ADMM from Operator Splitting

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Alternating direction method of multipliers (ADMM)

The required problem structure : $\begin{pmatrix} \gamma & f(x)+g(z) \\ \urcorner & Ax+Bz=c \end{pmatrix}$ ADMM problem structure

e.g. $\begin{pmatrix} \gamma & f(x) \\ \urcorner & x\in C \end{pmatrix} = \left(\gamma \;\; f(x)+I_C(x) \right) = \begin{pmatrix} \gamma & f(x)+\overset{g(z)}{\widetilde{I_C(z)}} \\ \urcorner & Ix-Iz=0 \end{pmatrix}$

ADMM algorithm: # $\lambda>0$

$$x^{k+1} = \underset{x}{\text{argmin}} \left(f(x) + \frac{\lambda}{2} \| Ax+Bz^{k}-c+u^{k} \|_2^2 \right)$$

$$z^{k+1} = \underset{z}{\text{argmin}} \left(g(z) + \frac{\lambda}{2} \| Ax^{k+1}+Bz-c+u^{k} \|_2^2 \right)$$

$$u^{k+1} = u^k + Ax^{k+1} + Bz^{k+1} - c$$ # $\lambda u^k \to \lambda u^*$ which is the optimal dual variable

*Assumption: • Argmin exists, unique

- (x^*, z^*) exists, unique
- optimal dual solution exists, though optimal dual variable might not be unique.

def: multiplier to residual mapping

* At first lets find out the MRM (Multiplier to Residual Mapping) to ADMM problem structure

We know that : in Default MRM :

$$\begin{pmatrix} \underset{x}{\gamma} & f(x) \\ \urcorner & Ax=b \end{pmatrix} \overset{L}{\leftrightarrow} L(x,y) = f(x)+y^T(Ax-b) = f(x)+(A^Ty)^Tx - y^Tb$$

MRM mapping. $F(y) = b - A\,\underset{\bar{x}}{\text{argmin}}\, L(\bar{x},y) \leftrightarrow F(y) = b - Ax \;\wedge\; \underbrace{x = \underset{\bar{x}}{\text{argmin}}\, L(\bar{x},y)}_{}$ with $\partial f(x)+A^Ty \ni 0$ above

$\therefore F : \left(F(y) = b - Ax \;\wedge\; \partial f(x) + A^Ty \ni 0 \right)$

So, for,

$$\begin{pmatrix} \gamma & f(x)+g(z) \\ \urcorner & Ax+Bz=c \end{pmatrix} = \begin{pmatrix} \gamma & \overbrace{f(x)+g(z)}^{\bar{f}(x,z)} \\ \urcorner & \underbrace{[A\;\;B]}_{\bar{A}} \underbrace{\begin{bmatrix} x \\ z \end{bmatrix}}_{\boxminus=(x,z)} = c \end{pmatrix}$$

, so $F : F(y) = c - [A\;\;B]\begin{bmatrix} x \\ z \end{bmatrix} \;\wedge\; \partial_{(x,z)} \bar{f}(x,z) + \overbrace{[A\;\;B]^T}^{\begin{bmatrix} A^T \\ B^T \end{bmatrix}} y \ni 0$

By def. $\nabla_{(x,z)} \tilde{f}(x) = \begin{bmatrix} \frac{d}{dx_1}\tilde{f}(x) \\ \vdots \\ \frac{d}{dx_n}\tilde{f}(x) \\ \frac{d}{dz_1}\tilde{f}(x) \\ \vdots \\ \frac{d}{dz_n}\tilde{f}(x) \end{bmatrix}$ with $\nabla_x \tilde{f}(x)$ for the first block and $\nabla_z \tilde{f}(x)=0$ for the second, $= \begin{bmatrix} \nabla_x \tilde{f}(x) \\ \nabla_z \tilde{f}(z) \end{bmatrix}$, similarly $\partial_{(x,z)} f(x) = \begin{bmatrix} \partial_x f(x) \\ 0 \end{bmatrix}$, $\partial_{(x,z)} g(z) = \begin{bmatrix} 0 \\ \partial_z f(z) \end{bmatrix}$

