatorname{epi} f$: closed.

(ii) $\Rightarrow$ (iii)

given : $\operatorname{epi} f$: closed $\Leftrightarrow \forall_{(y_a)_{a\in A} : \subseteq \operatorname{epi} f,\, y_a \to y}\ y \in \operatorname{epi} f$

goal : $\forall_{\xi \in \mathbb{R}}$ $\operatorname{lev}_{\leq \xi} f$: closed in $\mathcal{X}$

/* C : closed $\Leftrightarrow \forall_{(x_a)_{a\in A} : \subseteq C,\, x_a \to x}\ x \in C$ */

$\forall_{(x_a)_{a\in A} : \subseteq \operatorname{lev}_{\leq \xi} f,\, x_a \to x}$

$x_a \in \operatorname{lev}_{\leq \xi} f \Leftrightarrow f(x_a) \leq \xi \Leftrightarrow (x_a, \xi) \in \operatorname{epi} f$

now $(x_a, \xi) \to (x, \xi)$

$\Rightarrow (x, \xi) \in \operatorname{epi} f$

$\Leftrightarrow f(x) \leq \xi \Leftrightarrow x \in \operatorname{lev}_{\leq \xi} f$

$\therefore \forall_{\xi \in \mathbb{R}}$ $\operatorname{lev}_{\leq \xi} f$: closed in $\mathcal{X}$.

(iii) $\Rightarrow$ (i)

given : $\forall_{\xi \in \mathbb{R}}$ $\operatorname{lev}_{\leq \xi} f$: closed in $\mathcal{X} \Leftrightarrow \forall_{\xi \in \mathbb{R}} \forall_{(\tilde{x}_a)_{a\in A} : \subseteq \operatorname{lev}_{\leq \xi} f,\, \tilde{x}_a \to \tilde{x}}\ \tilde{x} \in \operatorname{lev}_{\leq \xi} f$

/*

* Definition 1.21.
[$\mathcal{X}$: Hausdorff space, $f : \mathcal{X} \to [-\infty, +\infty]$, $x \in \mathcal{X}$]
f : lower semicontinuous at $x \overset{\text{def}}{\Leftrightarrow} \forall_{(x_a)_{a\in A} : \text{net in } \mathcal{X},\, x_a \to x}\ \underline{\lim}\, f(x_a) \geq f(x)$ # definition in terms of a net

*/

take $(x_a)_{a \in A} : \subseteq \mathcal{X}$, $x_a \to x$

want to show $\underline{\lim}\, f(x_a) \geq f(x)$

set $\mu = \underline{\lim}\, f(x_a) \in [-\infty, +\infty]$

if $\mu = +\infty \Rightarrow$ goal is trivially proved.

now consider, $\mu \neq +\infty$. $\mu \in [-\infty, +\infty)$

Fact 1.15. ($\underline{\lim}$, $\overline{\lim}$ in different light) /* Conventionally this holds for bounded net */
[$(\xi_a)_{a\in A}$: net in $[-\infty, +\infty]$]

(i) $\underline{\lim}\, \xi_a = \lim \inf_{b \in A : b \geq a} \xi_b$; $\overline{\lim}\, \xi_a = \lim \sup_{b \in A : b \geq a} \xi_b$

(ii) $(\xi_a)_{a\in A}$: posseses subnets that converge to $\underline{\lim}\, \xi_a$ and $\overline{\lim}\, \xi_a$ respectively.

(iii) $(\xi_a)_{a\in A}$: converges $\Leftrightarrow \underline{\lim}\, \xi_a = \overline{\lim}\, \xi_a$ /* in this case, $\underline{\lim}\, \xi_a = \lim \xi_a = \overline{\lim}\, \xi_a$ */

$\exists_{(x_{k(b)})_{b \in B} : \text{subnet of } (x_a)_{a\in A}}\ f(x_{k(b)}) \to \mu$

set $\xi \in \,]\mu, +\infty[\, \subseteq \mathbb{R}$

then $f(x_{k(b)})_{b\in B}$: eventually lies in $[-\infty, \xi]$

/* $(x_a)_{a\in A}$: eventually in $A \overset{\text{def}}{\leftrightarrow} \exists_{c\in A} \forall_{a\in A : a\geq c} \; x_a \in Y$ */

$\leftrightarrow \exists_{c\in B} \forall_{b\in B : b\geq c} \quad f(x_{k(b)}) \in [-\infty, \xi]$

$$\updownarrow$$

$$-\infty \leq f(x_{k(b)}) \leq \xi$$

$$\updownarrow$$

$$x_{k(b)} \in \text{lev}_{\leq\xi} f$$

$\Rightarrow \{x_{k(b)} \mid b\geq c,\, b\in B\} \subseteq \text{lev}_{\leq\xi} f$

So. $(x_{k(b)})_{b\geq c;\, b\in B}$: $\subseteq \text{lev}_{\leq\xi} f$, $x_{k(b)} \to x$, as $\forall_{\xi\in\mathbb{R}}$ $\text{lev}_{\leq\xi} f$

using : $x \in \text{lev}_{\leq\xi} f \Leftrightarrow f(x) \leq \xi \in]\mu, +\infty[$

letting $\xi \downarrow \mu$ we have $f(x) \leq \mu = \underline{\lim}\, f(x_a)$

so f: lower semicontinuous at every point in $\mathcal{X}$.

$\boxtimes$

• Example: 1.25.

$$\iota_C : \mathcal{X} \to [-\infty, \infty] : x \to \begin{cases} 0, & \text{if } x\in C \\ +\infty, & \text{else} \end{cases} .$$

ι_C : lower semi continuous $\Leftrightarrow$ C : closed

Proof: $\forall_{\xi\in\mathbb{R}}$ /* Recall $\xi\in\mathbb{R} \leftrightarrow -\infty<\xi<\infty$ */

$\text{lev}_{\leq\xi}\, \iota_C = \{x\in\mathcal{X} \mid f(x)\leq\xi\}$

$\xi<0 \to \text{lev}_{\leq\xi}\, \iota_C = \{x\in\mathcal{X} \mid f(x)<0\} = \emptyset$: closed /* empty set is both open and closed */

$\xi\in[0,\infty) \to \text{lev}_{\leq\xi}\, \iota_C = \{x\in\mathcal{X} \mid f(x)\leq 0\} = \{x\in\mathcal{X} \mid f(x)=0\}$ /* by def $f(x)\geq 0$ */

$= \{x\in\mathcal{X} \mid x\in C\} = C$: closed

$\therefore \forall_{\xi\in\mathbb{R}}$ $\text{lev}_{\leq\xi} f$: closed in $\mathcal{X} \leftrightarrow \iota_C$: lower semicontinuous

* Lemma 1.26:

$\mathcal{X}$: Hausdorff space
$(f_i)_{i\in I}$: family of lower semicontinuous functions from $\mathcal{X}$ to $[-\infty, +\infty]$

$\Rightarrow$

• $\sup_{i\in I} f_i$: lower semicontinuous

• I : finite $\Rightarrow \min_{i\in I} f_i$: lower semi continuous

* Lemma 1.27.