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recall: $\partial_x f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix}$

$$\therefore \partial_{(x,z)} \bar{f}(x,z) = \begin{bmatrix} \frac{\partial}{\partial x_1}\bar{f}(x,z) \\ \vdots \\ \frac{\partial}{\partial x_n}\bar{f}(x,z) \\ \frac{\partial}{\partial z_1}\bar{f}(x,z) \\ \vdots \\ \frac{\partial}{\partial z_n}\bar{f}(x,z) \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1}\bar{f}(x,z) \\ \vdots \\ \frac{\partial}{\partial x_n}\bar{f}(x,z) \end{pmatrix} \\ \begin{pmatrix} \frac{\partial}{\partial z_1}\bar{f}(x,z) \\ \vdots \\ \frac{\partial}{\partial z_n}\bar{f}(x,z) \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \partial_x \bar{f}(x,z) \\ \partial_z \bar{f}(x,z) \end{bmatrix} = \begin{bmatrix} \partial_x f(x) \\ \partial_z g(z) \end{bmatrix}$$

$\partial_x \bar{f}(x,z) = \partial_x (f(x)+g(z)) = \partial_x f(x)$
$\partial_z \bar{f}(x,z) = \partial_z (f(x)+g(z)) = \partial_z g(z)$

$$\begin{bmatrix} \partial_x f(x) \\ \partial_z g(z) \end{bmatrix} + \begin{bmatrix} A^Ty \\ B^Ty \end{bmatrix} \ni 0 \leftrightarrow \begin{pmatrix} \partial_x f(x) + A^Ty \ni 0 \\ \partial_z g(z) + B^Ty \ni 0 \end{pmatrix}$$

Our next goal is to show F is a sum of two MRM relations.

(going to show below)

$\therefore F : F(y) = c - Ax - Bz \;\wedge\; \partial_x f(x) + A^Ty \ni 0 \;\wedge\; \partial_z g(z) + B^Ty \ni 0 \leftrightarrow F(y) = c - A\,x\{\partial f(x)+A^Ty \ni 0\} - B\,z\{\partial g(z) + B^Ty \ni 0\} = c - A\,\underset{x}{\text{argmin}}\, L_1(x,y) - B\,\underset{z}{\text{argmin}}\, L_2(z,y)$

$L_1(x,y) = f(x) + y^T(Ax-c)$
$L_2(x,y) = g(z) + y^T(Bz-0)$

Obviously the optimal (x,z,y) will satisfy $F(y) \ni 0$ and we know that MRM mapping is $\underbrace{\text{maximal}}_{}$ $\underbrace{\text{monotone}}_{}$

as takes entire R^n as input without any trouble

shown earlier

def: multiplier to residual mapping

Note that F(y) is continuous in y with domain Rⁿ. We have already know F: is monotone. Now we use the fact that any continuous function with domain Rⁿ is maximal. So, F is maximal monotone.

$\begin{pmatrix} \gamma & f(x) \\ \urcorner & Ax=c \end{pmatrix}$ # underlying optimization problem for F1

underlying optimization problem can be back calculated

$L_1(x,y) = f(x) + y^T(Ax-c)$

underlying Lagrangian will be :

For the optimal (x^*, z^*) , $F(y)=0$, with these two still satisfied

let us split $F = F_1 + F_2$ such that :

$\left(F_1(y) = c - Ax \text{ where } \partial f(x) + A^Ty \ni 0 \right) \leftrightarrow F_1(y) = c - A\,\underset{x}{\text{argmin}}\, L_1(x,y)$

$\left(F_2(y) = 0 - Bz \text{ where } \partial g(z) + B^Ty \ni 0 \right) \leftrightarrow F_2(y) = 0 - B\,\underset{z}{\text{argmin}}\, L_2(z,y)$

this is the compact form of this operators, note that both.
F_1 and F_2 are multiplier to residual mappings for different underlying optimization problem,

hence they are monotone

underlying Lagrangian

$L_2(z,y) = g(z) + y^T(Bz-0)$

def: multiplier to residual mapping

underlying optimization problem

$\begin{pmatrix} \gamma & g(z) \\ \urcorner & Bz=0 \end{pmatrix}$ # underlying optimization problem for F2

So the optimal solution will satisfy :

statement: overloaded sum operator for relations has additivity

$$(F_1+F_2)(y) = F_1(y)+F_2(y)=0$$

$$\rightarrow \quad F_1(y)+F_2(y) \ni 0 ,$$

thm: main theorem behind operator splitting

now we want to show that F_1, F_2 are maximally monotone, so that we can apply Operator splitting method, subsequently Douglas-Rachford splitting.

[# eq: Douglas-Rachford splitting (Ernest's notation

Proof that F_1 and F_2 are maximal monotone. First note that F_1 is the multiplier to residual mapping relation for the optimization problem minimize f(x)
subject to Ax=c

So F_1 is monotone (as MRM operator is monotone).

Now note that F_1 is also continuous in y with domain $F_1=R^n$. So F_1 is maximal.

So F_1 is maximal monotone.

Similarly F_2 is maximal monotone.

Old Proof:

From definition of maximal monotone operator (def: maximal monotone) থেকে জানি

$$\forall_{(x,u)} \left((x,u)\in F \leftrightarrow \forall_{(y,v)\in F} (v-u)^T(x-y)\geq 0 \right)$$

// কেবলমাত্র → হলে just monotone
// additionally ← হলে maximally minotone.

now, already জানা আছে, both F_1, F_2 are monotone so → direction prove করা লাগবে না.

lets prove ← direction for F_1:

Want to prove, $\forall_{(y_1,r_1)} \left(\forall_{(y_2,r_2)\in F_1} (r_2-r_1)^T(y_2-y_1)\geq 0 \Rightarrow (y_1,r_1)\in F_1 \right)$

any (y_1,r_1)
any (y_2,r_2)

for the statement to make sense $y_1\neq y_2$, otherwise প্রমাণ করার কিছু নাই

given	goal
$(y_2,r_2)\in F_1$	$(y_1,r_1)\in F_1$
$:(r_2-r_1)^T(y_2-y_1)\geq 0$	

$$(y_2,r_2)\in F_1 \leftrightarrow F_1(y_2)\ni r_2$$

$$\leftrightarrow \exists_{x^*(y_2)} \quad (-Ax^*(y_2)=r_2 \ , \ \partial f(x^*(y_2))+A^Ty_2\ni 0$$

now $(r_2-r_1)^T(y_2-y_1) = ((-Ax^*(y_2)-r_1)^T(y_2-y_1)\geq 0$

$$\rightarrow ((-Ax^*(y_2))^T y_2 \geq r_1^T y_1$$

Per contradiction,

$$(y_1,r_1)\notin F_1 \leftrightarrow \forall_{\bar{x}^*(y_1)} \left((-A\bar{x}^*(y_1)\neq r_1 \ \vee \ \partial f(\bar{x}^*(y_1)+A^Ty_1 \not\ni 0\right)$$

$$\leftrightarrow \forall_{\bar{x}^*(y_1)} \left(\partial f(\bar{x}^*(y_1)+A^Ty_1\ni 0 \Rightarrow r_1\neq(-A\bar{x}^*(y_1)\right)$$

$$\leftrightarrow \forall_{\bar{x}^*(y_1):\ \partial f(\bar{x}^*(y_1))+A^Ty_1\ni 0} \quad \underbrace{r_1\neq(-A\bar{x}^*(y_1)}_{r_1=(-A\bar{x}^*(y_1)+\underset{\neq 0}{d} = \tilde{r}_1+d}$$

But because, $L_1(\exists,\exists)$ has an argmin by assumption $\exists_{\tilde{r}_1} \ \tilde{r}_1=(-A\bar{x}^*(y_1))$, where $\partial f(\bar{x}^*(y_1))+A^Ty_1\ni 0$

so, $(y_1,\tilde{r}_1)\in F_1 \Rightarrow (r_2-\tilde{r}_1)^T(y_2-y_1)\geq 0$

$$\underbrace{(r_2-r_1)^T}_{(r_2-\tilde{r}_1-d)^T}(y_2-y_1)\geq 0$$

$\rightarrow (r_2-\tilde{r}_1)^T(y_2-y_1)\geq d^T(y_2-y_1)$ this holds for any y_2, lets $y_2=-d_2+y_1$, and as we have assumed the argmin exists, there will be a valid r_2 too.