$\mathcal{X}$: Hausdorff space
$(f_i)_{i\in I}$: finite family of lower semicontinus functions from $(-\infty, +\infty]$
$(\alpha_i)_{i\in I}$: $\alpha_i \in \mathbb{R}_{++}$

$\Rightarrow \sum_{i\in I} \alpha_i f_i$: lower semicontinuous

* Theorem 1.28 (Weierstrass) /* fundamental tool in proving existence of solutions of optimization problem */

$\mathcal{X}$: Hausdorff space
$f : \mathcal{X} \to [-\infty, +\infty]$, lower semicontinuous
C : compact subset of $\mathcal{X}$
$C \cap \text{dom} f \neq \emptyset$

$\Rightarrow$ f achieves its infimum over C

$\text{dom} f = \{x \mid f(x) < +\infty\}$

Proof:

$\inf \underbrace{f(C)}_{\text{a set of real numbers}} = \inf_{x\in C} f(x) + \iota_C(x) \in]-\infty, +\infty[$ as $C \cap \text{dom} f \neq \emptyset$

so $\inf f(C)$ will exist, the question is whether it will exist in C, i.e., $\overset{?}{\exists}\, x\in C \;\; f(x) = \inf f(C)$

$\Rightarrow$ $\exists_{\text{minimizing sequence}}$ of $\underbrace{f \text{ over } C}_{f+I_C}$; let the sequence be $\underbrace{(x_a)_{a\in A}}_{\text{also a net in } C}$ $\therefore f(x_a) \to \inf f(C)$ /* using *Fact 1.8.1. (Existence of a minimizing sequence for finite infimum) $\inf f(C) \in \mathbb{R} \Rightarrow \exists_{(x_n)_{n\in\mathbb{N}} \subseteq C}\ f(x_n) \to \inf f(C)$ */

C: compact $\Leftrightarrow$ every net in C has a subnet that converges to a point in C /* fact 1.11 */

we can extract subnet $(x_{k(b)})_{b\in B}$ from $(x_a)_{a\in A}$ such that $x_{k(b)} \to \underbrace{x}_{\text{some point in } C}$

then. $f(x_{k(b)}) \to \inf f(C) \leq f(x)$ $\forall$ /* $x_{k(b)}$ is a subnet of the minimizing sequence. $x\in C$ so $f(x_{k(b)}) \to \inf f(C)$ */

by definition of infimum $f(x) \geq \inf f(C)$ $\forall x \in C$

now f: lower semicontinuous /* f: lower-semicontinuous at x $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{(x_a)_{a\in A}\text{: subnet in } X,\ x_a \to x}$ $f(x) \leq \underline{\lim} f(x_a)$ */

as subnet is also a net, we have $f(x) \leq \underline{\lim}\, f(x_{k(b)})$ /* $x_{k(b)} \to x$. so $\underline{\lim}\, f(x_{k(b)}) = \lim f(x_{k(b)}) = \overline{\lim}\, f(x_{k(b)})$ as for a converging sequence $\underline{\lim}$, lim, $\overline{\lim}$ are sam */

$= \lim f(x_{k(b)})$ /* $\because f(x_{k(b)}) \to \inf f(C)$ $\therefore \lim f(x_{k(b)}) = \inf f(C)$

$= \inf f(C)$

$\therefore \inf f(C) \leq f(x) \leq \inf f(C) \Leftrightarrow f(\underset{\in C}{x}) = \inf f(C)$ $\therefore f$ achieves its infimum over C. ■

* Lemma 1.29.

$\begin{pmatrix} X\text{: Hausdorff space} \\ C\text{: compact Hausdorff space} \\ \varphi: X\times C \to [-\infty,+\infty],\ \text{lower semicontinuous} \end{pmatrix} \Rightarrow$

- $(f: X\to[-\infty,+\infty] : x \mapsto \inf \varphi(x, C))$: lower semicontinuous /* f is called the marginal function */
- $\forall_{x\in X}\ f(x) = \min \varphi(x, C)$

• Definition 1.30.

(X: Hausdorff space)

$\underset{:X\to[-\infty,\infty]}{\bar{f}}$: lower semicontinuous envelope $\overset{\text{def}}{\Leftrightarrow}$ $\bar{f} = \sup\{g: X\to[-\infty,+\infty] \mid g\leq f,\ g\text{: lower semicontinuous}\}$

$= (g: (\leq f, \text{lower semicontinuous}, : X\to[-\infty,\infty]) \,//\, \sup(\cdot))$

• Lemma 1.31.

$\begin{pmatrix} X\text{: Hausdorff space} \\ f: X\to[-\infty,\infty] \end{pmatrix} \Rightarrow$

(i) $\bar{f}$: largest lower semicontinuous function majorized by f.

(ii) epi $\bar{f}$: closed

(iii) dom $f \subseteq$ dom $\bar{f} \subseteq \overline{\text{dom}}\, f$, (iv) $\forall_{x\in X}\ \bar{f}(x) = \underline{\lim}_{y\to x} f(y)$

(v) $x\in X \Rightarrow (f$: lower semicontinuous at $f \Leftrightarrow \bar{f}(x) = f(x))$

(vi) epi $\bar{f} = \overline{\text{epi}}\, f$

Chapter 1: Part 3

6:50 AM

[Sequentially closed set]

1.11. Sequential Topological Notions

$\begin{bmatrix} \mathcal{X}: \text{Hausdorff space} \\ C: \text{subset of } \mathcal{X} \end{bmatrix}$ $(C: \text{sequentially closed}) \overset{\text{def}}{\leftrightarrow} \forall_{(x_n)_{n\in\mathbb{N}}: \text{convergent, lies in } C} \left(\lim_{n\to\infty} x_n\right): \text{lies in } C$

/* example : $C := (0,1]$, consider $(1/n)_{n\in\mathbb{N}}$ which converges to 0, and $\forall_n (1/n) \in C$, but 0, the limit itself does not. So, $C = (0,1]$ is not sequentially closed. */

- A closed set is sequentially closed, but converse is not true

- Definition 1.32.

[Sequentially compact set]

$[\ \mathcal{X}: \text{Hausdorff space},\ C \subseteq \mathcal{X}\ ]$

C: sequentially compact $\overset{\text{def}}{\leftrightarrow}$

$\forall_{\text{sequence in } C}\ \exists_{\text{subsequence}}$ subsequence converges to a point in C.