$y_2:=-d_2+y_1$

$$(r_2-\tilde{r}_1)^T(y_2-y_1)\geq d^T(-d)=-\|d\|_2^2<0 \quad (\text{as } d\neq 0)$$

$\therefore (r_2-\tilde{r}_1)(y_2-y_1)<0$ but this is not possible. as $(y_1,\tilde{r}_1),(y_2,r_2)\in F_1$ and $(r_2-\tilde{r}_1)(y_2-y_1)\geq 0$

$\therefore$ Contradiction so $\boxed{F_1 \text{ is maximal monotone.}}$

$\therefore$ Contradiction so $\boxed{F_1 \text{ is maximal monotone.}}$

- Similarly F_2 is also maximally monotone

[# eq: Douglas-Rachford splitting (Ernest's notation

So, both F_1, F_2 are maximal monotone → operator splitting applicable → Douglas-Rachford splitting applicable.

So, we are interested in finding y where

$$F(y) = F_1(y) + F_2(y) \ni 0 \leftrightarrow \zeta = C_{F_1}C_{F_2}(\zeta),\ y = R_{F_2}(\zeta)$$

Now the key result in operator splitting says:

$$A(x) + B(x) \ni 0 \leftrightarrow \underbrace{C_A C_B(z) = z}_{\text{intermediate variable}},\ \overset{\text{main variable}}{x} = R_B(z)$$

The associated D-R splitting will be:

$$x^{k+\frac{1}{2}} = R_B(z^k) \quad \text{1st intermediate variable}$$

$$\tilde{z}^{k+1/2} = 2x^{k+1/2} - z^k$$

Second Intermediate Variable

main variable

$$x^{k+1} = R_A(\tilde{z}^{k+1/2})$$

$$z^{k+1} = z^k + x^{k+1} - x^{k+1/2}$$

intermediate variable

main variable

Accordingly for the ADMM problem the D-R splitting should yield # $F_2 = B$, $F_1 = A$

$$y^{k+1/2} = R_{F_2}(\zeta^k) \qquad \{1\}$$

$$\tilde{\zeta}^{k+1/2} = 2y^{k+1/2} - \zeta^k \qquad \{2\}$$

$$y^{k+1} = R_{F_1}(\tilde{\zeta}^{k+1/2}) \qquad \{3\}$$

$$\zeta^{k+1} = \zeta^k + y^{k+1} - y^{k+1/2} \qquad \{4\}$$

Now, resolvent of the multiplier to residual mapping [Compact form: Resolvent of the multiplier to residual mapping

[illegible] :

Compact form for resolvent of the MRM mapping, $R = (I + \lambda F)^{-1}$

$R(\square)$ # underlying optimization problem $\forall f(\square) \rceil A\square = b$

Can be calculated by the equation:

$$\square = \underset{\square}{\operatorname{argmin}} \left(f(\square) + \square^T(A\square - b) + \frac{\lambda}{2}\|A\square - b\|_2^2\right)$$

$$R(\square) = \square + \lambda(A\square - b)$$

→ Anthromorphied form:

output $= R$(input) : R = resolvent(F) : $F(y) = b - A \underset{x}{\operatorname{argmin}} L(x,y)$

$L(x,y) = f(x) + y^T(Ax - b)$

$\forall f(x)$

$\rceil Ax = b$

minimizer $= \underset{\text{dummy}}{\operatorname{argmin}}$ $\left(f(\text{dummy}) + \text{input}^T(A\,\text{dummy} - b) + \frac{\lambda}{2}\|A\,\text{dummy} - b\|_2^2\right)$

output = input + λ(A minimizer $-b$)

remember for F_2 the underlying optimization problem is $\forall g(z) \rceil Bz = 0$ # underlying optimization problem for F2

$$y^{k+1/2} = R_{F_2}(\zeta^k) = \zeta^k + \lambda(B\square_\zeta^k - 0) \qquad \{1alt.2\}$$

$$\square_\zeta^k = \underset{\square}{\operatorname{argmin}}\left(g(\square) + \zeta^{k^T}(B\square) + \frac{\lambda}{2}\|B\square\|_2^2\right) \qquad \{1alt.1\}$$