$\overset{\text{def}}{\leftrightarrow} \forall_{\text{sequence in } C}$ the sequence has a sequential cluster point

/* $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ has a subsequence converging to $x \in \mathcal{X}$ $\overset{\text{def}}{\leftrightarrow}$ x: sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ */

In the notions of continuity and lower-semicontinuity replace nets by sequence, then we get sequential continuity and sequential lower semicontinuity.

[def: continuity using nets] [sec: Lower Semicontinuity]

[Sequentially Continuous operator]

- $[\ \mathcal{X}, \mathcal{Y}: \text{Hausdorff space},\ T: \mathcal{X} \to \mathcal{Y},\ x \in \mathcal{X}\ ]$

T: sequentially continuous at x $\overset{\text{def}}{\leftrightarrow} \forall_{(x_n)_{n\in\mathbb{N}}: \text{sequence in } \mathcal{X},\ x_n \to x}\ Tx_n \to Tx$

[Sequentially lower semicontinuous at x]

- f: sequentially lower semicontinuous at x $\overset{\text{def}}{\leftrightarrow} \forall_{(x_n)_{n\in\mathbb{N}}: \text{sequence in } \mathcal{X},\ x_n \to x}\ \underbrace{\underline{\lim} f(x_n)}_{\sup_{c\in\mathbb{N}} \inf_{b\in\mathbb{N},\, b\geq c} x_b} \geq f(x)$

Sequential versions of Lemma 1.12, Lemma 1.14, Lemma 1.24 :

[Lemma 1.24]

Lemma 1.33. /*[Sequentially compact set]*/

$[\ C: \text{sequentially compact subset of } \underbrace{\mathcal{X}}_{\text{Hausdorff space}}\ ]$

[Sequentially closed set]

C: sequentially closed, $\forall_{\tilde{C}: \text{sequentially closed subset of } C}\ \tilde{C}$: sequentially compact.

Lemma 1.34.

$[\ C: \text{sequentially compact subset of } \underset{\smile}{\mathcal{X}}\ ]$ /* Sequentially compact subset implies that any sequence in the subset will have a subsequence that converges to some point in that subset */

$\left[\begin{array}{l} C: \text{sequentially compact} \\ \text{subset of } \underbrace{\mathcal{X}}_{\text{Hausdorff space}} \end{array} \right]$ /* Sequentially compact subset implies that any sequence in the subset will have a subsequence that converges to some point in that subset */

$\underbrace{(x_n)_{n\in\mathbb{N}}}_{\text{sequence in } C}$: has unique sequential cluster point $x \Rightarrow x_n \to x$

[Sequential Cluster Point]

/* Recall that a sequential cluster point is the converging point of a subsequence of the original sequence. Now this theorem is saying that if we have a sequentially compact set C, in which the sequence in consideration has a unique sequential cluster point, then the sequence itself converges to that cluster point! */

<u>Lemma 1.35.</u>

$\left(\begin{array}{l} \mathcal{X}: \text{Hausdorff space} \\ f: \mathcal{X} \to [-\infty, +\infty] \end{array} \right) \Rightarrow$

(i) f: sequentially lower semicontinuous

(ii) epi f: sequentially closed in $\mathcal{X} \times \mathbb{R}$

(iii) $\forall_{\xi \in \mathbb{R}}$ $\text{lev}_{\leq \xi} f$: sequentially closed in $\mathcal{X}$

Chapter 1: Part 4

10:48 PM

1.12 Metric Spaces.

$\mathcal{X}$: metric space with metric (distance d)

$\underbrace{\dim C}_{\text{diameter of } C \subseteq \mathcal{X}} = \sup_{(x,y) \in C \times C} d(x,y)$

$\underbrace{d_C}_{\text{distance to a set } C \subseteq \mathcal{X}} : \mathcal{X} \to [0,+\infty] : x \mapsto \inf_{y \in C} d(x,C)$

$\underbrace{B(x;\rho)}_{\text{closed ball with center } x \in \mathcal{X}, \text{ radius } \rho \in \mathbb{R}_{++}} = \{y \in \mathcal{X} : d(x,y) \leq \rho\}$

$\underbrace{B(x;\rho)}_{\text{open ball}} = \{y \in \mathcal{X} : d(x,y) < \rho\}$

Metric topology of $\mathcal{X}$: Topology that admits the family of all open balls as a base

/* A subfamily $\mathcal{B}$ of topology $\mathcal{T}$ is a base $\overset{\text{def}}{\Leftrightarrow}$ $\forall_{x \in \mathcal{X}} \forall_{V \in \mathcal{V}(x)} \exists_{\tilde{B} \in \mathcal{B}} (x \in \tilde{B} \wedge \tilde{B} \subseteq V)$ */

Metrizable topological space: topology coincides with metrix topology

$\underbrace{(x_n)_{n\in\mathbb{N}}}_{\text{sequence in } \mathcal{X}}$: converges to $x \in \mathcal{X}$ $\overset{\text{def}}{\Leftrightarrow}$ $d(x_n, x) \to 0$

Fact 1.37.

[$\mathcal{X}$: metric space, $\mathcal{Y}$: Hausdorff space, $T: \mathcal{X} \to \mathcal{Y}$]

T: continuous $\overset{\text{def}}{\Leftrightarrow}$ T: sequentially continuous

[Sequentially Continuous operator]

Fact 1.38. [Sequentially compact set]

[C: subset of $\underbrace{\mathcal{X}}_{\text{metric space}}$] C: compact $\Leftrightarrow$ C: sequentially compact

Lemma 1.39.

$C : \subseteq \underbrace{\mathcal{X}}_{\text{metric space}}$, $\forall_{n\in\mathbb{N}}$ $C \cap B(0;n) \Rightarrow C$: closed

Lemma 1.40.

C: compact subset of $\underbrace{\mathcal{X}}_{\text{metric space}}$ $\overset{\text{def}}{\Leftrightarrow}$ C: closed, bounded

Lemma 1.41.