(1alt)

Later going to use:

$\square_\zeta^k := \tilde{z}^{k+1}$,

$\square_{\tilde{\zeta}}^{k+1/2} = \tilde{x}^{k+1}$

remember for F_1 the underlying optimization problem is: $\forall f(x) \rceil Ax = c$ # underlying optimization problem for F1

$$y^{k+1} = R_{F_1}(\tilde{\zeta}^{k+1/2}) = \tilde{\zeta}^{k+1/2} + \lambda(A\square_{\tilde{\zeta}}^{k+1/2} - c) \qquad \{3alt.2\}$$

$$\square_{\tilde{\zeta}}^{k+1/2} = \underset{\square}{\operatorname{argmin}}\left(f(\square) + \tilde{\zeta}^{k+1/2^T}(A\square - c) + \frac{\lambda}{2}\|A\square - c\|_2^2\right) \qquad \{3alt.1\}$$

(3 alt)

Now recollect every equations and write them again:

$$\square_\zeta^k = \underset{\square}{\operatorname{argmin}}\left(g(\square) + \zeta^{k^T}(B\square) + \frac{\lambda}{2}\|B\square\|_2^2\right) \qquad \{1alt.1\}$$

$$y^{k+1/2} = R_{F_2}(\zeta^k) = \zeta^k + \lambda(B\square_\zeta^k - 0) \qquad \{1alt.2\}$$

$$\tilde{\zeta}^{k+1/2} = 2y^{k+1/2} - \zeta^k \qquad \{2\}$$

$$\square_{\tilde{\zeta}}^{k+1/2} = \underset{\square}{\operatorname{argmin}}\left(f(\square) + \tilde{\zeta}^{k+1/2^T}(A\square - c) + \frac{\lambda}{2}\|A\square - c\|_2^2\right) \qquad \{3alt.1\}$$

$$y^{k+1} = R_{F_1}(\tilde{\zeta}^{k+1/2}) = \tilde{\zeta}^{k+1/2} + \lambda(A\square_{\tilde{\zeta}}^{k+1/2} - c) \qquad \{3alt.2\}$$

(equationlist 1)

$$= R_{F_1}\left(\tilde{\zeta}^{k+1/2}\right) = \tilde{\zeta}^{k+1/2} + \lambda\left(A\square^{k+1/2}_{\tilde{\zeta}} - c\right) \qquad \{3alt.2\}$$

$$\zeta^{k+1} = \zeta^k + y^{k+1} - y^{k+1/2} \qquad \{4\}$$

- $\xrightarrow[\{2\}]{\{1alt.2\}} \tilde{\zeta}^{k+1/2} = 2\left(\zeta^k + \lambda\left(B\square^k_\zeta - 0\right)\right) - \zeta^k = \zeta^k + 2\lambda\left(B\square^k_\zeta\right) \qquad \{mix1\}$

- $\xrightarrow[\{3alt.2\},\{4\}]{\{1alt.2\},} \zeta^{k+1} = \zeta^k + \tilde{\zeta}^{k+1/2} + \lambda\left(A\square^{k+1/2}_{\tilde{\zeta}} - c\right) - \left(\zeta^k + \lambda\left(B\square^k_\zeta - 0\right)\right)$

$$= \cancel{\zeta^k} + \tilde{\zeta}^{k+1/2} + \lambda\left(A\square^{k+1/2}_{\tilde{\zeta}} - c\right) - \cancel{\zeta^k} - \lambda B\square^k_\zeta$$

$$= \tilde{\zeta}^{k+1/2} + \lambda\left(A\square^{k+1/2}_{\tilde{\zeta}} - B\square^k_\zeta - c\right) \qquad \{mix2\}$$

So. {equationlist 1} থেকে {1alt·2}, {2}, (3alt·2}, {4} বাদ দিয়ে {mix1}, {mix2} দিয়ে replace করতে পারি:

$$\square^k_\zeta = \underset{\boxplus}{\text{argmin}}\left(g(\boxplus) + \zeta^{k^T}(B\boxplus) + \frac{\lambda}{2}\|B\boxplus\|_2^2\right) \qquad \{1alt.1\}$$

$$\tilde{\zeta}^{k+1/2} = \zeta^k + 2\lambda\left(B\square^k_\zeta\right) \qquad \{mix1\}$$

$$\square^{k+1/2}_{\tilde{\zeta}} = \underset{\boxplus}{\text{argmin}}\left(f(\boxplus) + \tilde{\zeta}^{k+1/2^T}(A\boxplus - c) + \frac{\lambda}{2}\|A\boxplus - c\|_2^2\right) \qquad \{3alt.1\}$$

$$\zeta^{k+1} = \tilde{\zeta}^{k+1/2} + \lambda\left(A\square^{k+1/2}_{\tilde{\zeta}} - B\square^k_\zeta - c\right) \qquad \{mix2\}$$

(equationlist 1)

{mix1} কে {3alt·1}, {mix2}) বসাতে পারি:

$$\square^{k+1/2}_{\tilde{\zeta}} = \underset{\boxplus}{\text{argmin}}\left(f(\boxplus) + \left(\zeta^k + 2\lambda\left(B\square^k_\zeta\right)\right)^T(A\boxplus - c) + \frac{\lambda}{2}\|A\boxplus - c\|_2^2\right) \qquad \{mix4\}$$

$$\zeta^{k+1} = \zeta^k + 2\lambda\left(B\square^k_\zeta\right) + \lambda\left(A\square^{k+1/2}_{\tilde{\zeta}} - B\square^k_\zeta - c\right)$$

$$= \zeta^k + \lambda\left(A\square^{k+1/2}_{\tilde{\zeta}} + B\square^k_\zeta - c\right) \qquad \{mix5\}$$

So, {mix1}, {3alt·1}, {mix2} এর পরিবর্তে {mix4}, {mix5} বসাতে পারি:

$$\square^k_\zeta = \underset{\boxplus}{\text{argmin}}\left(g(\boxplus) + \zeta^{k^T}(B\boxplus) + \frac{\lambda}{2}\|B\boxplus\|_2^2\right) \qquad \{1alt.1\}$$

$$\square^{k+1/2}_{\tilde{\zeta}} = \underset{\boxplus}{\text{argmin}}\left(f(\boxplus) + \left(\zeta^k + 2\lambda\left(B\square^k_\zeta\right)\right)^T(A\boxplus - c) + \frac{\lambda}{2}\|A\boxplus - c\|_2^2\right) \qquad (mix4)$$

$$\zeta^{k+1} = \zeta^k + \lambda\left(A\square^{k+1/2}_{\tilde{\zeta}} + B\square^k_\zeta - c\right) \qquad \{mix2\}$$

(equationlist 2)

#the iteration count is upto us (unless it overwrites some other valid counter, which this one doesnot)

now lets do some variable renaming: $\square^k_\zeta := \tilde{z}^{k+1}$, corresponding minimizing variable, $\boxplus = z$, # note the associated optimization problem is $\underset{Bz=0}{\min}\ g(z)$ with variable z

$\square^{k+1/2}_{\tilde{\zeta}} = \tilde{x}^{k+1}$, 〃 〃 〃 $\boxplus = x$ # 〃 〃 〃 〃 〃 is $\underset{Ax=c}{\min}\ f(x)$ with 〃 x

then (equationlist 2) becomes:

$$\tilde{z}^{k+1} = \underset{z}{\text{argmin}}\left(g(z) + \zeta^{k^T}(Bz) + \frac{\lambda}{2}\|Bz\|_2^2\right)$$

$$\tilde{x}^{k+1} = \underset{x}{\text{argmin}}\left(f(x) + \left(\zeta^k + 2\lambda\left(B\tilde{z}^{k+1}\right)\right)^T(Ax - c) + \frac{\lambda}{2}\|Ax - c\|_2^2\right)$$

$$\zeta^{k+1} = \zeta^k + \lambda\left(A\tilde{x}^{k+1} + B\tilde{z}^{k+1} - c\right)$$