[$\mathcal{X}$: metric space

$f: \mathcal{X} \to [-\infty, +\infty]$

$x \in \mathcal{X}$

$S(x)$: set of all sequences in $\mathcal{X}$ that converge to x

] $\Rightarrow$

$\underline{\lim}_{y \to x} f(y) = \min_{(x_n)_{n\in\mathbb{N}} \in S(x)} \underline{\lim} f(x_n)$

• Lemma 1.42. (Cantor)

A sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if $d(x_m, x_n) \to 0$ as $m,n \to \infty$

($\mathcal{X}$: complete metric space /* all Cauchy sequence converges */

$(C_n)_{n\in\mathbb{N}}$: sequence of nonempty closed sets, $\forall_{n\in\mathbb{N}} C_{n+1} \subseteq C_n$, $\operatorname{diam} C_n \to 0$)

$\Rightarrow \bigcap_{n\in\mathbb{N}} C_n$: singleton

Proof:

$C = \bigcap_{n\in\mathbb{N}} C_n \subseteq C_n \ \forall_{n\in\mathbb{N}} \Rightarrow \operatorname{diam} C \leqslant \operatorname{diam} C_n \ \forall_{n\in\mathbb{N}}$; now given $\operatorname{diam} C_n \to 0 \Rightarrow 0 \leqslant \operatorname{diam} C \leqslant \operatorname{diam} C_n \ \forall_{n\in\mathbb{N}}$

$\Rightarrow 0 \leqslant \operatorname{diam} C \leqslant \lim \operatorname{diam} C_n = 0 \Leftrightarrow \boxed{\operatorname{diam} C = 0}$

/* using Result 1.2.1. $(0 \leqslant a \leqslant b_n \to 0) \Rightarrow a = 0$ */

take $\forall_{n\in\mathbb{N}} \ x_n \in C_n$

$A_n = \{x_m\}_{m\in\mathbb{N}: m\geqslant n} = \{\underbrace{x_n}_{\in C_n}, \underbrace{x_{n+1}}_{\in C_{n+1} \subseteq C_n}, x_{n+2}, \dots\} \subseteq C_n$

as $\operatorname{diam} C_n \to 0 \Rightarrow \operatorname{diam} A_n \to 0$ // $\operatorname{diam} C = \sup_{(x,y)\in C\times C} d(x,y)$

$\Rightarrow \sup\{d(x_m, x_p) \mid m,p\in\mathbb{N},\ m,p \geqslant n\} \to 0$

$\Rightarrow d(x_m, x_p) \to 0$ as $m,p \to \infty$

$\Leftrightarrow (x_n)_{n\in\mathbb{N}}$: Cauchy sequence

As, $\mathcal{X}$ is a complete metric space $\exists_{x\in\mathcal{X}} \ x_n \to x$

$\forall_{n\in\mathbb{N}} \ \forall_{p\in\mathbb{N}} \ x_{n+p} \in C_{n+p} \subseteq C_n$, $x_{n+p} \to x$ as $p \to \infty$

as in a closed set every convergent net has its limit in the set. $x \in C_n \ \therefore \forall_{n\in\mathbb{N}} \ C_n \ni x \Leftrightarrow \bigcap_{n\in\mathbb{N}} C_n = C \ni x$ $\boxed{\therefore x \in C}$

$\Rightarrow \boxed{x \in C, \ \operatorname{diam} C = 0}$

So, $C = \{x\}$

■

Lemma 1.43 (Ursescu)

[$\mathcal{X}$: complete metric space]

(i) $(C_n)_{n\in\mathbb{N}}$: sequence of closed subsets of $\mathcal{X} \Rightarrow \overline{\bigcup_{n\in\mathbb{N}} \operatorname{int} C_n} = \overline{\operatorname{int} \bigcup_{n\in\mathbb{N}} C_n}$

(ii) $(C_n)_{n\in\mathbb{N}}$: sequence of open subsets of $\mathcal{X} \Rightarrow \operatorname{int} \bigcap_{n\in\mathbb{N}} \overline{C_n} = \operatorname{int} \overline{\bigcap_{n\in\mathbb{N}} C_n}$

Corollary 1.44.

[$\mathcal{X}$: complete metric space,

$(C_n)_{n\in\mathbb{N}}$: sequence of dense open subsets of $\mathcal{X}$

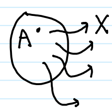
/* In topology and related areas of mathematics, a subset A of a topological space X is called dense (in X) if every point x in X either belongs to A or is a limit point of A.[1] Informally, for every point in X, the point is either in A or arbitrarily "close" to a member of A — for instance, every real number is either a rational number or has one arbitrarily close to it

From <https://en.wikipedia.org/wiki/Dense_set> */

]

$\Rightarrow \bigcap_{n\in\mathbb{N}} C_n$: dense $\underbrace{G_\delta}$ subset of $\mathcal{X}$

/* G_δ : countable intersection of open sets in Hausdorff space */



Theorem 1.45. (Ekeland)

[

$(\mathcal{X}, d)$: complete metric space

$f: \mathcal{X} \to (-\infty, +\infty]$, proper, $\underbrace{\text{lower semicontinuous}}_{\forall_{x\in\mathcal{X}} \ \forall_{(x_a)_{a\in A}: (\text{net in } \mathcal{X},\ x_a \to x)} \ \underline{\lim} f(x_a) \geqslant f(x)}$, bounded below

$\alpha \in \mathbb{R}_{++}$, $\beta \in \mathbb{R}_{++}$

$y \in \operatorname{dom} f$: $f(y) \leqslant \alpha + \inf f(\mathcal{X})$ /* y is an arbitrarily close point to the infimum */

]

$\Rightarrow$

$$\exists_{z\in\mathcal{X}} \begin{cases} \cdot\ f(z) + \frac{\alpha}{\beta} d(y,z) \leqslant f(y) \\ \cdot\ d(y,z) \leqslant \beta \\ \cdot\ \forall_{x\in\mathcal{X}\setminus\{z\}} \ f(z) < f(x) + \frac{\alpha}{\beta} d(x,z) \end{cases}$$

Definition 1.46. [Lipschitz Continuity]

[$(\mathcal{X}_1, d_1), (\mathcal{X}_2, d_2)$: metric spaces

$T: \mathcal{X}_1 \to \mathcal{X}_2$

C : subset of $\mathcal{X}_1$

]

• T: Lipschitz continuous with constant $\beta \in \mathbb{R}_+$ $\overset{\text{def}}{\longleftrightarrow}$ $\forall_{x \in \mathcal{X}_1} \forall_{y \in \mathcal{X}_1}$ $d_2(Tx, Ty) \leqslant \beta d_1(x, y)$

• T: (locally Lipschitz continuous near a point $x \in \mathcal{X}_1$) $\overset{\text{def}}{\longleftrightarrow}$ $\exists_{\rho \in \mathbb{R}_{++}}$ $T|_{B(x;\rho)}$: Lipschitz continuous

• T: (locally Lipschitz continuous near every point in C with constant $\beta \in \mathbb{R}_+$) $\overset{\text{def}}{\longleftrightarrow}$ $\forall_{x \in C} \forall_{y \in C}$ $d_2(Tx, Ty) \leqslant \beta d_1(x, y)$

Theorem 1.48 (Banach-Picard) [Convergence of contraction mapping iteration]

[$(\mathcal{X}, d)$: complete metric space

$T: \mathcal{X} \to \mathcal{X}$, β Lipschitz continuous with $\beta \in [0,1)$ /* it just means that T: contraction mapping */