(equationlist 3)

$$\boxed{\zeta^{k+1} = \zeta^k + \lambda(A\tilde{x}^{k+1} + B\tilde{z}^{k+1} - c)}$$

Now, let us take: $\zeta^k = \lambda u^k + \lambda(A\tilde{x}^k - c)$,

$$\lambda u^{k+1} + \lambda \cancel{A\tilde{x}^{k+1}} - \cancel{\lambda} c = \lambda u^k + \lambda A\tilde{x}^k - \cancel{\lambda} c + \lambda A \cancel{\tilde{x}^{k+1}} + \lambda B\tilde{z}^{k+1} - \lambda c$$

$$\Leftrightarrow u^{k+1} = u^k + A\tilde{x}^k + B\tilde{z}^{k+1} - c$$

We also note that:

$$\zeta^{k^T}(Bz) + \frac{\lambda}{2}\|Bz\|_2^2 = (\lambda u^k + \lambda(A\tilde{x}^k - c))^T(Bz) + \frac{\lambda}{2}\|Bz\|_2^2$$

$$= \lambda(u^k + A\tilde{x}^k - c)^T(Bz) + \frac{\lambda}{2}\|Bz\|_2^2$$

$$= 2\cdot\frac{\lambda}{2}(u^k + A\tilde{x}^k - c)^T(Bz) + \frac{\lambda}{2}\|Bz\|_2^2$$

$$\tilde{z}^{k+1} = \underset{z}{\operatorname{argmin}}\left(g(z) + \zeta^{k^T}(Bz) + \frac{\lambda}{2}\|Bz\|_2^2\right)$$

$$= \underset{z}{\operatorname{argmin}}\left(g(z) + \frac{\lambda}{2}\left(\|Bz\|_2^2 + 2(u^k + A\tilde{x}^k - c)^T(Bz)\right)\right)$$

$$= \underset{z}{\operatorname{argmin}}\left(g(z) + \frac{\lambda}{2}\left(\|Bz\|_2^2 + 2(u^k + A\tilde{x}^k - c)^T(Bz) + \underbrace{\|u^k + A\tilde{x}^k - c\|_2^2}\right)\right)$$

constant w.r.t z,
so add করলে argmin change হবে না.

$$= \underset{z}{\operatorname{argmin}}\left(g(z) + \frac{\lambda}{2}\|Bz + u^k + A\tilde{x}^k - c\|_2^2\right)$$

$$= \underset{z}{\operatorname{argmin}}\left(g(z) + \frac{\lambda}{2}\|A\tilde{x}^k + Bz - c + u^k\|_2^2\right)$$

$$\left(\zeta^k + 2\lambda(B\tilde{z}^{k+1})\right)^T(Ax - c) + \frac{\lambda}{2}\|Ax - c\|_2^2)$$

$$= \left(\lambda u^k + \lambda(A\tilde{x}^k - c) + 2\lambda B\tilde{z}^{k+1}\right)^T(Ax - c) + \frac{\lambda}{2}\|Ax - c\|_2^2$$

$$= \lambda(u^k + A\tilde{x}^k - c + 2B\tilde{z}^{k+1})^T(Ax - c) + \frac{\lambda}{2}\|Ax - c\|_2^2$$

$$= \lambda(u^k + A\tilde{x}^k + B\tilde{z}^{k+1} - c + B\tilde{z}^{k+1})^T(Ax - c) + \frac{\lambda}{2}\|Ax - c\|_2^2$$

$$= \underbrace{\lambda(u^{k+1} + B\tilde{z}^{k+1})^T(Ax - c) + \frac{\lambda}{2}\|Ax - c\|_2^2}$$

$$\tilde{x}^{k+1} = \underset{x}{\operatorname{argmin}}\left(f(x) + \overbrace{\left(\zeta^k + 2\lambda(B\tilde{z}^{k+1})\right)^T(Ax - c) + \frac{\lambda}{2}\|Ax - c\|_2^2}\right)$$