$\Updownarrow$

$\forall_{x,y \in \mathcal{X}}$ $d(Tx, Ty) \leqslant \beta d(x, y)$ $\leftrightarrow$ $\|Tx - Ty\| \leqslant \beta \|x - y\|$

$x_0 \in \mathcal{X}$

$\forall_{n \in \mathbb{N}}$ $x_{n+1} = T x_n$

] $\Rightarrow$

$\exists\, x \in \mathcal{X}$:

(i) x: unique fixed point of T

(ii) $\forall_{n \in \mathbb{N}}$ $d(x_{n+1}, x) \leqslant \beta d(x_n, x)$ /* $\beta \in [0,1)$ so, the distance from the optimal point will strictly decrease */

(iii) $\forall_{n \in \mathbb{N}}$ $d(x_n, x) \leqslant \beta^n d(x_0, x)$ /* converges linearly $\log \frac{d(x_n, x)}{d(x_0, x)} \leqslant n \log \beta$

(iv) A priori error estimate: $\forall_{n \in \mathbb{N}}$ $d(x_n, x) \leqslant \frac{\beta^n}{(1-\beta)} d(x_0, x_1)$

(v) A posteriori error estimate:

$$\forall_{n \in \mathbb{N}} \quad d(x_n, x) \leqslant \frac{d(x_n, x_{n+1})}{1-\beta}$$

(vi) $\frac{d(x_0, x_1)}{1+\beta} \leqslant d(x_0, x) \leqslant \frac{d(x_0, x_1)}{1-\beta}$ /* What a beautiful inequality! It shows how the distance from the original point is bounded */

/* Proof strategy:

We want to prove given $\Rightarrow$ (i) $\wedge$ (ii) $\ldots \wedge$ (v) $\leftrightarrow$ (given $\Rightarrow$ (i)) $\wedge \cdots \wedge$ (given $\Rightarrow$ (ii))

At first we prove $\underbrace{\text{given} \Rightarrow \text{(i), then (i)} \Rightarrow \text{(ii)}}_{\text{which implies given} \Rightarrow \text{(ii)}}$ ... and so on.

proof:

(i)

The triangle inequality says:

• $d(x,y) + d(y,z) \geqslant d(x,z)$

• $d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+m-1}, x_{n+m}) \geqslant d(x_n, x_{n+m})$

So, $\forall_{m \in \mathbb{N}} \forall_{n \in \mathbb{N}}$ $d(x_n, x_{n+m}) \leqslant d(x_n, x_{n+1}) + \underbrace{d(x_{n+1}, x_{n+2})}_{\leqslant \beta d(x_n, x_{n+1})} + \underbrace{d(x_{n+2}, x_{n+3})}_{\leqslant \beta d(x_{n+1}, x_{n+2}) \leqslant \beta^2 d(x_n, x_{n+1})} + \ldots + \underbrace{d(x_{n+m-1}, x_{n+m})}_{\leqslant \beta d(x_{n+m-2}, x_{n+m-1}) \ldots \leqslant \beta^{m-1} d(x_n, x_{n+1})}$

// $d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leqslant \beta (x_n, x_{n+1})$

$\leqslant d(x_n, x_{n+1}) \underbrace{(1 + \beta + \beta^2 + \ldots + \beta^{m-1})}_{\frac{1(1-\beta^m)}{1-\beta}} = \frac{1-\beta^m}{1-\beta} d(x_n, x_{n+1}) \leqslant \frac{1}{1-\beta} d(x_n, x_{n+1})$ ... (1.68)

/* $\beta \in [0,1)$ */

/* $(1-\beta^m) \leqslant 1$, so removing it will only make the term bigger */

/* $d(x_n, x_{n+1}) \leqslant \beta d(x_{n-1}, x_n)$
$\leqslant \beta^2 d(x_{n-2}, x_{n-1})$
$\vdots$
$\leqslant \beta^n d(\underbrace{x_0}_{n-n}, x_1)$ */

$\leqslant \frac{\beta^n}{1-\beta} d(x_0, x_1)$ ... (1.69)

So, we have,

$\forall_{m \in \mathbb{N}} \forall_{n \in \mathbb{N}}$ $0 \leqslant d(x_n, x_{n+m}) \leqslant \frac{\beta^n}{1-\beta} d(x_0, x_1)$

if $n \to \infty$, then $d(x_n, x_{n+m}) \to 0$ $\therefore (x_n)_{n \in \mathbb{N}}$: Cauchy sequence

/* Recall, $(x_n)_{n \in \mathbb{N}}$: Cauchy sequence $\overset{\text{def}}{\longleftrightarrow}$ $d(x_m, x_n) \to 0$ as $m, n \to \infty$ */

Now,

$(\mathcal{X}, d)$: complete metric space /* every Cauchy sequence converges */

$\therefore (x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$.

now T: Lipschitz continuous /* recall fact 1.19. $\left(\begin{array}{l} X: \text{Hausdorff space} \\ Y: \text{Hausdorff space} \\ T: X \to Y, \text{ continuous at } x \end{array}\right) \Rightarrow \forall (x_a)_{a\in A}$: net in X, converges to x $Tx_a \to Tx$ */

$\therefore \underbrace{Tx_n}_{x_{n+1}} \to Tx$ /* $\because x_{n+1} = Tx_n$ */

/* In a Hausdorff space convergence of a sequence is always to a unique point */

$\left.\begin{array}{l} x_{n+1} \to Tx \\ \text{but } x_{n+1} \to x \end{array}\right\} \Rightarrow Tx = x \Leftrightarrow x \in \mathrm{Fix}\,T.$

• (uniqueness)

Consider $y \in \mathrm{Fix}\,T \setminus \{x\} \Leftrightarrow x \neq y \wedge Ty = y$

$d(x,y) = d(Tx,Ty) \leq \beta\, d(x,y)$ /* [Lipschitz Continuity] */

$\Leftrightarrow \beta \geq 1$ [$\because x \neq y \Rightarrow d(x,y) \neq 0$, so we can cancel]

$\downarrow$

contradiction as $\beta \in [0,1)$

$\therefore y = x$.