$$= \underset{x}{\operatorname{argmin}}\left(f(x) + \lambda(u^{k+1} + B\tilde{z}^{k+1})^T(Ax - c) + \frac{\lambda}{2}\|Ax - c\|_2^2\right)$$

$$= \underset{x}{\operatorname{argmin}}\left(f(x) + \frac{\lambda}{2}\left(\|Ax - c\|_2^2 + 2\cdot(u^{k+1} + B\tilde{z}^{k+1})^T(Ax - c) + \underbrace{\|u^{k+1} + B\tilde{z}^{k+1}\|_2^2}_{\downarrow}\right)\right)$$

constant w.r.t x, so add করা যায়

$$= \underset{x}{\operatorname{argmin}}\left(f(x) + \frac{\lambda}{2}\|Ax - c + u^{k+1} + B\tilde{z}^{k+1}\|_2^2\right)$$

$$= \underset{x}{\operatorname{argmin}}\left(f(x) + \frac{\lambda}{2}\|Ax + B\tilde{z}^{k+1} + u^{k+1} - c\|_2^2\right)$$

So we arrive at the new iteration equations:

$$\boxed{\begin{aligned} \tilde{z}^{k+1} &= \underset{z}{\operatorname{argmin}}\left(g(z) + \frac{\lambda}{2}\|A\tilde{x}^k + Bz - c + u^k\|_2^2\right) \\ \tilde{x}^{k+1} &= \underset{x}{\operatorname{argmin}}\left(f(x) + \frac{\lambda}{2}\|Ax + B\tilde{z}^{k+1} + u^{k+1} - c\|_2^2\right) \\ u^{k+1} &= u^k + A\tilde{x}^k + B\tilde{z}^{k+1} - c \end{aligned}}$$

$$\boxed{u^{k+1} = u^k + A\tilde{x}^k + B\tilde{z}^{k+1} - c}$$

$\Big\downarrow$ = swap $u^{k+1}, \tilde{x}^{k+1}$ to get the correct dependency

$$\boxed{\begin{aligned}
\tilde{x}^k &= \operatorname*{argmin}_x \left(f(x) + \tfrac{\lambda}{2}\|Ax + B\tilde{z}^k + u^k - c\|_2^2\right.\\
\tilde{z}^{k+1} &= \operatorname*{argmin}_z \left(g(z) + \tfrac{\lambda}{2}\|A\tilde{x}^k + Bz - c + u^k\|_2^2\right)\\
u^{k+1} &= u^k + A\tilde{x}^k + B\tilde{z}^{k+1} - c\\
\tilde{x}^{k+1} &= \operatorname*{argmin}_x \left(f(x) + \tfrac{\lambda}{2}\|Ax + B\tilde{z}^{k+1} + u^{k+1} - c\|_2^2\right)
\end{aligned}}$$

→ we can consider only first three iterations

$\Big\Downarrow$ =

$$\boxed{\begin{aligned}
\tilde{x}^k &= \operatorname*{argmin}_x \left(f(x) + \tfrac{\lambda}{2}\|Ax + B\tilde{z}^k + u^k - c\|_2^2\right.\\
\tilde{z}^{k+1} &= \operatorname*{argmin}_z \left(g(z) + \tfrac{\lambda}{2}\|A\tilde{x}^k + Bz - c + u^k\|_2^2\right)\\
u^{k+1} &= u^k + A\tilde{x}^k + B\tilde{z}^{k+1} - c
\end{aligned}}$$

$\Big\downarrow$ = replace $\tilde{z}^k = z^k$, $\tilde{x}^k = x^{k+1}$ //as the iteration //number is upto us

$$\boxed{\begin{aligned}
x^{k+1} &= \operatorname*{argmin}_x \left(f(x) + \tfrac{\lambda}{2}\|Ax + Bz^k - c + u^k\|_2^2\right.\\
z^{k+1} &= \operatorname*{argmin}_z \left(g(z) + \tfrac{\lambda}{2}\|Ax^{k+1} + Bz - c + u^k\|_2^2\right)\\
u^{k+1} &= u^k + Ax^{k+1} + Bz^{k+1} - c
\end{aligned}}$$

∴ this is ADMM!

TADA!

Convergence follows immediately from convergence of ADMM.