/* Proof strategy: for (ii)-(v):

$(A \to B) \wedge (B \to C) \Rightarrow (A \to C)$

given (i) (i) (ii) given (iii)

*/

(ii)

By Lipschitz continuity, $\forall_{n\in\mathbb{N}}\ d(\underset{Tx_n}{x_{n+1}}, \underset{Tx}{x}) = d(Tx_n, Tx) \leq \beta\, d(x_n, x)$

(iii)

in (ii) we have shown, $d(x_{n+1}, x) \leq \beta\, d(x_n, x)$

so: $d(x_n, x) \leq \beta^1 d(x_{n-1}, x) \leq \beta^2 d(x_{n-2}, x) \ldots \leq \beta^n d(x_{\underbrace{n-n}_{0}}, x)$

$d(x_{n-1}, x) \leq \beta\, d(x_{n-2}, x)$

$\vdots \qquad \vdots$

$d(x_1, x) \leq \beta\, d(x_{\underbrace{0}_{n-n}}, x)$

$\therefore \boxed{d(x_n, x) \leq \beta^n (x_0, x)}$

(iv)

in (1.69) we have:

$\forall_{m,n\in\mathbb{N}}\ d(x_n, x_{n+m}) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1) \Rightarrow \forall_{n\in\mathbb{N}} \underline{\lim}\ d(x_n, x_{n+m}) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1)$

take $m \to \infty$ then $x_{n+m} \to x \Rightarrow d(x_n, x_{n+m}) \to d(x_n, x)$ $\therefore \lim d(x_n, x_{n+m}) = \underline{\lim}\ d(x_n, x_{n+m})$

$\therefore d(x_n, x) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1)$

(v) in (1.68) we have:

$\forall_{m,n\in\mathbb{N}}\ d(x_n, x_{n+m}) \leq \frac{1-\beta^m}{1-\beta} d(x_n, x_{n+1}) \Leftrightarrow \forall_{m,n\in\mathbb{N}} \left(\frac{1}{1-\beta^m}\right) d(x_n, x_{n+m}) \leq \frac{1}{1-\beta} d(x_n, x_{n+1})$

$\Rightarrow \forall_{n\in\mathbb{N}}\ \underline{\lim} \left(\frac{1}{1-\beta^m}\right) d(x_n, x_{n+m}) \leq \frac{1}{1-\beta} d(x_n, x_{n+1})$

take $d(x_n, x_{n+m}) \to d(x_n, x)$ $\therefore \lim d(x_n, x_{n+m}) = d(x_n, x)$; $\lim \left(\frac{1}{1-\beta^m}\right) = 1$ $\therefore \lim \left(\frac{1}{1-\beta^m}\right) d(x_n, x_{n+m}) = d(x_n, x) \leq \frac{1}{1-\beta} d(x_n, x_{n+1})$

$\underline{\lim} = \overline{\lim}$

$\therefore d(x_n, x) \leq \frac{1}{1-\beta} d(x_n, x_{n+1})$

(vi) we want to show:

$$\underbrace{\frac{d(x_0,x_1)}{1+\beta} \leq d(x_0,x)}\leq \frac{d(x_0,x_1)}{1-\beta}$$

By triangle inequality, $d(x_0,x_1) \leq d(x_0,x) + \underbrace{d(x,x_1)}_{= d(x_1,x) \leq \beta\, d(x_0,x)} \leq d(x_0,x) + \beta\, d(x_0,x) = (1+\beta)\, d(x_0,x)$

/* in (ii) : $\forall_{n\in\mathbb{N}}\ d(x_{n+1},x) \leq \beta\, d(x_n,x)$ */

from (iv) we have $\forall_{n\in\mathbb{N}}\ d(x_n,x) \leq \beta^n d(x_0,x_1)/(1-\beta)$

$n=0 \Rightarrow d(x_0,x) \leq d(x_0,x_1)/(1-\beta)$

∎

• Theorem 1.49. (Banach-Picard variant)

[(X,d): complete metric space

$T: X \to X$,

$$T^n = \begin{cases} \overset{\text{identity operator}}{\tilde{I}}, & \text{if } n=0 \\ n\text{-fold composition of } T, & \text{if } n>0 \end{cases}$$

$\exists_{(\beta_n)_{n\in\mathbb{N}}}$: summable sequence in $\mathbb{R}_+$ $\left(\forall_{x,y\in X}\ \forall_{n\in\mathbb{N}}\ d(T^n x, T^n y) \leq \beta_n d(x,y)\right)$ /* note the operator T is quite complicated here */

$x_0 \in X$

$\forall_{n\in\mathbb{N}}\ x_{n+1} = Tx_n$, $\alpha_n = \sum_{k=n}^{+\infty} \beta_k$

]

$\Rightarrow$

$$\exists_{x\in X} \begin{cases} \text{(i) } x: \text{unique fixed point of } T \\ \text{(ii) } x_n \to x \\ \text{(iii) } \forall_{n\in\mathbb{N}}\ d(x_n,x) \leq \alpha_n d(x_0,x_1) \end{cases}$$

/* Proof strategy: similar to Theorem 1.48 */

Proof:

given $\forall_{n\in\mathbb{N}}\ x_{n+1} = Tx_n$, $\alpha_n = \sum_{k=n}^{\infty} \beta_k$

Let's apply triangle inequality:

$\forall_{m\in\mathbb{N}}\ \forall_{n\in\mathbb{N}}\ d(x_n,x_{n+m}) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{n+m-1},x_{n+m})$

$= \sum_{k=n}^{n+m-1} d(\underbrace{x_k}_{T^k x_0}, \underbrace{x_{k+1}}_{T^{k+1}x_0})$

$T^{k+1}x_0 = T^k(Tx_0) = T^k(x_1)$

/* given, $x_{n+1} = Tx_n$

$x_1 = Tx_0$

$x_2 = Tx_1 = T^2 x_0$

$x_3 = Tx_2 = T^3 x_0$

$\vdots$

$x_k = T^k x_0$

*/

$= \sum_{k=n}^{n+m-1} \underbrace{d(T^k x_0, T^k x_1)}_{\leq \beta_k d(x_0,x_1)}$

/* By given, $\forall_{x,y\in X}\ \forall_{n\in\mathbb{N}}\ d(T^n x, T^n y) \leq \beta_n d(x,y)$ */

$\leq \sum_{k=n}^{n+m-1} \beta_k d(x_0,x_1) = d(x_0,x_1) \underbrace{\sum_{k=n}^{n+m-1} \beta_k}_{\leq \sum_{k=n}^{\infty} \beta_k = \alpha_n} \leq d(x_0,x_1)\alpha_n = \alpha_n d(x_0,x_1)$ . . . (1.72)

$\Rightarrow \boxed{\alpha_0 = \sum_{k=0}^{\infty} \beta_k}$

Proof of (i) & (ii) :

By given $(\beta_n)_{n\in\mathbb{N}}$: summable $\Rightarrow \exists_{b\in\mathbb{R}}\ \tilde{a}_k = \sum_{j=0}^{k} \beta_j \to b \Leftrightarrow$ as $k\to+\infty$, $\tilde{a}_k = \sum_{j=0}^{k} \beta_j \to b$ now $\alpha_0 = \sum_{K=0}^{\infty} \beta_K = \underbrace{\sum_{k=0}^{n} \beta_K}_{\tilde{a}_n} + \underbrace{\sum_{k=n+1}^{\infty} \beta_K}_{\alpha_{n+1}} \Leftrightarrow \alpha_0 = \tilde{a}_n + \alpha_{n+1}$; now as $n\to\infty$, $\tilde{a}_n \to \alpha_0$ $\therefore \alpha_0 = \alpha_0 + \alpha_{n+1}$, as $n\to\infty$

$\lim \tilde{a}_k = \sum_{j=0}^{\infty} \beta_j = b = \alpha_0$

$\Rightarrow \boxed{\alpha_n \to 0 \text{ as } n\to+\infty}$

From (1.72) we have:

$\forall_{m,n\in\mathbb{N}}\ d(x_n,x_{n+m}) \leq \alpha_n d(x_0,x_1)$

As $n\to\infty$,

$d(x_n,x_{n+m}) \to 0$ /* as $\alpha_n \to 0$ as $n\to\infty$ */

/* Recall, $(x_n)_{n\in\mathbb{N}}$: Cauchy sequence $\overset{\text{def}}{\Leftrightarrow} d(x_m,x_n)\to 0$ as $m,n\to\infty$ */

$\therefore (x_n)_{n\in\mathbb{N}}$: Cauchy sequence

Because (X,d) : complete metric space, so every Cauchy sequence converges

$(x_n)_{n\in\mathbb{N}}$ converges to some point x in X. i.e., $x_n \to x$

from given, set $n=1$. then, $\forall_{x\in\mathcal{X}} \forall_{y\in\mathcal{Y}}$ $d(Tx,Ty)\leq\beta_1 d(x,y)$

$\Rightarrow T$: Lipschitz continuous

now T: Lipschitz continuous /* recall fact 1.19. $\begin{pmatrix} \mathcal{X}: \text{Hausdorff space} \\ \mathcal{Y}: \text{Hausdorff space} \\ T:\mathcal{X}\to\mathcal{Y}, \text{ continuous at } x \end{pmatrix} \Rightarrow \forall_{(x_a)_{a\in A}: \text{ net in } \mathcal{X}, \text{ converges to } x}$ $Tx_a\to Tx$ */

$\therefore \underbrace{Tx_n}_{x_{n+1}}\to Tx$ /* $\because x_{n+1}=Tx_n$ */

$x_{n+1}\to Tx$, but $x_{n+1}\to x$ $\Rightarrow$ /* In a Hausdorff space convergence of a sequence is always to a unique point */ $Tx=x \Leftrightarrow x\in \text{Fix}\,T$.

• (Uniqueness)

Consider $y\in \text{Fix}\,T\setminus\{x\} \Leftrightarrow x\neq y \wedge Ty=y$

now in given, put $\underbrace{x, y}_{\text{fixed points}}$ $\forall_{n\in\mathbb{N}}$ $d(\underbrace{T^n x}_{x}, \underbrace{T^n y}_{y})\leq\beta_n d(x,y)$ /* $T^n x = T\ldots\underbrace{(Tx)}_{x}=\ldots=x$

$T^n y = T\ldots\underbrace{(Ty)}_{y}=y$ */

$\Leftrightarrow \forall_{n\in\mathbb{N}}$ $d(x,y)\leq\beta_n d(x,y)$

$\Leftrightarrow \forall_{n\in\mathbb{N}}$ $1\leq\beta_n \Leftrightarrow \lim_{n\to\infty}\beta_n\geq 1$, but if so. then $\sum_{n=0}^{\infty}\beta_n\to\infty$. but this is not possible as $(\beta_n)_{n\in\mathbb{N}}$: summable $\Rightarrow$ contradiction

$\therefore y=x$.

(iii) in 1.72 we have shown:

$\forall_{m\in\mathbb{N}}\forall_{n\in\mathbb{N}}$ $d(x_n,x_{n+m})\leq\alpha_n d(x_0,x_1)$

But $\lim_{m\to\infty} d(x_n,\underbrace{x_{n+m}}_{\downarrow\, x \text{ as } m\to\infty}) = d(x_n,x)\leq\alpha_n d(x_0,x_1)$ $\forall_{n\in\mathbb{N}}$

$\therefore \forall_{n\in\mathbb{N}}$ $d(x_n,x)\leq\alpha_n d(x_0,x_1)$

■

Chapter 1: Part 5

2:20 PM

Lemma 1.23. $**$

$[\mathcal{X}$: Hausdorff space ; $f: \mathcal{X} \to [-\infty, +\infty]$; $x \in \mathcal{X}]$

$$\underline{\lim}_{y \to x} f(y) = \min_{\substack{(x_a)_{a\in A}: \text{net in } \mathcal{X},\\ x_a \to x}} \underline{\lim}\, f(x_a)$$

Proof:

Define $N(x) = \{ (x_a)_{a\in A} \subseteq \mathcal{X} : x_a \to x \}$

take $(x_a)_{a\in A} \in N(x)$

$\forall_{a\in A}\ \mu_a = \inf\{ f(x_b) \mid b \geq a \}$ /*

$= \inf_{b\in A: b\geq a} f(x_b)$

**
Fact 1.15 ($\underline{\lim}$, $\overline{\lim}$ in different light) /* Conventionally this holds for bounded net */
$[(\xi_a)_{a\in A}$: net in $[-\infty, +\infty]$ $]$
(i) $\underline{\lim}\, \xi_a = \lim \inf_{b\in A: b\geq a} \xi_b$; $\overline{\lim}\, \xi_a = \lim \sup_{b\in A: b\geq a} \xi_b$
(ii) $(\xi_a)_{a\in A}$: possesses subnets that converge to $\underline{\lim}\, \xi_a$ and $\overline{\lim}\, \xi_a$ respectively.
(iii) $(\xi_a)_{a\in A}$: converges $\Leftrightarrow$ $\underline{\lim}\, \xi_a = \overline{\lim}\, \xi_a$ /* in this case, $\underline{\lim}\, \xi_a = \lim \xi_a = \overline{\lim}\, \xi_a$ */
*/

now $\underline{\lim}\, f(x_a) = \lim \underbrace{\inf_{b\in A: b\geq a} f(x_b)}_{\mu_a}$

$= \lim \mu_a$ // so, $(\mu_a)_{a\in A}$: converging net
(1)

recall $\mathcal{V}(x)$ = set of all neighborhoods (open) of x

now $x_a \to x$ /* recall that

* Convergence of a net
$[(x_a)_{a\in A}$: net in Hausdorff space $\mathcal{X}$ $]$
$x_a \to x \overset{\text{def}}{\Leftrightarrow} (x_a)_{a\in A}$ lies eventually in every neighborhood of x $\overset{\text{def}}{\Leftrightarrow}$
$\forall_{V_\varepsilon(x) \in \mathcal{V}(x)}\ \exists_{b\in A}\ \forall_{a\in A: a\geq b}\ x_a \in V_\varepsilon(x)$
*/

using this:

$\forall_{V\in\mathcal{V}(x)}\ \exists_{a_V\in A}\ \forall_{\tilde{a}\in A: \tilde{a}\geq a_V}\ x_{\tilde{a}} \in V$ then

so, $\forall_{a\geq a_V}\ \mu_a = \inf_{b\in A: b\geq a} f(x_b)$ // $a \geq a_V$, $b \geq a \Rightarrow b \geq a \geq a_V \Rightarrow x_b \in V$
// so, $\{ x_b \mid b\geq a, b\in A \} \subseteq V$

$= \inf f(\tilde{x}_b)$
s.t. $\tilde{x}_b \in \{ \tilde{x}_b \mid b\geq a, b\in A \}$ /* as $V \supseteq \{ \tilde{x}_b \mid b\geq a, b\in A \}$ we have $\inf f(V) \leq \inf f(\tilde{x}_b)$
s.t. $\tilde{x}_b \in \{ \tilde{x}_b \mid b\geq a, b\in A \}$

$\geq \inf f(x_b) = \inf f(V)$
s.t. $x_b \in V$

$\therefore\ \forall_{a\geq a_V}\ \mu_a \geq \inf f(V)$ /* as $(\mu_a)_{a\in A}$: converging net $\Rightarrow$ $(\mu_a)_{a\geq a_V}$: subnet also must be converging to the same point

$\Rightarrow \lim_{a\geq a_V} \mu_a \geq \inf f(V)$ /* using net version
• $\forall_{n\in\mathbb{N}}\ a_n \leq a \Rightarrow \lim a_n \leq a$ */
(2)

from (1), (2):
$\forall_{\substack{(x_a)_{a\in A}:\\ x_a \to x}}\ \forall_{V\in\mathcal{V}(x)}\ \underbrace{\underline{\lim}\, f(x_a)}_{\text{function of } x_a} = \lim \mu_a = \lim_{a\geq a_V} \mu_a \geq \inf f(V)$

$\Rightarrow \forall_{(x_a)_{a\in A}}\ \forall_{V\in\mathcal{V}(x)}\ \underbrace{\underline{\lim}\, f(x_a) + \mathbb{I}_{x_a\to x}}_{\text{function of } x_a} \geq \underbrace{\inf f(V)}_{\text{function of } V\in\mathcal{V}(x)}$
(indicator function on the condition $x_a \to x$)

$\Rightarrow \inf \{ \underline{\lim}\, f(x_a) \mid x_a \to x \} \geq \sup_{V\in\mathcal{V}(x)} \inf f(V)$ // using: Fact 1.7.2. $[f, g: \mathcal{H} \to [-\infty, +\infty]$, $\forall_{x,y\in\mathcal{H}}\ f(x) \leq g(y)]$ $\sup f(\mathcal{H}) \leq \inf g(\mathcal{H})$

... (1.37)

define: $B = \{ (y, V) \mid y\in V, V\in\mathcal{V}(x) \}$:
we impose order on the set elements, $(\tilde{y}, \tilde{V}), (\tilde{z}, \tilde{W})$ as follows: $(\tilde{y}, \tilde{V}) \preccurlyeq (\tilde{z}, \tilde{W}) \Leftrightarrow \tilde{W} \supseteq \tilde{V}$
($\tilde{V} \in \mathcal{V}(x)$, $\tilde{W} \in \mathcal{V}(x)$)
this makes B directed

Let A be a nonempty set and let $\preccurlyeq$ be a binary relation on $A \times A$. Consider the following statements:
① ... : self-inequality
② ... : transitivity
③ ... : existence of a larger value

which allows us to define a net on B. recall that for a net $(x_a)_{\underbrace{a\in A}}$ directed set is required

now denote: $b = (y, V)$

$\forall_{b=(y,V)\in B}\ x_b := y \Rightarrow x_b \in V, V\in\mathcal{V}(x) \Rightarrow \boxed{x_b \to x}$ // as this is a subnet of $(x_a)_{a\in A}$ and $x_a \to x$ using

Fact 1.9. ★ ✱
$[(x_a)_{a\in A}$: net in Hausdorff space $\mathcal{X}$, $x_a \to \underset{\in\mathcal{X}}{x}$
$(x_{k(b)})_{b\in A}$: subnet of $(x_a)_{a\in A}$ $] \Rightarrow x_{k(b)} \to x$

$y\in V$, $V\in\mathcal{V}(x)$; similarly $\forall_{c=(z,W)}\ x_c = z$

also: $\underline{\lim}\, f(x_b) = \sup_{b\in B} \inf_{c\geq b} f(x_c)$ // By definition: $\underline{\lim}\, \xi_a = \sup_{a\in A} \inf_{b\in A: b\geq a} \xi_b$ also
$(y, V): y\in V, V\in\mathcal{V}(x)$

/* now $b = (y, V)$: $y\in V$, $V\in\mathcal{V}(x)$
$c = (z, W)$: $z\in W$, $W\in\mathcal{V}(x)$
$c \geq b \Leftrightarrow (z, W) \geq (y, V) \Leftrightarrow W \supseteq V$ */

$= \sup_{y\in V: V\in\mathcal{V}(x)} \inf_{z\in W\subseteq V} f(z)$ /* using (·), (··), (·) */

$= \sup_{y\in V: V\in\mathcal{V}(x)} \underbrace{\inf f(W)}_{\leq \inf f(V)}$ // as $W \supseteq V \Rightarrow \inf f(W) \leq \inf f(V)$

$$\leq \sup_{y \in V : V \in \mathcal{V}(x)} \inf f(V) = \sup_{V \in \mathcal{V}(x)} \inf f(V)$$

$$\therefore \forall (x_b)_{b \in B} : x_b \to x \quad \underline{\lim}\, f(x_b) \leq \sup_{y \in V : V \in \mathcal{V}(x)} \inf f(V)$$

$$\Rightarrow \inf \{ \underline{\lim}\, f(x_b) \mid x_b \to x,\ b \in B \} \leq \sup_{V \in \mathcal{V}(x)} \inf f(V) \quad \ldots (1.38)$$

Finally recall that:

$$\underline{\lim}_{y \to x} f(y) = \sup_{V \in \mathcal{V}(x)} \inf f(V) \quad \ldots (1.34)$$

Combining (1.34), (1.37), (1.38) we have:

$$\underline{\lim}_{y \to x} f(y) = \min_{(x_a)_{a \in A} : x_a \to x} \underline{\lim}\, f(x_a)$$

∎